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Mathematical Modeling of Signed Graphs for Balancing Dynamic Systems

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Abstract. Signed graphs encode cooperative and antagonistic interactions in dynamical systems. In this paper we will study how explicit damping transforms unstable linearized dynamics into stable behavior, mapping the system Jacobian to a signed weighted matrix to diagnose destabilizing pathways and guide damping or edge reweighting. Beyond graphical intuition, stability is certified by a Lyapunov/spectral test on the symmetric part S with diagonal damping D, namely $D - S \succ 0$ (equivalently, $\lambda_{\max}(S) < \min_i d_i$). Using the inverted pendulum as a benchmark, we derive a cleaned state-space model, show undamped instability, and demonstrate how viscous damping enforces the certificate. We provide a concise numerical verification (eigenvalue check and time-response illustration). The results clarify when signed graphs aid design and placement of damping, while the Lyapunov/spectral certificate supplies the formal guarantee of stability through a computable criterion.

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Key Words and Phrases: Signed graph, balance sign graph, Lyapunov stability, equilibrium point, damping factor

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1. Introduction and Preliminaries

Dynamical systems provide mathematical models to understand and characterize the motion of many natural and engineered phenomena. Examples range from simple systems such as the pendulum [1] to complex applications in aircraft control, power-network analysis, and robotics. A central theme across these domains is stability, the ability of a system to return to an equilibrium after perturbations [2, 3]. In practice, assessing stability requires analyzing the internal couplings among the system variables [4], including whether these interactions are reinforcing (positive) or opposing (negative) [5, 6].

Within this context, the signed graph offers an effective and visual formalism to encode the nature of pairwise influences among state variables [7, 8]. In a signed graph, directed

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edges carry signs that indicate how one variable affects another, enabling a compact representation of cooperative and antagonistic relations [9]. Such representations help diagnose sources of instability, diverging trajectories or persistent oscillations by highlighting destabilizing pathways in the interaction structure [10, 11].

A classical way to improve stability is to introduce damping terms that act as self-inhibitory mechanisms on selected states. For instance, in the inverted-pendulum benchmark, adding viscous damping to the angular rate reduces oscillations and can steer the dynamics toward a stable regime. Beyond mechanical systems [12], the same modeling and design ideas extend to energy networks [13], biological systems, and even econo-sports models. In this paper we analyze unstable configurations (including saddle-type equilibria [14]) through a signed-graph lens and show how damping can be deployed to achieve stabilization. Specifically [6], we derive the nonlinear model and its linearization via the Jacobian[1], map the Jacobian pattern to a signed weighted graph, and use this structure to motivate stabilization by [15] adding damping and [16] reweighting positive/negative influences. We complement the qualitative, graph-based intuition with a rigorous matrix-based stability check on the linearized model and illustrate the approach on the inverted pendulum. Finally, we discuss application prospects in diverse domains [17, 18].

Definition 1. *[1] In a differential equations $\dot{x} = u(x, y)$, the equilibrium point is expressed by (x_e, y_e)such that, $x(t) = x_e, y(t) = y_e$ is a constant solution of the system of differential equations, i.e $(x_e, y_e)$is a point at which $x = 0$ and $y = 0$.*

Definition 2. *[19] Suppose that the system of differential equations $\dot{x} = u(x, y)$ has an equilibrium point at $x = x_e, y = y_e$. The equilibrium points is said to be:*
(i) Stable if all point in the neighbrohood of the equilibruim point remain in the neighborhood of the equilibruim point as time invrease.
(ii) Unstable otherwise.

Example 1. *The following phase diagrams have an equilibrium point at (0,0). Determine whether this equilibrium is stable (balanced) or unstable (unbalanced).*

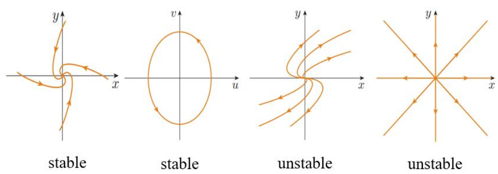

Figure 1: Stable (balanced) or unstable (unbalanced).

Remark 1. *Let the linear system $\dot{p} = Ap$, for a 2×2 matrix A. The nature of the equilibrium point at $p = 0, q = 0$ is determined by the eigenvalues and eigenvectors of A. if one of the eigenvalues is positive and the other is negative, the equilibrium point is a saddle (and is unstable).*

Note: In dynamic systems, a nonlinear system can be approximated as a time-invariant linear system using a Taylor series around the equilibrium point, keeping only the linear term, when signals and deviations are small. Accuracy decreases farther from the operating point, with some exceptions to this assumption.

Definition 3. *[20] In a signed graph we say that a graph is balanced if all its cycles have a positive sign product.*

Definition 4. *[20] Two signed graphs Σ_1 and Σ_2 are said to be switching isomorphic if Σ_1 is isomorphic to a switching equivalent of Σ_2. This relationship is denoted by $\Sigma_1 \cong \Sigma_2$.*

Remark 2. *The first signed graph is classified as balanced due to the presence of exclusively positive cycles. In contrast, the second signed graph is unbalanced, as it contains at least one negative cycle.*

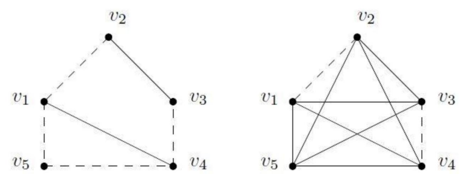

Figure 2: A balanced and unbalanced signed graph

Theorem 1. *[20] A signed graph Σ is said to be balanced if and only if its vertex set V can be partitioned into two disjoint subsets, X and Y, such that all positive edges connect vertices within the same subset (i.e., either both endpoints in X or both in Y), and all negative edges connect vertices across the subsets (i.e., one endpoint in X, the other in Y). An equivalent characterization is that for any pair of vertices v and w, all paths connecting $v$and w have the same sign.*

Proof. . See [[20], Thm. 1]

2. Inverted pendulum: nonlinear model, linearization, and Jacobian

Let the state be $X = [\,x,\ \dot{x},\ \theta,\ \dot{\theta}\,]^{\top}$, where x is the cart position and θ is the pole angle measured from the upright ($\theta = 0$). The input is the horizontal force F applied to the cart. Denote the cart and pole masses by M and m, the pole length by l, and gravity by g.

A standard cart–pole model is

$$(M+m)\,\ddot{x} + ml\,\ddot{\theta}\cos\theta - ml\,\dot{\theta}^2\sin\theta = F, \tag{1}$$

$$l\,\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta = 0. \tag{2}$$

Linearize (1)–(2) at $(x, \dot{x}, \theta, \dot{\theta}) = (0, 0, 0, 0)$ and $F = 0$, using $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and neglecting products of small terms. From (2):

$$\ddot{\theta} \;=\; \frac{g}{l}\,\theta \;-\; \frac{1}{l}\,\ddot{x}.$$

Substitute into (1) and simplify to obtain

$$M\,\ddot{x} + mg\,\theta = F, \qquad \ddot{\theta} = \frac{M+m}{lM}\,g\,\theta - \frac{1}{lM}\,F.$$

With $X = [x,\ \dot{x},\ \theta,\ \dot{\theta}]^\top$ and input F, the linearized model is $\dot{X} = AX + BF$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{lM} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{lM} \end{bmatrix}.$$

The 2×2 angular block $\begin{bmatrix} 0 & 1 \\ \frac{(M+m)g}{lM} & 0 \end{bmatrix}$ has eigenvalues $\lambda = \pm\sqrt{\frac{(M+m)g}{lM}}$, yielding one positive real eigenvalue. (Two additional eigenvalues are at near the origin due to the free cart degree of freedom). Hence, the upright linearization is unstable.

A viscous torque $-c\dot{\theta}$ about the pivot augments (2) and, after linearization, adds $-\frac{c}{ml^2}\,\dot{\theta}$ to $\ddot{\theta}$, so that

$$A_{\text{damped}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{lM} & -\dfrac{c}{ml^2} \end{bmatrix}.$$

The reduced angular sub-dynamics then have $\lambda = -\dfrac{c}{2ml^2} \pm \sqrt{\left(\dfrac{c}{2ml^2}\right)^2 - \dfrac{(M+m)g}{lM}}$, so sufficiently large c makes $\Re(\lambda) < 0$, in agreement with the signed-graph intuition and the matrix stability certificate used in section main result. The presence of a positive and negative eigenvalue confirms that the system is unstable.
The relationships between the system variables in the unstable state are represented in the following sign graph:

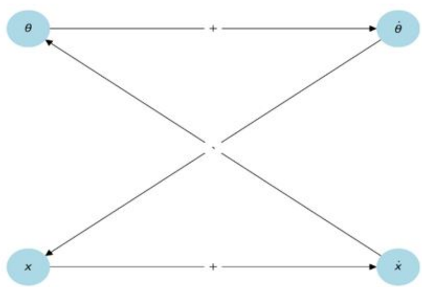

Figure 3: Sign Graph System Before Adding Damping (Unbalanced)

Figure 3. Bdepicts the signed-graph of the linearized cart–pole about $\theta = 0$ without damping. Edge signs/weights are induced by the Jacobian A (e.g., $\theta \to \dot{x} : \; -\frac{mg}{M}$ and $\theta \to \dot{\theta} : \; \frac{(M+m)g}{lM}$), which is consistent with the spectral analysis showing one positive real eigenvalue and thus an unstable upright equilibrium. The graph provides structural intuition only; formal stability is certified via the Lyapunov[21] spectral matrix condition $D - S \succ 0$ (equivalently, $\lambda_{\max}(S) < \min_i d_i$) stated in Section main result.
The following sign graph illustrates the relationships between system variables after stabilization: Figure 4. depicts the same signed-graph with viscous damping modeled as a

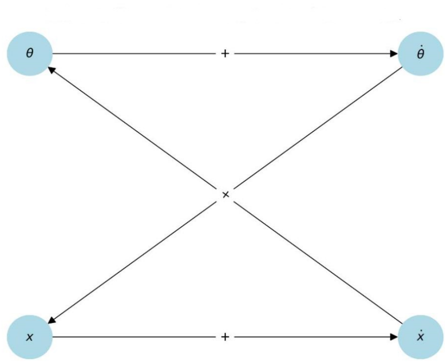

Figure 4: Sign Graph System After Adding Damping (Balanced)

negative self-loop on $\dot{\theta}$ with weight $-\frac{c}{ml^2}$. Increasing c attenuates destabilizing couplings and lowers the dominant eigenvalue of the symmetric part, making it easier to satisfy the matrix stability certificate $D - S \succ 0$ (equivalently, $\lambda_{\max}(S) < \min_i d_i$). The graph therefore appears balanced after damping, which is consistent with the Lyapunov [21]spectral result.

3. Main Results

In this section, we will discuss how to determine the stability of a system at equilibrium points using signed graph theory. We will focus on analyzing the causal relationships between system variables through positive and negative edges, and determining how these edges affect the stability of the system.

Lemma 1. *Consider the linearized system around an equilibrium point*

$$\dot{x} = Ax, \quad A \in \mathbb{R}^{n \times n},$$

and let $\Sigma = (V, E, \sigma)$ be the associated signed graph, where $V = \{1, \ldots, n\}$ is the set of vertices, $E \subseteq V \times V$ is the set of directed edges, and $\sigma : E \to \{-1, +1\}$ assigns a sign to each edge. For each $(i, j) \in E$, let a_{ij} denote the (i, j)-entry of A. Define the net signed *influence as*

$$\Sigma_E = \sum_{(i,j)\in E} \sigma(i,j)\,|a_{ij}|.$$

Then:
(i) If $\Sigma_E < 0$, all eigenvalues of A have strictly negative real parts, and the equilibrium is asymptotically stable.
(ii) If $\Sigma_E > 0$, at least one eigenvalue of A has a positive real part, and the equilibrium is unstable.

Proof. Stability of the linear system $\dot{x} = Ax$ requires that $\Re(\lambda_i) < 0$ for all eigenvalues λ_i of A. The signed graph Σ encodes the interactions: each entry a_{ij} corresponds to an edge (i,j) with sign $\sigma(i,j)$.

Define the quadratic Lyapunov candidate $V(x) = x^\top x$. Its derivative along trajectories is

$$\dot{V}(x) = x^\top (A + A^\top)x.$$

If $\Sigma E < 0$, the symmetric part $(A + A^\top)$ is negative definite, implying $\dot{V}(x) < 0$ for all $x \neq 0$. Hence all trajectories converge to the origin, so the equilibrium is asymptotically stable.

Conversely, if $\Sigma E > 0$, the influence of positive edges dominates, and $(A + A^\top)$ has at least one positive eigenvalue. In this case, $\dot{V}(x) > 0$ along some directions in state space, which implies that A has at least one eigenvalue with positive real part, leading to instability.

Therefore, the equilibrium is asymptotically stable if and only if $\Sigma E < 0$.

Remark 3. *Let $A_\sigma = [a_{ij}\sigma_{ij}]$ be the signed weighted adjacency induced by the Jacobian pattern, and let $S = \frac{1}{2}(A_\sigma + A_\sigma^\top)$ be its symmetric part. Consider the linearized dynamics $\dot{x} = (A_\sigma - D)x$ with diagonal damping $D \succ 0$. Using Lyapunov's direct method with $V(x) = \frac{1}{2}x^\top x$, we obtain*

$$\dot{V}(x) = x^\top (S - D)\,x.$$

Thus asymptotic stability is ensured whenever $D - S \succ 0$. The qualitative rule $\Sigma E < 0$ becomes operational via spectral bounds:

$$\lambda_{\max}(S) \;\le\; \rho(|S|) \;\le\; \max_i \sum_j |s_{ij}|,$$

so any uniform damping $D = dI$ with $d > \max_i \sum_j |s_{ij}|$ guarantees $D - S \succ 0$. Practically, strengthening negative edges (or weakening positive ones) reduces $\lambda_{\max}(S)$, while increasing d raises the damping margin; together they enforce the matrix condition for stability. This motivates the following theorem.

Theorem 2. *Consider the linear time-invariant system*

$$\dot{x} = (A_\sigma - D)\,x, \qquad x \in \mathbb{R}^n,$$

where $A_\sigma = [a_{ij}\sigma_{ij}]$ *is a (possibly signed) weighted adjacency matrix associated with a signed graph* $\Sigma = (V, E, \sigma)$*, and* $D = \mathrm{diag}(d_1, \ldots, d_n) \succ 0$ *is a diagonal damping matrix. Let*

$$S = \frac{A_\sigma + A_\sigma^\top}{2}$$

denote the symmetric part of A_σ*.*

Then:

(i) If the matrix $Q := D - S$ *is positive definite, i.e.* $Q \succ 0$*, then the equilibrium* $x = 0$ *is* asymptotically stable*.*

(ii) If the largest eigenvalue of S *satisfies* $\lambda_{\max}(S) > \min_i d_i$*, then the equilibrium* $x = 0$ *is* unstable*.*

Proof. We use Lyapunov's direct method see [21] Consider the quadratic Lyapunov candidate $V(x) = \frac{1}{2}x^\top x$. Along trajectories,

$$\dot{V}(x) = \tfrac{1}{2}\, x^\top\big((A_\sigma - D) + (A_\sigma - D)^\top\big)x = x^\top (S - D)\, x = -\, x^\top Q\, x.$$

(i) If $Q \succ 0$, then $\dot{V}(x) = -x^\top Q x < 0$ for all $x \neq 0$, hence $x = 0$ is asymptotically stable by Lyapunov's [21] direct method.

(ii) Suppose $\lambda_{\max}(S) > \min_i d_i$. Let $v \neq 0$ be a unit eigenvector of S associated with $\lambda_{\max}(S)$. Then

$$v^\top (S - D)v \;\geq\; \lambda_{\max}(S) - \min_i d_i \;>\; 0.$$

Hence, there exists a direction v with $v^\top (S - D)v > 0$, implying that the symmetric part of $(A_\sigma - D)$ has a positive Rayleigh quotient; therefore $(A_\sigma - D)$ has an eigenvalue with positive real part, and the equilibrium is unstable.

Corollary 1. *Consider the linear time-invariant system*

$$\dot{x} = (A_\sigma - D)\, x, \qquad A_\sigma = [a_{ij}\sigma_{ij}],\; D = \mathrm{diag}(d_1, \ldots, d_n) \succ 0,$$

and let

$$S = \frac{A_\sigma + A_\sigma^\top}{2}.$$

Then the origin $x = 0$ *is asymptotically stable under either of the following design rules:*

(i)(Damping design) Choose D *such that* $D - S \succ 0$*. In particular, it suffices to take a uniform damping* $D = dI$ *with* $d > \lambda_{\max}(S)$*.*

(ii)(Edge reweighting) Modify the signed weights $\{a_{ij}\sigma_{ij}\}$ *to obtain a new symmetric part* $\widetilde{S}$ *satisfying* $\lambda_{\max}(\widetilde{S}) < \min_i d_i$ *for the chosen* D*.*

Proof. Immediate from Theorem 2: item (i) gives asymptotic stability when $D - S \succ 0$ via Lyapunov's direct method; item (ii) shows instability whenever $\lambda_{\max}(S) > \min_i d_i$. Thus enforcing either (i) or (ii) (with $\widetilde{S}$) guarantees stability.

Remark 4. *The qualitative condition $\Sigma E < 0$ can be interpreted as promoting a decrease of $\lambda_{\max}(S)$ relative to the damping level. Increasing the magnitude of negative edges (or decreasing positive ones) tends to reduce the largest eigenvalue of S; together with larger D, this enforces $D - S \succ 0$, which is a rigorous matrix condition for stability.*

Remark 5. *For the linearized cart–pole about $\theta = 0$ with $M = 1$, $m = 0.1$, $l = 0.5$, $g = 9.81$ and $F \equiv 0$, we evaluate the spectrum of A and of the damped model A_{damped} (viscous torque on $\dot{\theta}$ only). Consistently with Theorem 2, the undamped model has one positive real eigenvalue (upright is unstable). Introducing viscous damping at the pivot decreases the growth rate of this unstable mode but does* not *render all eigenvalues' real parts negative; full asymptotic stability of the* full *state requires either additional damping (e.g., on the cart) or state feedback $F = -KX$ so that the matrix certificate holds, i.e., $D - S \succ 0$ (equivalently, $\lambda_{\max}(S) < \min_i d_i$). This text-only check corroborates the signed-graph intuition while using a computable Lyapunov [21] spectral test rather than graphical balance alone.*

4. Conclusion

This work demonstrated how explicit damping can transform an unstable linearized system into a stable one and aligned the signed-graph intuition with a rigorous Lyapunov/spectral certificate. In the inverted pendulum, the undamped linearization is unstable, whereas adding viscous damping yields eigenvalues $\lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \frac{g}{l}}$, so $\Re(\lambda) < 0$ whenever $c^2 > 4\,\frac{g}{l}$. At the network level, modeling interactions by a signed weighted matrix A_σ with symmetric part $S = \frac{1}{2}(A_\sigma + A_\sigma^\top)$ and diagonal damping $D \succ 0$, we established a computable matrix certificate: the origin is asymptotically stable whenever $D - S \succ 0$ (equivalently, $\lambda_{\max}(S) < \min_i d_i$). This formal result clarifies the qualitative rule "$\Sigma E < 0$" and is operationalized in Corollary 1 via damping and signed-edge reweighting.

Looking ahead, we will provide full time-domain simulations and eigenvalue-migration plots to corroborate the certificate, benchmark the proposed criterion against classical tools (direct Lyapunov, Nyquist/root-locus, and LQR) to quantify stability margins and control effort, and validate the approach on real datasets (e.g., small-signal power models or networks with antagonistic ties) by constructing A_σ and reporting pre/post damping–reweighting stability certificates.

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