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The Double Sawi-Shehu Transform

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Abstract. This research combines the Sawi and Shehu transforms into a unified framework called the Double Sawi-Shehu Transform. We study its core features such as when it exists and how to recover the original function. We also develop improved methods for solving partial differential equations in multiple dimensions and extend the double convolution theorem to two-dimensional problems. Practical examples show how this approach simplifies complex calculations in physics and other sciences proving its effectiveness.

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1. Introduction

Integral transforms simplify mathematical problems by converting functions into easier forms. Engineers and physicists widely use these tools to study complex phenomena leading researchers to create new transforms like the Sawi [1] and Shehu [2] transforms in recent years.

For equations with multiple variables specialized double transforms are needed. Examples include the Double Laplace Transform [3] Double Shehu Transform [**?**] and others, see [4–10]. In this work, we introduce a new transform that merges the strengths of the Sawi and Shehu transforms. This combination solves a wider range of partial and integral differential equations. Its simplicity makes it particularly useful in physics saving time and effort compared to traditional methods.

2. Sawi and Shehu transforms

This section gives a brief description and some basic properties of the single transforms: Sawi, and Shehu transforms.

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2.1. Sawi transform

Definition 1. *The Sawi transform of a continuous function* $b(\varepsilon)$ *on* $[0,\infty)$ *is defined as follows*

$$B(\sigma) = W(b(\varepsilon)) = \frac{1}{\sigma^2}\int_0^\infty e^{-\sigma\varepsilon} b(\varepsilon) d\varepsilon, \sigma > 0.$$

Some basic properties of the Sawi transform are now given.

Let $B(\sigma) = W(b(\varepsilon))$, then for nonzero constants β and γ, we have

$$W(\beta b_1(\varepsilon) + \gamma b_2(\varepsilon)) = \beta W(b_1(\varepsilon)) + \gamma W(b_2(\varepsilon)), \tag{1}$$

where $b_1(\varepsilon)$ and $b_2(\varepsilon)$ are continuous functions on $[0,\infty)$.

$$W(\varepsilon^\beta) = \Gamma(\beta+1)\sigma^{\beta-1}, \tag{2}$$

$$W(e^{\beta\varepsilon}) = \frac{1}{\sigma\left(1-\sigma\beta\right)}, \ \beta \in \mathbb{R}, \tag{3}$$

$$W(b'(\varepsilon)) = \frac{B(\sigma)}{\sigma} - \frac{b(0)}{\sigma^2}, \tag{4}$$

$$W(b''(\varepsilon)) = \frac{B(\sigma)}{\sigma^2} - \frac{b(0)}{\sigma^3} - \frac{b'(0)}{\sigma^2}. \tag{5}$$

2.2. The Shehu transform

Definition 2. *The Shehu transform of a continuous function* $p(\zeta)$ *on* $[0,\infty)$ *is defined as follows*

$$P(\phi,\delta) = H(p(\zeta)) = \int_0^\infty e^{-\frac{\phi\zeta}{\delta}} p(\zeta) d\zeta.$$

We now outline the fundamental properties of the Shehu transform.

Suppose that $P_1(\phi,\delta) = H(p_1(\zeta))$ and $P_2(\phi,\delta) = H(p_2(\zeta))$, and β and γ are nonzero real numbers, then the following properties hold:

$$H(\beta p_1(\zeta) + \gamma p_2(\zeta)) = \beta H(p_1(\zeta)) + \gamma H(p_2(\zeta)), \tag{6}$$

$$H(\zeta^\beta) = \Gamma(\beta+1)\left(\frac{\delta}{\phi}\right)^{\beta+1}, \tag{7}$$

$$H(e^{\gamma\zeta}) = \frac{\delta}{\phi - \gamma\delta}, \tag{8}$$

$$H(p'(\zeta)) = \frac{\phi}{\delta}P(\phi,\delta) - p(0), \tag{9}$$

$$H(p''(\zeta)) = \frac{\phi^2}{\delta^2}P(\phi,\delta) - \frac{\phi}{\delta}p(0) - p'(0). \tag{10}$$

3. The Double Sawi-Shehu transform

This section introduces the Double Sawi-Shehu Transformation (DSW-SHT), which combines the Sawi and Shehu transforms. The fundamental properties of this new double transform, including linearity and inversion, are presented. Additionally, new results related to partial derivatives and the convolution theorem are established. These results are implemented to compute the DSW-SHT for some basic functions. We define the DSW-SHT transform as follows:

$$U(\sigma,\phi,\delta) = W_\varepsilon H_\zeta(u(\varepsilon,\zeta)) = \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon,\zeta)\ d\varepsilon d\zeta, \tag{11}$$

where $u(\varepsilon,\zeta)$ is a continuous function on $[0,\infty)\times[0,\infty)$.

If $u(\varepsilon,\zeta)$ can be written as $u(\varepsilon,\zeta) = w(\varepsilon)x(\zeta)$ for some continuous functions w and σ, then $W_\varepsilon H_\zeta(u(\varepsilon,\zeta)) = W(w(\varepsilon))H(x(\zeta))$. In fact

$$\begin{aligned} W_\varepsilon H_\zeta(u(\varepsilon,\zeta)) &= W_\varepsilon H_\zeta(w(\varepsilon)x(\zeta)) \\ &= \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}w(\varepsilon)x(\zeta)d\varepsilon d\zeta \\ &= \left(\frac{1}{\sigma^2}\int_0^\infty e^{-\frac{\varepsilon}{\sigma}}w(\varepsilon)d\varepsilon\right)\left(\int_0^\infty e^{-\frac{\phi\zeta}{\delta}}x(\zeta)d\zeta\right) \\ &= W(w(\varepsilon))H(x(\zeta)). \end{aligned}$$

3.1. The Double Sawi-Shehu Transform for some basic functions

(i)

$$W_\varepsilon H_\zeta(1) = \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}d\varepsilon d\zeta$$

$$= \left(\frac{1}{\sigma^2}\int_0^\infty e^{-\frac{\varepsilon}{\sigma}}\right)\left(\int_0^\infty e^{-\frac{\phi\zeta}{\delta}}d\zeta\right) = \frac{1}{\sigma}\times\frac{\delta}{\phi} = \frac{\delta}{\sigma\phi},\ \mathrm{Re}(\sigma) > 0.$$

(ii)

$$W_\varepsilon H_\zeta(\varepsilon^\beta \zeta^\gamma) = \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\varepsilon^\beta\zeta^\gamma d\varepsilon d\zeta$$

$$= \left(\frac{1}{\sigma^2}\int_0^\infty \varepsilon^\beta e^{-\frac{\varepsilon}{\sigma}}d\varepsilon\right)\left(\int_0^\infty \zeta^\gamma e^{-\frac{\phi\zeta}{\delta}}d\zeta\right)$$

$$= \Gamma(\beta+1)\sigma^{\beta-1}\times\Gamma(\gamma+1)\left(\frac{\delta}{\phi}\right)^{\gamma+1}$$

$$= \frac{\sigma^{\beta-1}\delta^{\gamma+1}}{\phi^{\gamma+1}}\Gamma(\beta+1)\Gamma(\gamma+1),\ \mathrm{Re}(\sigma)>0 \text{ and } \mathrm{Re}(\beta)>-1.$$

(iii)

$$W_\varepsilon H_\zeta(e^{\beta\varepsilon+\gamma\zeta}) = \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}e^{\beta\varepsilon+\gamma\zeta}d\varepsilon d\zeta$$

$$= \left(\frac{1}{\sigma^2}\int_0^\infty e^{\beta\varepsilon-\frac{\varepsilon}{\sigma}}d\varepsilon\right)\left(\int_0^\infty e^{\gamma\zeta-\frac{\phi\zeta}{\delta}}d\zeta\right) = \frac{1}{\sigma\left(1-\sigma\beta\right)}\times\frac{\delta}{\phi-\gamma\delta}$$

$$= \frac{\delta}{\sigma\left(1-\sigma\beta\right)\left(\phi-\gamma\delta\right)},\ \mathrm{Re}(\frac{1}{\sigma})>\mathrm{Re}(\beta).$$

3.2. Existence condition for Double Sawi-Shehu Transform

Definition 3. *A function $u(\varepsilon,\zeta)$ is said to be of exponential orders β and γ on $0\le\varepsilon<\infty$ and $0\le\zeta<\infty$. If there exist $B,X,Y>0$ such that $|u(\varepsilon,\zeta)|\le Be^{\beta\varepsilon+\gamma\zeta}$, for all $\varepsilon>X$, $\zeta>Y$.*

Theorem 1. *Let $u(\varepsilon,\zeta)$ be a continuous function on the region $[0,\infty)\times[0,\infty)$ of exponential orders β and γ. Then $U(\sigma,\phi,\delta)$ exists for σ,ϕ and δ whenever $\mathrm{Re}(\frac{1}{\sigma})>\beta$ and $\mathrm{Re}\left(\frac{\phi}{\delta}\right)>\gamma$.*

Proof.

$$\begin{aligned}
|U(\sigma,\phi,\delta)| &= \left|\frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon,\zeta)\ d\varepsilon d\zeta\right| \le \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\left|u(\varepsilon,\zeta)\right|d\varepsilon d\zeta \\
&\le \frac{B}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}e^{\beta\varepsilon+\gamma\zeta}d\varepsilon d\zeta = \frac{B}{\sigma^2}\int_0^\infty e^{-(\frac{1}{\sigma}-\beta)\varepsilon}d\varepsilon\int_0^\infty e^{-(\frac{\phi}{\delta}-\gamma)\zeta}d\zeta
\end{aligned}$$

$$= \frac{B}{\sigma(1-\sigma\beta)(\frac{\phi}{\delta}-\gamma)} = \frac{B\delta}{\sigma(1-\sigma\beta)(\phi-\frac{\gamma}{\delta})}$$

where $\mathrm{Re}(\frac{1}{\sigma}) > \beta$ and $\mathrm{Re}\left(\frac{\phi}{\delta}\right) > \gamma$.

3.3. Linearity

The transform $W_\varepsilon H_\zeta(u(\varepsilon,\zeta))$ is linear transformation. In fact, for nonzero constants β and γ, we have

$W_\varepsilon H_\zeta(\beta u_1(\varepsilon,\zeta)+\gamma u_2(\varepsilon,\zeta))$

$$\begin{aligned}
&= \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\left(\beta u_1(\varepsilon,\zeta)+\gamma u_2(\varepsilon,\zeta)\right)\, d\varepsilon d\zeta,\\
&= \beta\times\frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u_1(\varepsilon,\zeta)\, d\varepsilon d\zeta + \gamma\times\frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u_2(\varepsilon,\zeta)\, d\varepsilon d\zeta\\
&= \beta W_\varepsilon H_\zeta(u_1(\varepsilon,\zeta)) + \gamma W_\varepsilon H_\zeta(u_2(\varepsilon,\zeta)).
\end{aligned}$$

4. Properties of the Double Sawi-Shehu Transform

Now, we present some basic properties of the DSW-SHT

4.1. Derivatives properties

Let $U(\sigma,\phi,\delta) = W_\varepsilon H_\zeta(u(\varepsilon,\zeta))$. Then

(i)
$$W_\varepsilon H_\zeta\left(\frac{\partial u(\varepsilon,\zeta)}{\partial\varepsilon}\right) = \frac{U(\sigma,\phi,\delta)}{\sigma} - \frac{H(u(0,\zeta))}{\sigma^2}, \tag{12}$$

(ii)
$$W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon^2}\right) = \frac{U(\sigma,\phi,\delta)}{\sigma^2} - \frac{H(u(0,\zeta))}{\sigma^3} - \frac{H(u_\varepsilon(0,\zeta))}{\sigma^2}, \tag{13}$$

(iii)
$$W_\varepsilon H_\zeta\left(\frac{\partial u(\varepsilon,\zeta)}{\partial\zeta}\right) = \frac{\phi}{\delta}U(\sigma,\phi,\delta) - W(u(\varepsilon,0)), \tag{14}$$

(iv)
$$W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\zeta^2}\right) = \frac{\phi^2}{\delta^2}U(\sigma,\phi,\delta) - \frac{\phi}{\delta}W(u(\varepsilon,0)) - W(u_\zeta(\varepsilon,0)), \tag{15}$$

(v)

$$W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon\partial\zeta}\right)=\frac{\phi}{\sigma\delta}U(\sigma,\phi,\delta)-\frac{1}{\sigma}W(u(\varepsilon,0))-\frac{\phi}{\sigma^2\delta}H(u(0,\zeta))+\frac{1}{\sigma^2}u(0,0). \quad (16)$$

Proof. (1) $W_\varepsilon H_\zeta\left(\frac{\partial u(\varepsilon,\zeta)}{\partial\varepsilon}\right)=\frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\frac{\partial u(\varepsilon,\zeta)}{\partial\varepsilon}\ d\varepsilon d\zeta$
$=\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}\frac{\partial u(\varepsilon,\zeta)}{\partial\varepsilon}d\varepsilon d\zeta.$
By integrating by parts, we get
$W_\varepsilon H_\zeta\left(\frac{\partial u(\varepsilon,\zeta)}{\partial\varepsilon}\right)=\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}\left(-u(0,\zeta)+\frac{1}{\sigma}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}u(\varepsilon,\zeta)\ d\varepsilon\right)d\zeta$
$=-\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}u(0,\zeta)d\zeta+\frac{1}{\sigma^2}\times\frac{1}{\sigma}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon,\zeta)\ d\varepsilon d\zeta$
$=\frac{U(\sigma,\phi,\delta)}{\sigma}-\frac{H(u(0,\zeta))}{\sigma^2}.$

(2) $W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon^2}\right)=\frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon^2}\ d\varepsilon d\zeta=\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon^2}d\varepsilon d\zeta.$

By integrating by parts, we get

$W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon^2}\right)=\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}\left(-u_\varepsilon(0,\zeta)-\frac{1}{\sigma}u(0,\zeta)+\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}u(\varepsilon,\zeta)d\varepsilon\right)d\zeta$
$=-\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}u_\varepsilon(0,\zeta)d\zeta-\frac{1}{\sigma^3}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}u(0,\zeta)d\zeta+\frac{1}{\sigma^2}\times\frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon,\zeta)d\varepsilon d\zeta$
$=\frac{U(\sigma,\phi,\delta)}{\sigma^2}-\frac{H(u(0,\zeta))}{\sigma^3}-\frac{H(u_\varepsilon(0,\zeta))}{\sigma^2}.$

(3) $W_\varepsilon H_\zeta\left(\frac{\partial u(\varepsilon,\zeta)}{\partial\zeta}\right)=\frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\frac{\partial u(\varepsilon,\zeta)}{\partial\zeta}\ d\varepsilon d\zeta=\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}\frac{\partial u(\varepsilon,\zeta)}{\partial\zeta}d\zeta d\varepsilon.$

By integrating by parts, we get

$W_\varepsilon H_\zeta\left(\frac{\partial u(\varepsilon,\zeta)}{\partial\zeta}\right)=\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}\left(-u(\varepsilon,0)+\frac{\phi}{\delta}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}u(\varepsilon,\zeta)d\zeta\right)d\varepsilon$
$=-\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}u(\varepsilon,0)d\varepsilon+\frac{\phi}{\delta}\times\frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon,\zeta)\ d\zeta d\varepsilon$
$=\frac{\phi}{\delta}U(\sigma,\phi,\delta)-W(u(\varepsilon,0)).$

(4) $W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\zeta^2}\right)=\frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\zeta^2}\ d\varepsilon d\zeta=\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\zeta^2}d\zeta d\varepsilon.$

By integrating by parts, we get

$W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\zeta^2}\right) = \frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}\left(-u_\zeta(\varepsilon,0) - \frac{\phi}{\delta}u(\varepsilon,0) + \frac{\phi^2}{\delta^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}u(\varepsilon,\zeta)d\zeta\right)d\varepsilon$

$= -\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}u_\zeta(\varepsilon,0)d\varepsilon - \frac{\phi}{\delta}\times\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}u(\varepsilon,0)d\varepsilon + \frac{\phi^2}{\delta^2}\times\frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon,\zeta)d\zeta d\varepsilon$

So, $W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\zeta^2}\right) = \frac{\phi^2}{\delta^2}U(\sigma,\phi,\delta) - \frac{\phi}{\delta}W(u(\varepsilon,0)) - W(u_\zeta(\varepsilon,0))$.

(5) $W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon\partial\zeta}\right) = \frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon\partial\zeta}d\varepsilon d\zeta = \frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon\partial\zeta}d\varepsilon d\zeta$

By integrating by parts, we get

$W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon\partial\zeta}\right) = \frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}\left(-u_\zeta(0,\zeta) + \frac{1}{\sigma}\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}}u_\zeta(\varepsilon,\zeta)\ d\varepsilon\right)d\zeta$

$= -\frac{1}{\sigma^2}\int\limits_0^\infty e^{-\frac{\phi\zeta}{\delta}}u_\zeta(0,\zeta)d\zeta + \frac{1}{\sigma^2}\times\frac{1}{\sigma}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u_\zeta(\varepsilon,\zeta)d\varepsilon d\zeta$

$= -\frac{1}{\sigma^2}H(u_\zeta(0,\zeta)) + \frac{1}{\sigma}W_\varepsilon H_\zeta\left(u_\zeta(\varepsilon,\zeta)\right)$

Using Equations 9 and 14, we get

$W_\varepsilon H_\zeta\left(\frac{\partial^2 u(\varepsilon,\zeta)}{\partial\varepsilon\partial\zeta}\right) = \frac{\phi}{\sigma\delta}U(\sigma,\phi,\delta) - \frac{1}{\sigma}W(u(\varepsilon,0)) - \frac{\phi}{\sigma^2\delta}H(u(0,\zeta)) + \frac{1}{\sigma^2}u(0,0)$.

4.2. Convolution Theorem of DSW-SHT

The Heaviside unit step function $M(\varepsilon,\zeta)$ defined as

$$M(\varepsilon-\beta,\zeta-\gamma) = \begin{cases} 1, & \varepsilon > \beta \text{ and } \zeta > \gamma \\ 0, & \text{otherwise} \end{cases}$$

Then we have the following lemma

Lemma 1. $W_\varepsilon H_\zeta(u(\varepsilon-\beta,\zeta-\gamma)M(\varepsilon-\beta,\zeta-\gamma)) = e^{-\frac{\beta}{\sigma}-\frac{\phi\gamma}{\delta}}W_\varepsilon H_\zeta(u(\varepsilon,\zeta)$

Proof. We have

$$W_\varepsilon H_\zeta(u(\varepsilon-\beta,\zeta-\gamma)M(\varepsilon-\beta,\zeta-\gamma))$$

$$= \frac{1}{\sigma^2}\int\limits_0^\infty\int\limits_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon-\beta,\zeta-\gamma)M(\varepsilon-\beta,\zeta-\gamma)d\varepsilon d\zeta$$

$$= \frac{1}{\sigma^2}\int\limits_\beta^\infty\int\limits_\gamma^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon-\beta,\zeta-\gamma)d\varepsilon d\zeta. \tag{17}$$

Now, by making the substitution $s = \varepsilon - \beta$ and $r = \zeta - \gamma$, equation (17) becomes:

$$\begin{aligned} W_\varepsilon H_\zeta(u(\varepsilon-\beta,\zeta-\gamma)M(\varepsilon-\beta,\zeta-\gamma)) &= \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{(s+\beta)}{\sigma}-\frac{\phi(r+\gamma)}{\delta}}u(s,r)dsdr \\ &= e^{-\frac{\beta}{\sigma}-\frac{\phi\gamma}{\delta}}W_\varepsilon H_\zeta(u(\varepsilon,\zeta)). \end{aligned}$$

Definition 4. *Let $u(\varepsilon,\zeta)$ and $p(\varepsilon,\zeta)$ be continuous functions. We define the convolution in the DSW-SHT as*

$$(u**p)(\varepsilon,\zeta) = \int_0^\varepsilon\int_0^\zeta u(\varepsilon-\beta,\zeta-\gamma)p(\beta,\gamma))d\beta d\gamma.$$

In the following theorem, we compute DSW-SHT of the convolution of two functions

Theorem 2. *Let $U(\sigma,\phi,\delta) = W_\varepsilon H_\zeta(u(\varepsilon,\zeta))$ and $P(\sigma,\phi,\delta) = W_\varepsilon H_\zeta(p(\varepsilon,\zeta))$. Then*

$$W_\varepsilon H_\zeta((u**p)(\varepsilon,\zeta)) = \sigma^2 U(\sigma,\phi,\delta)P(\sigma,\phi,\delta).$$

Proof.

$W_\varepsilon H_\zeta((u**p)(\varepsilon,\zeta))$

$$\begin{aligned} &= \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}(u**p)(\varepsilon,\zeta)d\varepsilon d\zeta \\ &= \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\left(\int_0^\varepsilon\int_0^\zeta u(\varepsilon-\beta,\zeta-\gamma)p(\beta,\gamma))d\beta d\gamma\right)d\varepsilon d\zeta. \end{aligned} \tag{18}$$

Using the Heaviside unit step function, we can write equation (18) as

$W_\varepsilon H_\zeta((u**p)(\varepsilon,\zeta))$

$$\begin{aligned} &= \frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}\left(\int_0^\infty\int_0^\infty u(\varepsilon-\beta,\zeta-\gamma)M(\varepsilon-\beta,\zeta-\gamma)p(\beta,\gamma))d\beta d\gamma\right)d\varepsilon d\zeta \\ &= \int_0^\infty\int_0^\infty p(\beta,\gamma)\left(\frac{1}{\sigma^2}\int_0^\infty\int_0^\infty e^{-\frac{\varepsilon}{\sigma}-\frac{\phi\zeta}{\delta}}u(\varepsilon-\beta,\zeta-\gamma)M(\varepsilon-\beta,\zeta-\gamma)d\varepsilon d\zeta\right)d\beta d\gamma \end{aligned}$$

So by Lemma 1, we have

$$W_\varepsilon H_\zeta((u**p)(\varepsilon,\zeta)) = U(\sigma,\phi,\delta)\int_0^\infty\int_0^\infty p(\beta,\gamma)e^{-\frac{\beta}{\sigma}-\frac{\phi\gamma}{\delta}}d\beta d\gamma$$

$$= \sigma^2 U(\sigma,\phi,\delta)P(\sigma,\phi,\delta).$$

In Table 1, we have the DSW-SHT of some basic functions

Table 1: Table of the Double Sawi-Shehu Transform

$u(\varepsilon,\zeta)$	$W_\varepsilon H_\zeta(u(\varepsilon,\zeta))$
$w(\varepsilon)x(\zeta)$	$W(w(\varepsilon))H(x(\zeta))$
1	$\frac{\delta}{\sigma\phi}$, $\text{Re}(\sigma)>0$
$\varepsilon^\beta\zeta^\gamma$	$\frac{\sigma^{\beta-1}\delta^{\gamma+1}}{\phi^{\gamma+1}}\Gamma(\beta+1)\Gamma(\gamma+1)$, $\text{Re}(\sigma)>0$ and $\text{Re}(\beta)>-1$
$e^{\beta\varepsilon+\gamma\zeta}$	$\frac{\delta}{\sigma(1-\sigma\beta)(\phi-\gamma\delta)}$, $\text{Re}(\frac{1}{\sigma})>\text{Re}(\beta)$
$e^{i(\beta\varepsilon+\gamma\zeta)}$	$\frac{\delta}{\sigma(1-i\sigma\beta)(\phi-i\gamma\delta)}$, $\text{Im}(\beta)+\text{Re}(\frac{1}{\sigma})>0$
$\sin(\beta\varepsilon+\gamma\zeta)$	$\frac{\delta(\sigma\phi\beta+\delta\gamma)}{\sigma(1+\sigma^2\beta^2)(\phi^2+\gamma^2\delta^2)}$, $\lvert\text{Im}(\beta)\rvert<\text{Re}(\frac{1}{\sigma})$
$\cos(\beta\varepsilon+\gamma\zeta)$	$\frac{\delta(\phi-\sigma\delta\beta\gamma)}{\sigma(1+\sigma^2\beta^2)(\phi^2+\gamma^2\delta^2)}$, $\lvert\text{Im}(\beta)\rvert<\text{Re}(\frac{1}{\sigma})$
$\sinh(\beta\varepsilon+\gamma\zeta)$	$\frac{\delta(\sigma\phi\beta+\delta\gamma)}{\sigma(\sigma^2\beta^2-1)(\phi^2-\gamma^2\delta^2)}$, $\text{Re}(\frac{1}{\sigma})>\text{Re}(\beta)$ and $\text{Re}(\frac{1}{\sigma}+\beta)>0$
$\cosh(\beta\varepsilon+\gamma\zeta)$	$\frac{\delta(\phi+\sigma\delta\beta\gamma)}{\sigma(\sigma^2\beta^2-1)(\phi^2-\gamma^2\delta^2)}$, $\text{Re}(\frac{1}{\sigma})>\text{Re}(\beta)$ and $\text{Re}(\frac{1}{\sigma}+\beta)>0$
$J_0\left(c\sqrt{\varepsilon\zeta}\right)$	$\frac{4\delta}{\sigma(4\phi+c^2\sigma\delta)}$, $\text{Re}\left(\frac{1}{\sigma}+\frac{c^2\delta}{4\phi}\right)>0$
$u(\varepsilon-\beta,\zeta-\gamma)M(\varepsilon-\beta,\zeta-\gamma)$	$e^{-\frac{\beta}{\sigma}-\frac{\phi\gamma}{\delta}}W_\varepsilon H_\zeta(u(\varepsilon,\zeta))$
$(u**p)(\varepsilon,\zeta)$	$\sigma^2 W_\varepsilon H_\zeta(u(\varepsilon,\zeta))W_\varepsilon H_\zeta(p(\varepsilon,\zeta))$

5. Applications

In this section, we use the DSW-SHT for solving PDEs and Integro PDEs

Example 1. *Consider the Advection-Diffusion equation*

$$u_\zeta + u_{\varepsilon\varepsilon} = u_\varepsilon + 2, \text{ where } \varepsilon,\zeta \geq 0, \tag{19}$$

With initial conditions(ICs)
$u(\varepsilon,0) = -e^\varepsilon$,
and boundary conditions(BCs)
$u(0,\zeta) = 2\zeta - 1$, $u_\varepsilon(0,\zeta) = -1$.

Solution 1. *By applying the single Sawi transform to the ICs and the single Shehu transform to the BCs, we get*
$W(u(\varepsilon,0)) = \frac{-1}{\sigma(1-\sigma)}$, $H(u(0,\zeta)) = \frac{2\delta^2}{\phi^2} - \frac{\delta}{\phi}$, $H(u_\varepsilon(0,\zeta)) = -\frac{\delta}{\phi}$.
Apply the DSW-SHT to Equation 19*, we get*

$$\frac{\phi}{\delta}U(\sigma,\phi,\delta) - W(u(\varepsilon,0)) + \frac{U(\sigma,\phi,\delta)}{\sigma^2}$$

$$
\begin{aligned}
&-\frac{H(u(0,\zeta))}{\sigma^3}-\frac{H(u_\varepsilon(0,\zeta))}{\sigma^2}\\
=\quad &\frac{U(\sigma,\phi,\delta)}{\sigma}-\frac{H(u(0,\zeta))}{\sigma^2}+\frac{2\delta}{\sigma\phi}.
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{\sigma^2\phi+\delta-\sigma\delta}{\sigma^2\delta}\times U(\sigma,\phi,\delta) \quad = &\\
&\frac{-1}{\sigma\left(1-\sigma\right)}+\frac{1}{\sigma^3}\times\left(\frac{2\delta^2}{\phi^2}-\frac{\delta}{\phi}\right)-\frac{\delta^2}{\phi}\\
&-\frac{1}{\sigma^2}\times\left(\frac{2\delta^2}{\phi^2}-\frac{\delta}{\phi}\right)+\frac{2\delta}{\sigma\phi}.
\end{aligned}
$$

By simplifying, we get,

$$
U(\sigma,\phi,\delta)=\frac{2\delta^2}{\sigma\phi^2}-\frac{\delta}{\sigma\left(1-\sigma\right)\phi}.
$$

Therefore,

$$
u(\varepsilon,\zeta)=W_\varepsilon^{-1}H_\zeta^{-1}\left(\frac{2\delta^2}{\sigma\phi^2}-\frac{\delta}{\sigma\left(1-\sigma\right)\phi}\right)=2\zeta-e^{\varepsilon}.
$$

Its graph is

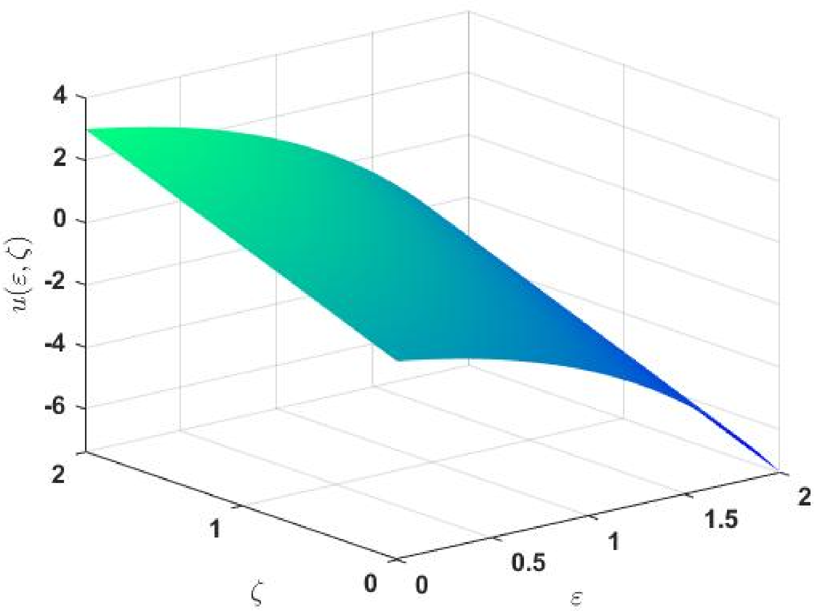

Figure 1: The solution of Example 1

Example 2. *Consider the telegraph equation*

$$u_{\varepsilon\varepsilon} - u_{\varepsilon} + u_{\zeta\zeta} = u(\varepsilon, \zeta), \text{ where } \varepsilon, \zeta \geq 0, \tag{20}$$

With ICs
$u(\varepsilon, 0) = 0,\ u_{\zeta}(\varepsilon, 0) = e^{2\varepsilon}$,
and BCs
$u\left(0, \zeta\right) = \sin\zeta,\ u_{\varepsilon}\left(0, \zeta\right) = 2\sin\zeta$.

Solution 2. *By applying the single Sawi transform and the single Shehu transform to the ICs, we get*
$W\left(u(\varepsilon, 0)\right) = 0,\ W\left(u_{\zeta}(\varepsilon, 0)\right) = \frac{1}{\sigma(1-2\sigma)},\ H\left(u\left(0, \zeta\right)\right) = \frac{\delta^2}{\phi^2+\delta^2}$,
$H\left(u_{\varepsilon}\left(0, \zeta\right)\right) = \frac{2\delta^2}{\phi^2+\delta^2}$.

Apply the DSW-SHT to Equation 20, *we get*

$$\begin{aligned}
&\frac{U(\sigma, \phi, \delta)}{\sigma^2} - \frac{H(u(0, \zeta))}{\sigma^3} - \frac{H(u_{\varepsilon}(0, \zeta))}{\sigma^2} - \frac{U(\sigma, \phi, \delta)}{\sigma} \\
&+\frac{H(u(0, \zeta))}{\sigma^2} + \frac{\phi^2}{\delta^2}U(\sigma, \phi, \delta) - \frac{\phi}{\delta}W(u(\varepsilon, 0)) \\
&-W(u_{\zeta}(\varepsilon, 0)) = U.
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\delta^2 - \sigma\delta^2 + \sigma^2\phi^2 - \sigma^2\delta^2}{\sigma^2\delta^2} \times U(\sigma, \phi, \delta) &= \\
&\frac{\delta^2}{\sigma^3\left(\phi^2+\delta^2\right)} + \frac{\delta^2}{\sigma^2\left(\phi^2+\delta^2\right)} + \frac{1}{\sigma\left(1-2\sigma\right)}.
\end{aligned}$$

By simplifying, we get,

$$U(\sigma, \phi, \delta) = \frac{\delta^2}{\sigma\left(1-2\sigma\right)\left(\phi^2+\delta^2\right)}.$$

Therefore,

$$u(\varepsilon, \zeta) = W_{\varepsilon}^{-1}H_{\zeta}^{-1}\left(\frac{\delta^2}{\sigma\left(1-2\sigma\right)\left(\phi^2+\delta^2\right)}\right) = e^{2\varepsilon}\sin\zeta.$$

Its graph is

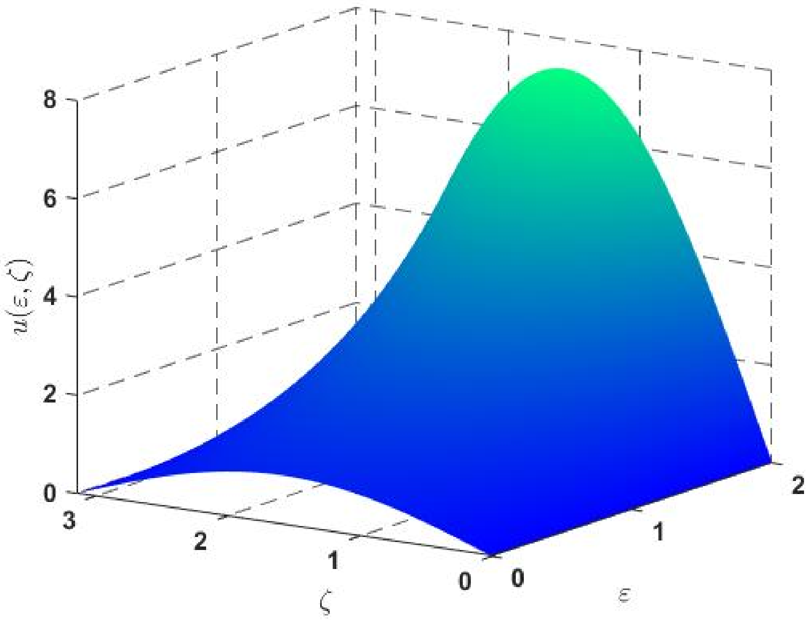

Figure 2: The solution of Example 2

Example 3. *Consider the equation of Volterra Integro PDE.*

$$u_\varepsilon + u_\zeta - \cosh\varepsilon\cos\zeta + e^\varepsilon\sin\zeta - \sin\zeta = \int_0^\varepsilon\int_0^\zeta u(\gamma,\delta))d\gamma d\delta, \text{ where } \varepsilon,\zeta \geq 0, \tag{21}$$

With ICs
$u(\varepsilon,0) = \sinh\varepsilon,\ u(0,\zeta) = 0.$

Solution 3. *By applying the single Sawi transform and the single Shehu transform to the ICs, we get*
$W\left(u(\varepsilon,0)\right) = \frac{1}{1-\sigma^2},\ H\left(u\left(0,\zeta\right)\right) = 0.$
By Definition 4 and Theorem 2, we have

$$\int_0^\varepsilon\int_0^\zeta u(\gamma,\delta))d\gamma d\delta = \left(1 ** u\right)\left(\varepsilon,\zeta\right). \tag{22}$$

Apply the DSW-SHT to Equation 21*, we get*

$$\frac{U(\sigma,\phi,\delta)}{\sigma} - \frac{H(u(0,\zeta))}{\sigma^2} + \frac{\phi}{\delta}U(\sigma,\phi,\delta)$$

$$\begin{aligned}
& -W(u(\varepsilon,0)) - \frac{\delta\phi}{\phi^2+\delta^2} \times \frac{1}{\sigma\left(1-\sigma^2\right)} \\
& + \frac{\delta^2}{\phi^2+\delta^2} \times \frac{1}{\sigma\left(1-\sigma\right)} - \frac{1}{\sigma} \times \frac{\delta^2}{\phi^2+\delta^2} \\
= \quad & \sigma^2 \times \frac{\delta}{\sigma\phi} \times U(\sigma,\phi,\delta).
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\delta\phi+\sigma\phi^2-\sigma^2\delta^2}{\sigma\delta} \times U(\sigma,\phi,\delta) \quad = & \\
& \frac{1}{1-\sigma^2} + \frac{\delta\phi}{\sigma\left(1-\sigma^2\right)\left(\phi^2+\delta^2\right)} \\
& - \frac{\delta^2}{\sigma\left(1-\sigma\right)\left(\phi^2+\delta^2\right)} + \frac{\delta^2}{\sigma\left(\phi^2+\delta^2\right)}.
\end{aligned}$$

By simplifying, we get,

$$U(\sigma,\phi,\delta) = \frac{\delta\phi}{\left(1-\sigma^2\right)\left(\phi^2+\delta^2\right)}.$$

Therefore,

$$u(\varepsilon,\zeta) = W_{\varepsilon}^{-1} H_{\zeta}^{-1}\left(\frac{\delta\phi}{\left(1-\sigma^2\right)\left(\phi^2+\delta^2\right)}\right) = \sinh\varepsilon\cos\zeta.$$

Its graph is

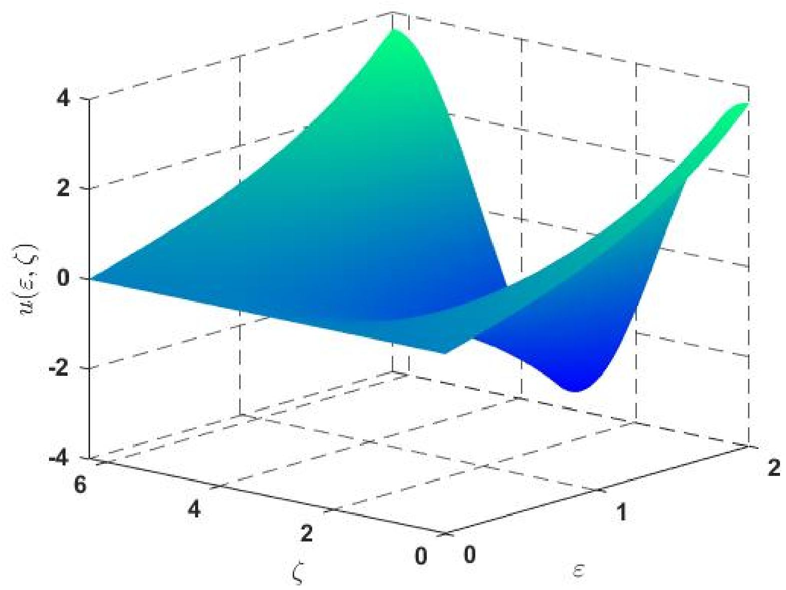

Figure 3: The solution of Example 3

6. Conclusion

This paper introduced DSW-SHT and explored its main properties while establishing the conditions needed for its existence. The results showed that this hybrid double transform can serve as a powerful tool in convolution theory and in dealing with derivative operations. The theoretical framework that has been developed confirms the robustness of the transform and demonstrates that it can be applied to a wide range of mathematical problems.

The study also connected the proposed transform with earlier numerical procedures and results from related works to underline its practical relevance. Through these connections it became clear that the DSW-SHT does not only extend the family of integral transforms but also provides more flexibility in analyzing equations that are otherwise difficult to handle. The advantages of the DSW-SHT appear in its ability to simplify calculations, unify different approaches, and improve the analysis of complex models.

Looking forward, the DSW-SHT can be a basis for further research directions. It shows strong potential in the study of fractional and conformable partial differential equations and in integro-partial differential equations that involve variable coefficients. These areas remain rich with open problems where new approaches are still needed. We believe that the extension of the DSW-SHT to conformable PDEs and its applications in other branches of applied mathematics will lead to deeper insights and more effective methods for solving challenging equations.

Future studies may also focus on numerical implementations and computational aspects of the transform to test its efficiency in real applications. Another promising direction is to investigate how the DSW-SHT interacts with other transforms and whether hybrid structures can be created to address specialized problems. Such efforts will strengthen the role of the DSW-SHT in both theoretical and applied mathematics and confirm its place as a valuable tool for ongoing and future research. Further developments and applications in this field, including extensions to conformable PDEs, are available in [11–13].

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