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A Robust Decision-Making Framework for Recruitment Optimization via Hybrid Dombi–Archimedean Operators in Complex q-Rung Orthopair Fuzzy Sets

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Abstract. Complex q-Rung Orthopair Fuzzy Sets (Cq-ROFSs) are a powerful tool for handling two-dimensional, periodic uncertain information. The Dombi operator provides a highly flexible parameterized framework for combining fuzzy information through smooth and adjustable t-norm and t-conorm functions. Similarly, the Archimedean operator offers a generalized and consistent mechanism for modeling nonlinear interactions in fuzzy aggregation processes. This paper introduces a novel hybrid approach by combining the Dombi and Archimedean t-norm and t-conorm operations under the Cq-ROF environment. Based on these new operational laws, we develop a suite of aggregation operators: the complex q-Rung orthopair fuzzy Dombi-Archimedean weighted averaging (Cq-ROFDAWA) operator, the complex q-Rung orthopair fuzzy Dombi-Archimedean ordered weighted averaging (Cq-ROFDAOWA) operator, the complex q-Rung orthopair fuzzy Dombi-Archimedean weighted geometric (Cq-ROFDAWG) operator, and the complex q-Rung orthopair fuzzy Dombi-Archimedean ordered weighted geometric (Cq-ROFDAOWG) operator. Key properties of these operators, including idempotency, monotonicity, and boundedness, are rigorously investigated. Furthermore, a robust Multi-Criteria Decision-Making (MCDM) framework is established based on the proposed operators. The model's reliability and consistency are demonstrated through two distinct scenarios: one with fully unknown criteria weights and another with partially known weights. A numerical example concerning human resource selection is presented to validate the method's effectiveness and applicability. Finally, a comparative analysis with existing methods underscores the advantages and superiority of the proposed approach.

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1. Introduction

Uncertainty is an inherent part of real-world decision-making. To model this uncertainty, Zadeh introduced Fuzzy Sets (FSs) in 1965 [1], a revolutionary concept where elements have degrees of membership in the interval [0,1]. This framework has been extensively applied in fields such as linguistics [2], decision-making [3], and data clustering [4]. Subsequent extensions of FSs have been developed to capture more complex forms of uncertainty. Atanassov's Intuitionistic Fuzzy Sets (IFSs) [5] incorporated a non-membership degree, with the constraint that their sum is less than or equal to 1. The enhanced ability of IFSs to capture hesitation has enabled their effective application in image classification [6], medical diagnosis [7], environmental assessment [8], and MCDM [9]. Star coloring and its variants extend classical coloring by introducing additional structural constraints [10].

Yager further generalized this model with Pythagorean Fuzzy Sets (PFSs) [11], relaxing the condition to the sum of squares of membership and non-membership degrees being less than or equal to 1. PFSs have proven highly effective in areas like pattern recognition [12] and MCDM [13, 14]. The most general form in this hierarchy is the q-Rung Orthopair Fuzzy Set (q-ROFS) [15], which requires that the sum of the q-th powers of the membership and non-membership degrees is bounded by 1. A higher value of the parameter q provides greater flexibility in representing ambiguous and conflicting information. q-ROFSs have been effectively applied in areas such as image analysis [16], clinical decision support [17], engineering [18], and MCDM [19].

To handle two-dimensional, periodic information, the concept of a Complex Fuzzy Set (CFS) was proposed by Ramot et al. [20], where the membership degree is a complex-valued function within the unit disc. This was later extended to Complex Intuitionistic Fuzzy Sets (CIFSs) [21]. CIFSs have been effectively applied in areas such as image analysis [22], clinical decision support [23], environmental assessment [24], and MCDM [25]. Complex Pythagorean Fuzzy Sets (CPFSs) [26]. However, CIFSs and CPFSs can fail to model certain decision scenarios where the information is highly inconsistent. For instance, with a membership grade of $0.9e^{i2\pi(0.8)}$ and a non-membership grade of $0.8e^{i2\pi(0.9)}$, the squared sum of the real parts exceeds 1, violating CPFS constraints. CPFSs have found valuable applications in computational intelligence [27], medical assessment [28], and MCDM [29].

To overcome this limitation, the Complex q-Rung Orthopair Fuzzy Set (Cq-ROFS) was introduced [30]. A Cq-ROFS imposes the conditions $0 \leq (\mu)^q + (\nu)^q \leq 1$ and $0 \leq (\Re(\mu))^q + (\Re(\nu))^q \leq 1$ on the complex-valued membership and non-membership grades. For the previous example, setting $q = 5$ gives $0.9^5 + 0.8^5 = 0.91817$, which satisfies the condition. This makes Cq-ROFS a significantly more powerful and versatile tool for capturing two-dimensional uncertainty in complex decision-making problems [31]. Cq-ROFSs have shown effectiveness in domains including data analysis [32], medical evaluation [33], and MCDM [34].

Triangular norms (t-norms) and fuzzy operators play a vital role in defining aggregation and logical operations within fuzzy systems. Klement and Mesiar [35] provided a comprehensive study on the logical and algebraic properties of triangular norms, forming the foundation for many fuzzy aggregation techniques. Dombi [36] further generalized these operators through the De Morgan class, establishing a broader structure for measuring fuzziness. Aczél and Alsina [37] explored quasilinear functions to characterize such operators, offering analytical tools for synthesizing judgments. The theoretical insights of Nguyen and Walker [38] helped formalize fuzzy logic principles that underpin most modern fuzzy decision models. These mathematical bases have been effectively applied in real-world decision problems such as personnel selection using intuitionistic fuzzy sets [39] and multi-criteria evaluation through ELECTRE-based approaches [40]. Together, these studies support the continued refinement of fuzzy aggregation and decision-making methods. A thorough analysis of the literature, summarized in Table 1, reveals that while aggregation operators based on Dombi or Archimedean operations have been studied independently in various fuzzy environments, their hybrid combination has not been explored within the Cq-ROFS context. This hybrid approach, leveraging the parameterized flexibility of Dombi operations and the generalizability of Archimedean operations, promises to create a more powerful and adaptable aggregation framework.
The absence of such Dombi-Archimedean (DA) operators for Cq-ROFS means that a significant class of complex, two-dimensional decision-making problems remains unaddressed. This gap is the primary motivation for our work. By fusing these two powerful operational paradigms, we aim to develop a robust set of aggregation tools that can handle the intricate uncertainty captured by Cq-ROFSs more effectively than existing methods.

Table 1: Summary of Aggregation Operators in Various Fuzzy Environments

Fuzzy Environment	**Dombi AOs**	**Archimedean AOs**	**Dombi-Archimedean**
Intuitionistic FS (IFS)	✓	✓	×
Pythagorean FS (PFS)	✓	✓	×
q-Rung Orthopair FS (q-ROFS)	✓	✓	✓
Complex IFS (CIFS)	✓		×
Complex PFS (CPFS)	✓		×
Complex q-ROFS (Cq-ROFS)			×
This Paper	✓	✓	**Proposed Work**

The principal contributions of this paper are as follows:

(i) To establish novel operational laws for Cq-ROFSs by hybridizing Dombi and Archimedean (DA) t-norms and t-conorms.

(ii) To develop a new family of aggregation operators based on these DA operations:

- Complex q-Rung Orthopair Fuzzy Dombi-Archimedean Weighted Averaging (Cq-ROFDAWA) operator.
- Complex q-Rung Orthopair Fuzzy Dombi-Archimedean Ordered Weighted Averaging (Cq-ROFDAOWA) operator.
- Complex q-Rung Orthopair Fuzzy Dombi-Archimedean Weighted Geometric (Cq-ROFDAWG) operator.
- Complex q-Rung Orthopair Fuzzy Dombi-Archimedean Ordered Weighted Geometric (Cq-ROFDAOWG) operator.

(iii) To investigate the essential mathematical properties of these operators, such as idempotency, monotonicity, and boundedness.

(iv) To design a comprehensive MCDM framework utilizing the proposed operators to solve complex decision-making problems.

(v) To validate the proposed framework through a practical numerical example in human resource selection and to demonstrate its robustness and superiority via a comparative analysis with existing methods.

The remainder of this paper is organized as follows. Section 2 covers the necessary preliminaries, including Cq-ROFSs, Dombi operations, and Archimedean t-norms and t-conorms. Section 3 introduces the new Dombi-Archimedean operational laws for Cq-ROFSs. Section 4 presents the proposed aggregation operators and explores their key properties. Section 5 outlines the MCDM algorithm based on these operators. A detailed case study on recruitment optimization is presented in Section 6, with implementation scenarios in same section . A sensitivity analysis is conducted in Section 7, followed by a comparative analysis in Section 8. A general discussion on comparative analysis is discussed in section 9. Finally, Section 10 concludes the paper and suggests directions for future research.

2. Preliminaries

This section outlines the fundamental concepts necessary for understanding the subsequent developments in this work. We briefly revisit the definitions of Complex q-Rung Orthopair Fuzzy Sets (Cq-ROFSs), Archimedean operations, and Dombi operations, which form the foundational building blocks for the new methodologies introduced later.

Definition 1. *[28] A $Cq-ROFS$ is defined as:*

$$\widetilde{D} = \{\left(x, M_{\widetilde{D}}\left(x\right), N_{\widetilde{D}}\left(x\right)/x \in X\right)\} \tag{1}$$

Where $M_{\widetilde{D}}$ represent the MD and $N_{\widetilde{D}}$ denote the NMD and $0 \leq |z_1^q| + |z_2^q| \leq 1$. $M_{\widetilde{D}}\left(x\right) = \mu_{\widetilde{D}}\left(x\right)e^{i2\pi(\phi\mu(x))}, N\widetilde{D}\left(x\right) = \nu_{\widetilde{D}}\left(x\right)e^{i2\pi(\phi_\nu(x))}$ must satisfying the condition $0 \leq \mu^q_{\widetilde{D}}\left(x\right) + \nu^q_{\widetilde{D}}\left(x\right) \leq 1$ and $0 \leq \phi^q_{\mu_{\widetilde{D}}}(x) + \phi^q_{\nu_{\widetilde{D}}}(x) \leq 1$.

Definition 2. *[24] The score function of C_{qORFS} denoted by $\widetilde{s}$ is defined as:*

$$\widetilde{s} = \frac{1}{2} \left| (\mu^q - \nu^q) + (\phi_{\mu^q} - \phi_{\nu^q}) \right| \tag{2}$$

Definition 3. *[41]Let us pretend that there is a strictly decreasing function $\emptyset$ with the following properties: $\emptyset(1) = 0$ and that $\emptyset : [0, 1)$ is continuous. A purely Archimedean TN is expressed by the following equation:*

$$\delta(\hbar, \hbar) = \emptyset^{-1} \left(\emptyset(\hbar) + \emptyset(\hbar) \right) \ for\ \hbar, \hbar \in (0, 1] \tag{3}$$

Definition 4. *[38] For any $l \in [0, 1,$ let $\Theta(l) = \Theta(1 - l)$, and let $\Theta : [0, 1) \to R$ denote a strictly rising and continuous function. A Archimedean TCN is expressed by the equation.*

$$\rho(\hbar, \hbar) = \Theta^{-1} \left(\Theta(\hbar) + \Theta(\hbar) \right) \ for\ \hbar, \hbar \in (0, 1] \tag{4}$$

Definition 5. *[36]The Dombi TN operator, $T_{d(\hbar,\hbar)}$, is defined as:*

$$T_{d(\hbar,\hbar)} = \frac{1}{1 + \left\langle \ \left(\frac{1-\hbar}{\hbar}\right)^k + \left(\frac{1-\hbar}{\hbar}\right)^k \ \right\rangle^{\frac{1}{k}}} \tag{5}$$

The degrees of MD of two fuzzy sets, $\hbar$ and $\hbar$, are controlled by the variable k, which also indicates the degree of intersection.

Definition 6. *[36]The Dombi TCN operator, $D_{d(\hbar,\hbar)}$, is defined as:*

$$D_{d(\hbar,\hbar)} = 1 - \frac{1}{1 + \left\langle \ \left(\frac{\hbar}{1-\hbar}\right)^k + \left(\frac{\hbar}{1-\hbar}\right)^k \ \right\rangle^{\frac{1}{k}}} \tag{6}$$

the degrees of MD of two fuzzy sets, $\hbar$ and $\hbar$, and k is the degree of union of the controls.

3. DOMBI–ARCHIMEDEAN OPERATIONS ON $Cq - ROFEs$

Recent analyses have combined Dombi TN and TCN operators with Archimedean TN and TCN operators to create a new class of operators known as Dombi-Archimedean operators. When examined inside the $Cq - ROF$ framework, these operators have shown promising results in a number of fuzzy reasoning tasks. By combining the characteristics and properties of Dombi and Archimedean operations, Dombi-Archimedean operators provide a comprehensive approach to managing uncertainty and unpredictability in the context of $Cq - ROF$. These operators enhance thinking abilities and are helpful for a variety of fuzzy reasoning applications. Given the potential benefits and real-world applications of this integrated method, Dombi-Archimedean operators are being studied in the context of $Cq - ROF$. Thus far, studies have shown the effectiveness of these operators in resolving $Cq - ROFS$ fuzzy reasoning problems.

Definition 7. *Consider the functions:* $\Re_k(\vartheta) = \left(\frac{\vartheta}{1-\vartheta}\right)^k$ *for* $\vartheta \in [0,1)$ *and* $k \geq 1$, $\Im_k(\vartheta') = \left(\frac{1-\vartheta'}{\vartheta'}\right)^k$ *for* $\vartheta' \in (0,1]$. *For two* $Cq-ROFNs$, *denoted by* $\Delta_1 = (\hbar_1, \hbar_1, \theta_1, \phi_1)$ *and* $\Delta_2 = (\hbar_2, \hbar_2, \theta_2, \phi_2)$, *the Dombi-Archimedean operations are defined as follows.*

- $$\Delta_1 \bigoplus^{D\hbar} \Delta_2 = \left\langle \sqrt[q]{1 - \left\langle \begin{array}{c} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{array}{c} \Theta^{-1}(\Re_k(\hbar_1^q))+ \\ \Theta^{-1}(\Re_k(\hbar_2^q)) \end{array} \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{array}{c} 1+ \\ \left\langle \emptyset \left\langle \begin{array}{c} \emptyset^{-1}(\Im_k(\hbar_1^q))+ \\ \emptyset^{-1}(\Im_k(\hbar_2^q)) \end{array} \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \right.$$
$$\left. \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{array}{c} \Theta^{-1}(\Re_k(\theta_1^q))+ \\ \Theta^{-1}(\Re_k(\theta_2^q)) \end{array} \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}} \sqrt[q]{\left(\begin{array}{c} 1+ \\ \left\langle \emptyset \left\langle \begin{array}{c} \emptyset^{-1}(\Im_k(\phi_1^q))+ \\ \emptyset^{-1}(\Im_k(\phi_2^q)) \end{array} \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right)^{-1}} \right\rangle$$

- $$\Delta_1 \bigotimes^{D\hbar} \Delta_2 = \left\langle \sqrt[q]{\left\langle \begin{array}{c} 1+ \\ \left\langle \emptyset \left\langle \begin{array}{c} \emptyset^{-1}(\Im_k(\hbar_1^q))+ \\ \emptyset^{-1}(\Im_k(\hbar_2^q)) \end{array} \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{array}{c} \Theta^{-1}(\Im_k(\hbar_1^q))+ \\ \Theta^{-1}(\Im_k(\hbar_2^q)) \end{array} \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \right.$$
$$\left. \sqrt[q]{\left\langle \begin{array}{c} 1+ \\ \left\langle \emptyset \left\langle \begin{array}{c} \emptyset^{-1}(\Im_k(\phi_1^q))+ \\ \emptyset^{-1}(\Im_k(\phi_2^q)) \end{array} \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \sqrt[q]{1 - \left(\begin{array}{c} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{array}{c} \Theta^{-1}(\Im_k(\theta_1^q)) \\ +\Theta^{-1}(\Im_k(\theta_2^q)) \end{array} \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right)^{-1}} \right\rangle$$

- $$\lambda *^{D\hbar} \Delta_1 = \left\langle \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \left\langle \begin{array}{c} \Theta \\ \left\langle \lambda \Theta^{-1}(\Re_k(\hbar_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \left\langle \begin{array}{c} \emptyset \\ \left\langle \lambda \emptyset^{-1}(\Im_k(\hbar_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}} \right.$$
$$\left. \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \left\langle \begin{array}{c} \Theta \\ \left\langle \lambda \Theta^{-1}(\Re_k(\theta_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \sqrt[q]{\left(\begin{array}{c} 1+ \\ \left\langle \begin{array}{c} \emptyset \\ \left\langle \lambda \emptyset^{-1}(\Im_k(\phi_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \end{array} \right)^{-1}} \right\rangle, (\lambda > 0)$$

- $$\lambda \circ^{D\hbar} \Delta_1 = \left\langle \sqrt[q]{\left\langle \begin{array}{c} 1+ \\ \left\langle \begin{array}{c} \emptyset \\ \left\langle \lambda \emptyset^{-1}(\Im_k(\hbar_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \left\langle \begin{array}{c} \Theta \\ \left\langle \lambda \Theta^{-1}(\Re_k(\hbar_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}} \right.$$

$$\sqrt[q]{\left\langle \left\langle \begin{matrix} 1+ \\ \emptyset \\ \langle \lambda \emptyset^{-1}(\Im_k(\phi_1^q)\rangle \end{matrix} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{matrix} 1- \\ \left\langle \begin{matrix} 1+ \\ \Theta \\ \langle \lambda \Theta^{-1}(\Re_k(\theta_1^q))\rangle \end{matrix} \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}} \quad (\lambda > 0)$$

Theorem 1. *Let $\Delta_1 = (\hbar_1, \hbar_1, \theta_1, \phi_1)$ and $\Delta_2 = (\hbar_2, \hbar_2, \theta_2, \phi_2)$ be two Cq–ROFEs such that*

$$0 \leq \hbar_i^q + \hbar_i^q \leq 1, \quad 0 \leq \theta_i^q + \phi_i^q \leq 1, \quad i = 1, 2.$$

Then, for any $\lambda > 0$, the results of the Dombi–Archimedean operations $\Delta_1 \bigoplus^{DA} \Delta_2$, $\Delta_1 \bigotimes^{DA} \Delta_2$, $\lambda *^{DA} \Delta_1$, and $\lambda \circ^{DA} \Delta_1$ are also valid Cq–ROFEs.

Where:
$\hbar$ represents the amplitude term of the membership degree,
$\hbar$ represents the amplitude term of the non-membership degree,
θ represents the phase term of the membership degree, and
ϕ represents the phase term of the non-membership degree.

Proof. We prove that the Cq-ROFDAWA operator preserves the fundamental constraints of Cq-ROFSs. Let $\Delta_j = (\hbar_j, \hbar_j, \theta_j, \phi_j)_{j=1}^r$ be a collection of Cq-ROFNs, which by definition satisfy:

$$0 \leq \hbar_j^q + \hbar_j^q \leq 1 \qquad 0 \leq \theta_j^q + \phi_j^q \leq 1 \quad \text{for all } j = 1, 2, \ldots, r$$

Let $\Delta = (\hbar, \hbar, \theta, \phi) = \text{Cq-ROFDAWA}(\Delta_1, \Delta_2, \ldots, \Delta_r)$ be the aggregated value. From the definition of Cq-ROFDAWA, we have:

$$\hbar^q = 1 - \left\langle 1 + \left\langle \Theta \left\langle \sum_{j=1}^r \omega_j \Theta^{-1}(\Re_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1} \hbar^q = \left\langle 1 + \left\langle \emptyset \left\langle \sum_{j=1}^r \omega_j \emptyset^{-1}(\Im_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1}$$

Step 1: Non-negativity and Boundedness of Individual Terms First, we show that $0 \leq \hbar^q \leq 1$ and $0 \leq \hbar^q \leq 1$. Since $\hbar_j^q \in [0, 1]$ for all j, and $\Re_k(\vartheta) = \left(\frac{\vartheta}{1-\vartheta}\right)^k$ is a strictly increasing function mapping $[0, 1)$ to $[0, \infty)$, we have $\Re_k(\hbar_j^q) \geq 0$. The function Θ and its inverse Θ^{-1} preserve non-negativity due to their Archimedean properties. Therefore:
$A = \sum_{j=1}^r \omega_j \Theta^{-1}(\Re_k(\hbar_j^q)) \geq 0$
$\Theta(A) \geq 0$
$\langle \Theta(A) \rangle^{\frac{1}{k}} \geq 0$
$1 + \langle \Theta(A) \rangle^{\frac{1}{k}} \geq 1$
$\left\langle 1 + \langle \Theta(A) \rangle^{\frac{1}{k}} \right\rangle^{-1} \in (0, 1]$

$\hbar^q = 1 - \left\langle 1 + \langle \Theta(A) \rangle^{\frac{1}{k}} \right\rangle^{-1} \in [0,1)$ Similarly, for $\hbar^q$, since $\hbar^q_j \in [0,1]$ and $\Im_k(\vartheta') = \left(\frac{1-\vartheta'}{\vartheta'}\right)^k$ maps $(0,1]$ to $[0,\infty)$, we have:

$$B = \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\hbar^q_j)) \geq 0 \emptyset(B) \geq 0 \, \langle \emptyset(B) \rangle^{\frac{1}{k}} \geq 0 1 + \langle \emptyset(B) \rangle^{\frac{1}{k}} \geq 1 \hbar^q = \left\langle 1 + \langle \emptyset(B) \rangle^{\frac{1}{k}} \right\rangle^{-1} \in (0,1]$$

Step 2: Sum Constraint for Amplitude Terms Now we prove the critical constraint: $0 \leq \hbar^q + \hbar^q \leq 1$. From the Dombi-Archimedean structure with the specific function forms $\Theta(t) = \left(\frac{1-t}{t}\right)^k$ and $\emptyset(t) = \left(\frac{t}{1-t}\right)^k$, we derive:

$$\Theta^{-1}(\Re_k(\hbar^q_j)) = \Theta^{-1}\left(\left(\frac{\hbar^q_j}{1-\hbar^q_j}\right)^k\right) = 1 - \hbar^q_j \emptyset^{-1}(\Im_k(\hbar^q_j)) = \emptyset^{-1}\left(\left(\frac{1-\hbar^q_j}{\hbar^q_j}\right)^k\right) = 1 - \hbar^q_j$$

Therefore:

$$A = \sum_{j=1}^{r} \omega_j (1 - \hbar^q_j) B = \sum_{j=1}^{r} \omega_j (1 - \hbar^q_j)$$

Now, using the Dombi function properties:

$$\hbar^q = 1 - \frac{1}{1 + \left(\frac{1-A}{A}\right)} = 1 - A \hbar^q = \frac{1}{1 + \left(\frac{B}{1-B}\right)} = 1 - B$$

Thus:

$$\hbar^q + \hbar^q = (1-A) + (1-B) = 2 - (A+B) = 2 - \sum_{j=1}^{r} \omega_j (2 - (\hbar^q_j + \hbar^q_j))$$

Since each Δ_j is a valid Cq-ROFN, we have $\hbar^q_j + \hbar^q_j \leq 1$, which implies $2 - (\hbar^q_j + \hbar^q_j) \geq 1$. Therefore:

$$A + B = \sum_{j=1}^{r} \omega_j (2 - (\hbar^q_j + \hbar^q_j)) \geq \sum_{j=1}^{r} \omega_j = 1 \hbar^q + \hbar^q = 2 - (A+B) \leq 1$$

The non-negativity $\hbar^q + \hbar^q \geq 0$ follows from $\hbar^q, \hbar^q \geq 0$. **Step 3: Phase Terms Constraint** The proof for phase terms (θ, ϕ) follows identically to the amplitude terms, since the aggregation operators for phase terms have the same functional form and the input phase terms satisfy the same constraints $0 \leq \theta^q_j + \phi^q_j \leq 1$. Therefore, the aggregated value $\Delta = \text{Cq-ROFDAWA}(\Delta_1, \Delta_2, \ldots, \Delta_r)$ satisfies both Cq-ROFS constraints:

(i) $0 \leq \hbar^q + \hbar^q \leq 1$

(ii) $0 \leq \theta^q + \phi^q \leq 1$

This completes the proof that the Cq-ROFDAWA operator is closed under the Cq-ROFS domain.

Example 1. *Given two Cq−ROFNs on U:* $\Delta_1 = (0.6, 0.7), (0.7, 0.6)$, $\Delta_2 = (0.4, 0.8), (0.6, 0.5)$, *and parameters* $q = 4$, $\lambda = 0.3$, $k = 2$, *and functions:* $\Theta(t) = -\ln(1-t)$, $\Theta^{-1}(t) = 1-e^{-t}$, $\emptyset(t) = -i\ln(t)$, $\emptyset^{-1}(t) = e^{-t}$. *The expressions are then defined accordingly.*

- $\Delta_1 \bigoplus^{D\hbar} \Delta_2 = \left\langle \sqrt[q]{1 - \left\langle \begin{matrix} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{matrix} \Theta^{-1}(\Re_k(\hbar_1^q))+ \\ \Theta^{-1}(\Re_k(\hbar_2^q)) \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{matrix} 1+ \\ \left\langle \emptyset \left\langle \begin{matrix} \emptyset^{-1}(\Im_k(\hbar_1^q))+ \\ \emptyset^{-1}(\Im_k(\hbar_2^q)) \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}}\right.$

- $\sqrt[q]{\left\langle \begin{matrix} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{matrix} \Theta^{-1}(\Re_k(\theta_1^q))+ \\ \Theta^{-1}(\Re_k(\theta_2^q)) \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}} \sqrt[q]{\left(\begin{matrix} 1+ \\ \left\langle \emptyset \left\langle \begin{matrix} \emptyset^{-1}(\Im_k(\phi_1^q))+ \\ \emptyset^{-1}(\Im_k(\phi_2^q)) \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right)^{-1}} \left.\right\rangle$

$$\Delta_1 \bigoplus^{D\hbar} \Delta_2 = \left\langle \sqrt[q]{\left(\begin{matrix} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{matrix} 1 - e^{-\left(\frac{\hbar_1^q}{1-\hbar_1^q}\right)^k} + \\ 1 - e^{-\left(\frac{\hbar_2^q}{1-\hbar_2^q}\right)^k} \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right)^{-1}}, \sqrt[q]{\left(\begin{matrix} 1+ \\ \left\langle \emptyset \left\langle \begin{matrix} 1 - e^{-\left(\frac{1-\hbar_1^q}{\hbar_1^q}\right)^k} + \\ 1 - e^{-\left(\frac{\hbar_2^q}{1-\hbar_2^q}\right)^k} \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right)^{-1}} \right.$$

$$\sqrt[q]{\left(\begin{matrix} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{matrix} 1 - e^{-\left(\frac{\theta_1^q}{1-\theta_1^q}\right)^k} + \\ 1 - e^{-\left(\frac{\theta_2^q}{1-\theta_2^q}\right)^k} \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right)^{-1}}, \sqrt[q]{\left(\begin{matrix} 1+ \\ \left\langle \emptyset \left\langle \begin{matrix} 1 - e^{-\left(\frac{1-\phi_1^q}{\phi_1^q}\right)^k} + \\ 1 - e^{-\left(\frac{\phi_2^q}{1-\phi_2^q}\right)^k} \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right)^{-1}}$$

Placing the values yields
$=\{(0.0629, 0.9743), (0.0267, 0.9977)\}$

- $\Delta_1 \bigotimes^{D\hbar} \Delta_2 = \left\langle \sqrt[q]{\left\langle \begin{matrix} 1+ \\ \left\langle \emptyset \left\langle \begin{matrix} \emptyset^{-1}(\Im k(\hbar_1^q))+ \\ \emptyset^{-1}(\Im k(\hbar_2^q)) \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}}, \sqrt[q]{\begin{matrix} 1- \\ \left\langle 1 + \left\langle \Theta \left\langle \begin{matrix} \Theta^{-1}(\Re_k(\hbar_1^q))+ \\ \Theta^{-1}(\Re_k(\hbar_2^q)) \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1} \end{matrix}} \right.$

$$\sqrt[q]{\left\langle \begin{matrix} 1+ \\ \left\langle \emptyset \left\langle \begin{matrix} \emptyset^{-1}(\Im k(\phi_1^q))+ \\ \emptyset^{-1}(\Im k(\phi_2^q)) \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{matrix} 1- \\ 1+ \\ \left\langle \Theta \left\langle \begin{matrix} \Theta^{-1}(\Re_k(\theta_1^q))+ \\ \Theta^{-1}(\Re_k(\theta_2^q)) \end{matrix} \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}} \left.\right\rangle$$

On substituting the numerical values

$\Delta_1 \bigotimes^{D\hbar} \Delta_2 = \{(0.9370, 0.0256), (0.9732, 0.0022)\}$

- $$\lambda *^{D\hbar} \Delta_1 = \left\langle \sqrt[q]{\begin{matrix} 1- \\ \left\langle \begin{matrix} 1+ \\ \left\langle \Theta \left\langle \lambda \Theta^{-1} (\Re_k(\hbar_1^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1} \end{matrix}}, \sqrt[q]{\begin{matrix} , \\ \left\langle \begin{matrix} 1+ \\ \left\langle \emptyset \left\langle \lambda \emptyset^{-1} (\Im_k(\hbar_1^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1} \end{matrix}} \right.$$
$$\left. \sqrt[q]{\begin{matrix} 1- \\ \left\langle \begin{matrix} 1+ \\ \left\langle \Theta \left\langle \lambda \Theta^{-1} (\Re_k(\theta_1^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1} \end{matrix}}, \sqrt[q]{\begin{matrix} , \\ \left\langle \begin{matrix} 1+ \\ \left\langle \emptyset \left\langle \lambda \emptyset^{-1} (\Im_k(\phi_1^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1} \end{matrix}} \right\rangle \quad (\lambda > 0)$$

Substituting the values

$=\{(0.5595, 0.4407), (0.5592, 0.4409)\}$

- $$\lambda \circ^{D\hbar} \Delta_1 = \left\langle \sqrt[q]{\left\langle \begin{matrix} 1+ \\ \left\langle \begin{matrix} \emptyset \\ \left\langle \lambda \emptyset^{-1} (\Im_k(\hbar_1^q)) \right\rangle \end{matrix} \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}}, \sqrt[q]{\begin{matrix} 1- \\ \left\langle \begin{matrix} 1+ \\ \left\langle \begin{matrix} \Theta \\ \left\langle \lambda \Theta^{-1} (\Re_k(\hbar_1^q)) \right\rangle \end{matrix} \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1} \end{matrix}} \right.$$
$$\left. \sqrt[q]{\left\langle \begin{matrix} 1+ \\ \left\langle \begin{matrix} \emptyset \\ \left\langle \lambda \emptyset^{-1} (\Im_k(\phi_1^q) \right\rangle \end{matrix} \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}}, \sqrt[q]{\begin{matrix} 1- \\ \left\langle \begin{matrix} 1+ \\ \left\langle \begin{matrix} \Theta \\ \left\langle \left(\lambda \Theta^{-1} (\Re_k(\theta_1^q)) \right) \right\rangle \end{matrix} \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1} \end{matrix}} \right\rangle$$

Substituting the values

$=\{(0.4404, 0.5592), (0.4409, 0.5593)\}$

Theorem 2. *Let* $\Delta_1 = (\hbar_1, \hbar_1, \theta_1, \phi_1)$ *and* $\Delta_2 = (\hbar_2, \hbar_2, \theta_2, \phi_2)$*be the* $Cq - ROFEs$*,where* $\hbar_{1,2}, \hbar_{1,2}$ *denotes the amplitude terms such that* $\hbar_{1,2}$ *stand for MD while the* $\hbar_{1,2}$ *stand for NMD and* $\theta_{1,2}, \phi_{1,2}$ *represent the phase term such that the* $\theta_{1,2}$ *denote the MD while* $\phi_{1,2}$ *is for NMD with* $\lambda > 0$*,* $\lambda_1 > 0$*,* $\lambda_3 > 0$

(i) $\Delta_1 \bigoplus^{DA} \Delta_2 = \Delta_2 \bigoplus^{DA} \Delta_1$

(ii) $\Delta_1 \bigotimes^{DA} \Delta_2 = \Delta_2 \bigotimes^{DA} \Delta_1$

(iii) $\lambda *^{DA} (\Delta_1 \bigoplus^{DA} \Delta_2) = \left(\lambda *^{DA} \Delta_1\right) \bigoplus^{DA} \left(\lambda *^{DA} \beta^h_{\zeta 2}\right)$

(iv) $\lambda \circ^{DA} (\Delta_1 \bigoplus^{DA} \Delta_2) = \left(\lambda \circ^{DA} \Delta_1\right) \bigotimes^{DA} \left(\lambda \circ^{DA} \beta^h_{\zeta 2}\right)$

(v) $(\lambda_1 + \lambda_2) *^{DA} \Delta_1 = \left(\lambda_1 *^{DA} \Delta_1\right) \bigoplus^{DA} \left(\lambda_2 *^{DA} \beta^h_{\zeta 2}\right)$

(vi) $(\lambda_1+\lambda_2)*^{DA}\Delta_1 = \left(\lambda_1 *^{DA}\Delta_1\right)\bigotimes^{DA}\left(\lambda_2 *^{DA}\beta^h_{\zeta 2}\right)$

Proof. The proof of (i) and (ii).is not nessecary.

(iii) $\lambda *^{D\hbar}(\Delta_1 \bigoplus^{D\hbar}\Delta_2)$

$$= \lambda *^{D\hbar} \left\langle \sqrt[q]{\begin{matrix}1- \\ \left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\Theta \\ \left\langle \begin{matrix}\Theta^{-1}(\Re_k(\hbar_1^q))+ \\ \Theta^{-1}(\Re_k(\hbar_2^q))\end{matrix}\right\rangle^k\end{matrix}\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}\end{matrix}}, \sqrt[q]{\left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\emptyset \\ \left\langle \begin{matrix}\emptyset^{-1}(\Im_k(\hbar_1^q))+ \\ \emptyset^{-1}(\Im_k(\hbar_2^q))\end{matrix}\right\rangle^k\end{matrix}\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}}\right.$$

$$\left. \sqrt[q]{\begin{matrix}1- \\ \left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\Theta \\ \left\langle \begin{matrix}\Theta^{-1}(\Re_k(\theta_1^q))+ \\ \Theta^{-1}(\Re_k(\theta_2^q))\end{matrix}\right\rangle\end{matrix}\right\rangle^{k\frac{1}{k}}\end{matrix}\right\rangle^{-1}\end{matrix}}, \sqrt[q]{\left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\emptyset \\ \left\langle \begin{matrix}\emptyset^{-1}(\Im_k(\phi_1^q))+ \\ \emptyset^{-1}(\Im_k(\phi_2^q))\end{matrix}\right\rangle\end{matrix}\right\rangle^{k\frac{1}{k}}\end{matrix}\right\rangle^{-1}}\right\rangle$$

$$= \left\langle \sqrt[q]{\begin{matrix}1- \\ \left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\Theta \\ \left\langle \begin{matrix}\lambda \\ \left\langle \begin{matrix}\Theta^{-1}(\Re_k(\hbar_1^q))+ \\ \Theta^{-1}(\Re_k(\hbar_2^q))\end{matrix}\right\rangle^k\end{matrix}\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}\end{matrix}}, \sqrt[q]{\left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\emptyset \\ \left\langle \begin{matrix}\lambda \\ \left\langle \begin{matrix}\emptyset^{-1}(\Im_k(\hbar_1^q))+ \\ \emptyset^{-1}(\Im_k(\hbar_2^q))\end{matrix}\right\rangle^k\end{matrix}\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}}\right.$$

$$\left. \sqrt[q]{\begin{matrix}1- \\ \left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\Theta \\ \left\langle \begin{matrix}\lambda \\ \left\langle \begin{matrix}\Theta^{-1}(\Re_k(\theta_1^q))+ \\ \Theta^{-1}(\Re_k(\theta_2^q))\end{matrix}\right\rangle^k\end{matrix}\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}\end{matrix}}, \sqrt[q]{\left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\emptyset \\ \left\langle \begin{matrix}\lambda \\ \left\langle \begin{matrix}\emptyset^{-1}(\Im_k(\phi_1^q))+ \\ \emptyset^{-1}(\Im_k(\phi_2^q))\end{matrix}\right\rangle^k\end{matrix}\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}}\right\rangle$$

and also,

$= \lambda *^{D\hbar}(\Delta_1 \bigoplus^{D\hbar}\Delta_2) = \left(\lambda *^{D\hbar}\Delta_2\right)\bigoplus^{D\hbar}\left(\lambda *^{D\hbar}\Delta_2\right)$

$$= \left\langle \sqrt[q]{\begin{matrix}1- \\ \left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\Theta \\ \left\langle \lambda_1 \Theta^{-1}(\Re_k(\hbar_1^q))\right\rangle^k\end{matrix}\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}\end{matrix}}, \sqrt[q]{\left\langle \begin{matrix}1+ \\ \left\langle \begin{matrix}\emptyset \\ \left\langle \lambda_1 \emptyset^{-1}(\Im_k(\hbar_1^q))\right\rangle^k\end{matrix}\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}}\right.$$

$$\sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle \lambda_1\Theta^{-1}(\Re_k(\theta_1^q))\right\rangle^k\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle},\ \sqrt[q]{\left\langle\left\langle 1+\left\langle \emptyset\left\langle \lambda_1\emptyset^{-1}(\Im_k(\phi_1^q))\right\rangle^k\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}\ \Bigg\rangle$$

$$\bigoplus^{D\hbar}$$

$$\left\langle \sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle \lambda_2\Theta^{-1}(\Re_k(\hbar_2^q))\right\rangle^k\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle},\ \sqrt[q]{\left\langle\left\langle 1+\left\langle \emptyset\left\langle \lambda_2\emptyset^{-1}(\Im_k(\hbar_2^q))\right\rangle^k\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}\right.$$

$$\left.\sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle \lambda_2\Theta^{-1}(\Re_k(\theta_2^q))\right\rangle^k\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle},\ \sqrt[q]{\left\langle\left\langle 1+\left\langle \emptyset\left\langle \lambda_2\emptyset^{-1}(\Im_k(\phi_2^q))\right\rangle^k\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}\right\rangle$$

$$=\left\langle \sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle \lambda\left\langle \Theta^{-1}(\Re_k(\hbar_1^q))+\Theta^{-1}(\Re_k(\hbar_2^q))\right\rangle^k\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle},\ \sqrt[q]{\left\langle\left\langle 1+\left\langle \emptyset\left\langle \lambda\left\langle \emptyset^{-1}(\Im_k(\hbar_1^q))+\emptyset^{-1}(\Im_k(\hbar_2^q))\right\rangle^k\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}\right.$$

$$\left.\sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle \lambda\left\langle \Theta^{-1}(\Re_k(\theta_1^q))+\Theta^{-1}(\Re_k(\theta_2^q))\right\rangle^k\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle},\ \sqrt[q]{\left\langle\left\langle 1+\left\langle \emptyset\left\langle \lambda\left\langle \emptyset^{-1}(\Im_k(\phi_1^q))+\emptyset^{-1}(\Im_k(\phi_2^q))\right\rangle^k\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}\right\rangle$$

Getting,

$\lambda *^{D\hbar} (\Delta_1 \bigoplus^{D\hbar} \Delta_2) = \left(\lambda *^{D\hbar} \Delta_1\right) \bigoplus^{D\hbar} \left(\lambda *^{D\hbar} \Delta_2\right)$

(iv) See the (iii),

(v) From (i) and (iii), we have,

$$(\lambda_1+\lambda_2) *^{D\hbar} \Delta_1 =$$

$$\left\langle \sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle (\lambda_1+\lambda_2)\,\Theta^{-1}(\Re_k(\hbar_1^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle},\ \sqrt[q]{\left\langle\left\langle 1+\left\langle \emptyset\left\langle (\lambda_1+\lambda_2)\,\emptyset^{-1}(\Im_k(\hbar_1^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}\right.$$

$$\sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle (\lambda_1+\lambda_2)\,\Theta^{-1}(\Re_k(\theta_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle (\lambda_1+\lambda_2)\,\emptyset^{-1}(\Im_k(\phi_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \Bigg\rangle$$

and also

$$\left(\lambda *^{D\hbar} \Delta_1\right) \bigoplus^{D\hbar} \left(\lambda *^{D\hbar} \Delta_2\right)$$

$$\left\langle \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \lambda_1\Theta^{-1}(\Re_k(\hbar_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \lambda_1\emptyset^{-1}(\Im_k(\hbar_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right.$$

$$\left. \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \lambda_1\Theta^{-1}(\Re_k(\theta_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \lambda_1\emptyset^{-1}(\Im_k(\phi_1^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right\rangle$$

$$\bigoplus^{D\hbar}$$

$$= \left\langle \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \lambda_2\Theta^{-1}(\Re_k(\hbar_2^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \lambda_2\emptyset^{-1}(\Im_k(\hbar_2^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right.$$

$$\left. \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \lambda_2\Theta^{-1}(\Re_k(\theta_2^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \lambda_2\emptyset^{-1}(\Im_k(\phi_2^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right\rangle$$

$$= \left\langle \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \begin{array}{c} \lambda\Theta^{-1}(\Re_k(\hbar_1^q))+ \\ \lambda\Theta^{-1}(\Re_k(\hbar_2^q)) \end{array} \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \begin{array}{c} \lambda\emptyset^{-1}(\Im_k(\hbar_1^q))+ \\ \lambda\emptyset^{-1}(\Im_k(\hbar_2^q)) \end{array} \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right.$$

$$\left. \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \begin{array}{c} \lambda\Theta^{-1}(\Re_k(\theta_1^q))+ \\ \lambda\Theta^{-1}(\Re_k(\theta_2^q)) \end{array} \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \begin{array}{c} \lambda\emptyset^{-1}(\Im_k(\phi_1^q))+ \\ \lambda\emptyset^{-1}(\Im_k(\phi_2^q)) \end{array} \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right\rangle$$

Getting

$$(\lambda_1+\lambda_2) *^{D\hbar} \Delta_1 = \left(\lambda_1 *^{D\hbar} \Delta_1\right) \bigoplus^{D\hbar} \left(\lambda_2 *^{D\hbar} \beta^{h}_{\zeta_2}\right)$$

(vi) See (v)

4. $Cq - ROF$ Dombi–Archimedean Weighted Aggregation operators ($Cq - ROFDAWA$)

In order to aggregate $Cq-ROF$ information, the authors provide $Cq-ROF$ operators that combine Dombi-Archimedean procedures with weighting systems. Weighted aggregation taking element significance into account is made possible by the $Cq - ROF$ Dombi-Archimedean weighted average $Cq - ROFDAWA$ operator and the Cq-ROF Dombi-Archimedean weighted geometric CqROFDAWG operator. A more sophisticated depiction of preferences is provided by an alternative version, the $Cq - ROF$ Dombi-Archimedean ordered weighted average $Cq - ROFDAOWA$, and the $Cq - ROFDAOWG$ operator, which takes item order into account. In the $Cq - ROF$ setting, these variants allow for a range of decision-making situations that are sensitive to ambiguity and uncertainty. The operators make the $Cq - ROF$ and Dombi-Archimedean framework more flexible, which helps practitioners with complicated decision-making problems and gives them accurate $Cq - ROF$ data representations.

Definition 8. *Let $\Delta_j = (\hbar_j, \hbar_j, \theta_j, \phi_j)$ be the $Cq - ROFEs$,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD. Then the mapping $ROFDAWA$) is as follows:$Cq - ROFNs^U \longrightarrow Cq - ROFNs^U$ and as*

$$Cq - ROFDAWA(\Delta_1, \Delta_2, ..., \Delta_r) = \bigoplus_{j=1}^{r^{D\hbar}} \left(\omega_j *^{D\hbar} \Delta_j\right) \tag{7}$$

Here $\omega_{j(j=1,2,...,r)}$ represents the weight of Δ_j with the condition as $\omega_j > 0$

Theorem 3. *Let $\Delta_j = (\hbar_j, \hbar_j, \theta_j, \phi_j)$ be the $Cq - ROFEs$,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD. Then the aggregated value of $Cq - ROFDAWA(\Delta_1, \Delta_2, ..., \Delta_r)$ as:*

$Cq - ROFDAWA(\Delta_1, \Delta_2, ..., \Delta_r) =$

$$\left\langle \sqrt[q]{\begin{matrix} 1- \\ 1+ \\ \Theta \\ \left\langle \left\langle \left\langle \sum_{j=1}^{r+1} \omega_j \Theta^{-1}(\Re_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1} \end{matrix}}, \sqrt[q]{\begin{matrix} 1+ \\ \emptyset \\ \left\langle \left\langle \left\langle \sum_{j=1}^{r+1} \omega_j \emptyset^{-1}(\Im_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1} \end{matrix}} \right\rangle$$

$$\sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle},\ \sqrt[q]{\left\langle\left\langle 1+\left\langle \emptyset\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}\right\rangle \tag{8}$$

As $\sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}$, $\sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}$ called amplitude term ,

and $\sqrt[q]{\left\langle 1-\left\langle 1+\left\langle \Theta\left\langle\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}\right\rangle}$, $\sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}$ represent a phase term and also $\omega_{j(j=1,2,\ldots,r)}$ denotes the weights of Δ_j with conditions and $\sum_{j=1}^{r}\omega_j = 1$. Here $\omega_{j(j=1,2,\ldots,r)}$ denoted weights of Δ_j with the conditions and $\sum_{j=1}^{r}\omega_j = 1$.

Proof. By mathematical induction, when $n = 1$, it is true,

for $n = 2$,

$Cq - ROFDAWA(\Delta_1, \Delta_2) = (\omega_1 *^{D\hbar} \Delta_1) \bigoplus^{D\hbar}(\omega_2 *^{D\hbar} \Delta_2)$

$$\left\langle\sqrt[q]{1-\left\langle 1+\left\langle \Theta\left(\omega_1\Theta^{-1}(\Re_k(\hbar_1^q))\right)\right\rangle^{\frac{1}{k}}\right\rangle^{-1}},\ \sqrt[q]{\left\langle 1+\left\langle \emptyset\left(\omega_1\emptyset^{-1}(\Im_k(\hbar_1^q))\right)\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\sqrt[q]{1-\left\langle 1+\left\langle \Theta\left(\omega_1\Theta^{-1}(\Re_k(\theta_1^q))\right)\right\rangle^{\frac{1}{k}}\right\rangle^{-1}},\ \left.\sqrt[q]{\left\langle 1+\left\langle \emptyset\left(\omega_1\emptyset^{-1}(\Im_k(\phi_1^q))\right)\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right\rangle$$

$$\bigoplus^{D\hbar} = \left\langle \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left(\omega_2\Theta^{-1}(\Re_k(\hbar_2^q))\right) \end{array} \right\rangle^{\frac{1}{k}} }^{-1}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left(\omega_2\emptyset^{-1}(\Im_k(\hbar_2^q))\right) \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}$$

$$\sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left(\omega_2\Theta^{-1}(\Re_k(\theta_2^q))\right) \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left(\omega_2\emptyset^{-1}(\Im_k(\phi_2^q))\right) \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right\rangle$$

or

$$= \left\langle \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \begin{array}{c} \omega_1\Theta^{-1}(\Re_k(\hbar_1^q))+ \\ \omega_2\Theta^{-1}(\Re_k(\hbar_2^q)) \end{array} \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \begin{array}{c} \omega_1\emptyset^{-1}(\Im_k(\hbar_1^q))+ \\ \omega_2\emptyset^{-1}(\Im_k(\hbar_2^q)) \end{array} \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}$$

$$\sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \begin{array}{c} \omega_1\Theta^{-1}(\Re_k(\theta_1^q))+ \\ \omega_2\Theta^{-1}(\Re_k(\theta_2^q)) \end{array} \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \begin{array}{c} \omega_1\emptyset^{-1}(\Im_k(\phi_1^q))+ \\ \omega_2\emptyset^{-1}(\Im_k(\phi_2^q)) \end{array} \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right\rangle$$

$$= \left\langle \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \sum_{j=1}^{2} \omega_j\Theta^{-1}(\Re_k(\hbar_j^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \sum_{j=1}^{2} \omega_j\emptyset^{-1}(\Im_k(\hbar_j^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}$$

$$\sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \Theta \\ \left\langle \sum_{j=1}^{2} \omega_j\Theta^{-1}(\Re_k(\theta_j^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \emptyset \\ \left\langle \sum_{j=1}^{2} \omega_j\emptyset^{-1}(\Im_k(\phi_j^q)) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right\rangle$$

getting, Eq.(4.2) is valid.for n=2 Assume for the moment that $n = r$ is the case and Equation (4.2) is valid. Then,

$Cq - ROFDAWA(\Delta_1, \Delta_2, ..., \Delta_r) =$

$$\left\langle \sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\left.\sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right\rangle$$

We now have $n = r + 1$ for which
$Cq - ROFDAWA(\Delta_1, \Delta_2, ..., \Delta_{r+1}) =$

$$\left\langle \sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset^{-1}\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\left.\sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset^{-1}\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right\rangle$$

$\bigoplus^{D\hbar}(\omega_{k+1} *^{D\hbar} \Delta_{r+1})$
resulting

$$= \left\langle \sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle + \left\langle\Theta\left(\omega_{r+1}\Theta^{-1}(\Re_k(\hbar_{r+1}^q))\right)\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}} + \left\langle\emptyset\left(\omega_{r+1}\emptyset^{-1}(\Im_k(\hbar_{r+1}^q))\right)\right\rangle\right\rangle^{-1}}\right.$$

$$\sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle \left(\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right)+ \left\langle \Theta\left(\omega_{r+1}\Theta^{-1}(\Re_k(\theta_{r+1}^q))\right)\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle + \left\langle\emptyset\left(\omega_{r+1}\emptyset^{-1}(\Im_k(\phi_{r+1}^q))\right)\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}$$

$$= \left\langle \sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle \sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle \sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\left.\sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle \sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle \sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right\rangle$$

So, for $n = r + 1$, Equation (4.2) is still valid. So, all naturally occurring numbers n are covered by Equation (4.2).

Theorem 4. *Let $\Delta_j = (\hbar_j, \hbar_j, \theta_j, \phi_j)$ be the $Cq-ROFEs$,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD with $\Delta_o \in Cq-ROFNs$. Then*

$$Cq-ROFDAWA(\Delta_0 \bigoplus^{DA} \Delta_r) = \Delta_0 \bigoplus^{DA} Cq-ROFDAWA\,(\Delta_1, \Delta_2, ..., \Delta_r) \tag{9}$$

Proof. we know, $\Delta_0 \bigoplus^{D\hbar} \Delta_r =$

$$\left\langle \sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle \begin{matrix}\lambda\Theta^{-1}(\Re_k(\hbar_0^q))+\\ \lambda\Theta^{-1}(\Re_k(\hbar_j^q))\end{matrix}\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle \begin{matrix}\lambda\emptyset^{-1}(\Im_k(\hbar_0^q))+\\ \lambda\emptyset^{-1}(\Im_k(\hbar_j^q))\end{matrix}\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\left.\sqrt[q]{1-\left\langle 1+\left\langle \Theta\left\langle \begin{matrix}\lambda\Theta^{-1}(\Re_k(\theta_0^q))+\\ \lambda\Theta^{-1}(\Re_k(\theta_j^q))\end{matrix}\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\left\langle \emptyset\left\langle \begin{matrix}\lambda\emptyset^{-1}(\Im_k(\phi_0^q))+\\ \lambda\emptyset^{-1}(\Im_k(\phi_j^q))\end{matrix}\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right\rangle$$

$$=\left\langle \sqrt[q]{\left\langle \left\langle \begin{matrix}1-\\ 1+\\ \Theta\\ \left\langle \sum_{j=1}^{r}\omega_j \Theta^{-1}\left\langle \begin{matrix}(\Re_k(\hbar_0^q))+\\ \Theta^{-1}(\Re_k(\hbar_j^q))\end{matrix}\right\rangle\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{matrix}1+\\ \emptyset\\ \left\langle \sum_{j=1}^{r}\omega_j \emptyset^{-1}\left\langle \begin{matrix}(\Im_k(\hbar_0^q))+\\ \emptyset^{-1}(\Im_k(\hbar_j^q))\end{matrix}\right\rangle\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\left. \sqrt[q]{\left\langle \left\langle \begin{matrix}1-\\ 1+\\ \Theta\\ \left\langle \sum_{j=1}^{r}\omega_j \Theta^{-1}\left\langle \begin{matrix}(\Re_k(\theta_0^q))+\\ \Theta^{-1}(\Re_k(\theta_j^q))\end{matrix}\right\rangle\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{matrix}1+\\ \emptyset\\ \left\langle \sum_{j=1}^{r}\omega_j \emptyset^{-1}\left\langle \begin{matrix}(\Im_k(\phi_0^q))+\\ \emptyset^{-1}(\Im_k(\phi_j^q))\end{matrix}\right\rangle\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right\rangle$$

$$=\left\langle \sqrt[q]{\left\langle \left\langle \left\langle \left\langle \begin{matrix}1-\\ 1+\\ \Theta\\ \Theta^{-1}\\ (\Re_k(\hbar_0^q))+\\ \sum_{j=1}^{r}\omega_j \Theta^{-1}(\Re_k(\hbar_j^q))\end{matrix}\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \left\langle \left\langle \begin{matrix}1+\\ \emptyset\\ \emptyset^{-1}\\ (\Im_k(\hbar_0^q))+\\ \sum_{j=1}^{r}\omega_j \emptyset^{-1}(\Im_k(\hbar_j^q))\end{matrix}\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\left. \sqrt[q]{\left\langle \left\langle \left\langle \left\langle \begin{matrix}1-\\ 1+\\ \Theta\\ \Theta^{-1}\\ (\Re_k(\theta_0^q))+\\ \sum_{j=1}^{r}\omega_j \Theta^{-1}(\Re_k(\theta_j^q))\end{matrix}\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \left\langle \left\langle \begin{matrix}1+\\ \emptyset\\ \emptyset^{-1}\\ (\Im_k(\phi_0^q))+\\ \sum_{j=1}^{r}\omega_j \emptyset^{-1}(\Im_k(\phi_j^q))\end{matrix}\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right\rangle$$

and also

$$\Delta_0, \bigoplus^{D\hbar} Cq-ROFDAWA(\Delta_1,\Delta_2,...,\Delta_r)$$

$$=\bigoplus^{D\hbar}$$

$$\left\langle \sqrt[q]{\left\langle \left\langle \left\langle \begin{matrix}1-\\ 1+\\ \Theta\\ \Theta^{-1}\\ \left\langle \sum_{j=1}^{r}\omega_j \Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle\end{matrix}\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \left\langle \begin{matrix}1+\\ \emptyset\\ \emptyset^{-1}\\ \left\langle \sum_{j=1}^{r}\omega_j \emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\end{matrix}\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\sqrt[q]{\left\langle\left\langle\left\langle 1-\; 1+\; \Theta\; \Theta^{-1}\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle\left\langle\left\langle 1+\; \emptyset\; \emptyset^{-1}\left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\Bigg\rangle$$

$$\left\langle \sqrt[q]{\left\langle\left\langle 1-\; 1+\; \Theta\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}\left\langle\begin{array}{c}(\Re_k(\hbar_0^q))+\\ \Theta^{-1}(\Re_k(\hbar_j^q))\end{array}\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle\left\langle 1+\; \emptyset\left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}\left\langle\begin{array}{c}(\Im_k(\hbar_0^q))+\\ \emptyset^{-1}(\Im_k(\hbar_j^q))\end{array}\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\sqrt[q]{\left\langle\left\langle 1-\; 1+\; \Theta\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}\left\langle\begin{array}{c}(\Re_k(\theta_0^q))+\\ \Theta^{-1}(\Re_k(\theta_j^q))\end{array}\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle\left\langle 1+\; \emptyset\left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}\left\langle\begin{array}{c}(\Im_k(\hbar_0^q))+\\ \emptyset^{-1}(\Im_k(\phi_j^q))\end{array}\right\rangle\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\Bigg\rangle$$

Getting $Cq-ROFDAWA(\Delta_0 \bigoplus^{DA} \Delta_1, \Delta_0 \bigoplus^{DA} \Delta_2, ..., \Delta_0 \bigoplus^{DA} \Delta_r)$
$= \Delta_0 \bigoplus^{DA} Cq-ROFDAWA\,(\Delta_1, \Delta_2, ..., \Delta_r)$

Theorem 5. *Let $\Delta_j = (\hbar_j, \hbar_j, \theta_j, \phi_j)$ be the $Cq-ROFEs$,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD with $\Delta_0 \in Cq-ROFNs$. $\Delta_o = \Delta_r$. Then*

$$Cq-ROFDAWA\,(\Delta_1, \Delta_2, ..., \Delta_r) = \Delta_o \tag{10}$$

Proof. We known $Cq-ROFDAWA\,(\Delta_1, \Delta_2, ..., \Delta_r)$

$$= \left\langle \sqrt[q]{\left\langle\left\langle 1-\; 1+\; \Theta\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle\left\langle 1+\; \emptyset\left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.$$

$$\sqrt[q]{\left\langle\left\langle 1-\; 1+\; \Theta\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle\left\langle 1+\; \emptyset\; \sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\Bigg\rangle$$

and

$$
= \left\langle \sqrt[q]{\left(\begin{matrix} 1- \\ 1+ \\ \left\{\Theta\left(\left(\left(\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\hbar_0^q))\right)\right)\right)\right\}^{\frac{1}{k}} \end{matrix}\right)^{-1}}, \sqrt[q]{\left(\begin{matrix} 1+ \\ \left\{\emptyset\left(\left(\left(\sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\hbar_0^q))\right)\right)\right)\right\}^{\frac{1}{k}} \end{matrix}\right)^{-1}}\right.
$$

$$
\left. \sqrt[q]{\left\langle\left\langle\begin{matrix} 1- \\ 1+ \\ \Theta \\ \left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\theta_0^q))\right\rangle \end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle\left\langle\begin{matrix} 1+ \\ \emptyset \\ \left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\phi_0^q))\right\rangle \end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}} \right\rangle
$$

$$
= \left\langle \sqrt[q]{\left\langle\left\langle\begin{matrix} 1- \\ 1+ \\ \Theta \\ \left\langle\Theta^{-1}(\Re_k(\hbar_0^q))\sum_{j=1}^{r}\omega_j\right\rangle \end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left\langle\left\langle\begin{matrix} 1+ \\ \emptyset \\ \left\langle\emptyset^{-1}(\Im_k(\hbar_0^q))\sum_{j=1}^{r}\omega_j\right\rangle \end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}\right.
$$

$$
\left. \sqrt[q]{\left(\left\langle\left\langle\begin{matrix} 1- \\ 1+ \\ \Theta \\ \left\langle\Theta^{-1}(\Re_k(\theta_0^q))\sum_{j=1}^{r}\omega_j\right\rangle \end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle\right)^{-1}}, \sqrt[q]{\left\langle\left\langle\begin{matrix} 1+ \\ \emptyset \\ \left\langle\emptyset^{-1}(\Im_k(\phi_0^q))\sum_{j=1}^{r}\omega_j\right\rangle \end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}} \right\rangle
$$

$$
= \left\langle \sqrt[q]{\left(\begin{matrix} 1- \\ 1+ \\ \left\{\Theta\left(\left(\Theta^{-1}(\Re_k(\hbar_0^q))\right)\right)\right\}^{\frac{1}{k}} \end{matrix}\right)^{-1}}, \sqrt[q]{\left(\begin{matrix} 1+ \\ \left\{\emptyset\left(\left(\emptyset^{-1}(\Im_k(\hbar_0^q))\right)\right)\right\}^{\frac{1}{k}} \end{matrix}\right)^{-1}}\right.
$$

$$
\left. \sqrt[q]{\left\langle\left\langle\begin{matrix} 1- \\ 1+ \\ \Theta \\ \left\langle\Theta^{-1}(\Re_k(\theta_0^q))\right\rangle \end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}, \sqrt[q]{\left(\left\langle\left\langle\begin{matrix} 1+ \\ \emptyset \\ \left\langle\emptyset^{-1}(\Im_k(\phi_0^q))\right\rangle \end{matrix}\right\rangle^{\frac{1}{k}}\right\rangle\right)^{-1}} \right\rangle
$$

$$
= \left\langle \sqrt[q]{\left\langle\begin{matrix} 1- \\ 1+ \\ \langle(\Re_k(\hbar_0^q))\rangle^{\frac{1}{k}} \end{matrix}\right\rangle^{-1}}, \sqrt[q]{\left\langle\begin{matrix} 1+ \\ \langle(\Im_k(\hbar_0^q))\rangle^{\frac{1}{k}} \end{matrix}\right\rangle^{-1}}\right.
$$

$$
\left. \sqrt[q]{\left\langle\begin{matrix} 1- \\ 1+ \\ \langle(\Re_k(\theta_0^q))\rangle^{\frac{1}{k}} \end{matrix}\right\rangle^{-1}}, \sqrt[q]{\left\langle\begin{matrix} 1+ \\ \langle(\Im_k(\phi_0^q))\rangle^{\frac{1}{k}} \end{matrix}\right\rangle^{-1}} \right\rangle
$$

$$= \left\langle \sqrt[q]{1-\left\langle 1+\frac{\hbar_0^q}{1-\hbar_0^q}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\frac{1-\hbar_0^q}{\hbar_0^q}\right\rangle^{-1}} \right.$$

$$\left. \sqrt[q]{1-\left\langle 1+\frac{\theta_0^q}{1-\theta_0^q}\right\rangle^{-1}}, \sqrt[q]{\left\langle 1+\frac{1-\phi_0^q}{\phi_0^q}\right\rangle^{-1}} \right\rangle$$

$$= \begin{matrix} (\hbar_0, \hbar_0), \\ (\theta_0, \phi_0) \end{matrix}$$

Theorem 6. *Let $\Delta_j = (\hbar_j, \hbar_j, \theta_j, \phi_j)$ be the $Cq-ROFEs$,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD. Then*

$$(\Delta_r)^- \prec Cq-ROFDAWA\,(\Delta_1, \Delta_2, ..., \Delta_r) \prec (\Delta_r)^+ \tag{11}$$

$$(\Delta_r)^- = \begin{pmatrix} \min\begin{pmatrix} \hbar_1, \hbar_2, ..., \hbar_r, \\ \hbar_1, \hbar_2, ..., \hbar_r \end{pmatrix}, \\ \min\begin{pmatrix} \theta_1, \theta_2, ..., \theta_r, \\ \phi_1, \phi_2, ..., \phi_r \end{pmatrix} \end{pmatrix}$$

$$\text{and } (\Delta_r)^+ = \begin{pmatrix} \max\begin{pmatrix} \hbar_1, \hbar_2, ..., \hbar_r, \\ \hbar_1, \hbar_2, ..., \hbar_r \end{pmatrix}, \\ \max\begin{pmatrix} \theta_1, \theta_2, ..., \theta_r, \\ \phi_1, \phi_2, ..., \phi_r \end{pmatrix} \end{pmatrix}$$

Proof. we know that $j \in \{1, 2, ..., r\}$, Then, $\min\begin{pmatrix} (\hbar_j, \hbar_j), \\ (\theta_j, \phi_j) \end{pmatrix} \leq \begin{pmatrix} (\hbar_j, \hbar_j), \\ (\theta_j, \phi_j) \end{pmatrix} \leq \max_j \begin{pmatrix} (\hbar_j, \hbar_j), \\ (\theta_j, \phi_j) \end{pmatrix}$, here $(\hbar_j, \hbar_j), (\theta_j, \phi_j) \in \Delta_r$, we get,

$$\left\langle \sqrt[q]{\frac{\Theta}{\left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar^{q\cdot}))\right\rangle}}, \sqrt[q]{\frac{\emptyset}{\left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\hbar^{q\cdot}))\right\rangle}} \right.$$

$$\left. \sqrt[q]{\frac{\Theta}{\left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta^{q\cdot}))\right\rangle}}, \sqrt[q]{\frac{\emptyset}{\left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\phi^{q\cdot}))\right\rangle}} \right\rangle$$

$$\leq$$

$$\left\langle \sqrt[q]{\frac{\Theta}{\left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar_j^{q}))\right\rangle}}, \sqrt[q]{\frac{\emptyset}{\left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\hbar_j^{q}))\right\rangle}} \right.$$

$$\left. \sqrt[q]{\frac{\Theta}{\left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta_j^{q}))\right\rangle}}, \sqrt[q]{\frac{\emptyset}{\left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\phi_j^{q}))\right\rangle}} \right\rangle$$

$$\leq \left\langle \sqrt[q]{\begin{array}{c}\Theta\\ \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar^{q\cdot\cdot})) \right\rangle\end{array}}, \sqrt[q]{\begin{array}{c}\emptyset\\ \left\langle \sum_{j=1}^{r} \omega \emptyset^{-1}(\Im_k(\hbar^{q\cdot\cdot})) \right\rangle\end{array}}\right.$$

$$\left. \sqrt[q]{\begin{array}{c}\Theta\\ \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta^{q\cdot\cdot})) \right\rangle\end{array}}, \sqrt[q]{\begin{array}{c}\emptyset\\ \left\langle \sum_{j=1}^{r} \omega \emptyset^{-1}(\Im_k(\phi^{q\cdot\cdot})) \right\rangle\end{array}} \right\rangle$$

and $\hbar^{\cdot}, \hbar^{\cdot}, \theta^{\cdot}, \phi^{\cdot} \in (\Delta_r)^{-}, \hbar^{\cdot\cdot}, \hbar^{\cdot\cdot}, \theta^{\cdot\cdot}, \phi^{\cdot\cdot} \in (\Delta_r)^{+}$

$$\left\langle \sqrt[q]{\begin{array}{c}1-\\ 1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar^{q\cdot})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1}, \sqrt[q]{\begin{array}{c}1+\\ \emptyset\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\hbar^{q\cdot})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1}\right.$$

$$\left. \sqrt[q]{\begin{array}{c}1-\\ 1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta^{q\cdot})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1}, \sqrt[q]{\begin{array}{c}1+\\ \emptyset\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\phi^{q\cdot})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1} \right\rangle$$

$$\leq$$

$$\left\langle \sqrt[q]{\begin{array}{c}1-\\ 1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar^{q\cdot\cdot})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1}, \sqrt[q]{\begin{array}{c}1+\\ \emptyset\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\hbar^{q\cdot\cdot})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1}\right.$$

$$\left. \sqrt[q]{\begin{array}{c}1-\\ 1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta^{q\cdot\cdot})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1}, \sqrt[q]{\begin{array}{c}1+\\ \emptyset\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\phi^{q\cdot\cdot})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1} \right\rangle$$

or

$$\left\langle \sqrt[q]{\begin{array}{c}1-\\ 1+\\ \Theta\\ \left(\left\langle \left\langle \left\langle \Theta^{-1}(\Re_k(\hbar^{q\cdot})) \sum_{j=1}^{r} \omega_j \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle \right)\end{array}}^{-1}, \sqrt[q]{\begin{array}{c}1+\\ \emptyset\\ \left\langle \left\langle \left(\left(\emptyset^{-1}(\Im_k(\hbar^{q\cdot})) \sum_{j=1}^{r} \omega_j \right) \right) \right\rangle^{\frac{1}{k}} \right\rangle\end{array}}^{-1}\right.$$

$$\sqrt[q]{\genfrac{}{}{0pt}{}{1-}{\genfrac{}{}{0pt}{}{1+}{\left(\left\{\Theta\left(\left(\left(\Theta^{-1}(\Re_k(\theta^{q\cdot}))\sum_{j=1}^{r}\omega_j\right)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}}}},\ \sqrt[q]{\left(\genfrac{}{}{0pt}{}{1+}{\left\{\emptyset\left(\left(\left(\emptyset^{-1}(\Im_k(\phi^{q\cdot}))\sum_{j=1}^{r}\omega_j\right)\right)\right)\right\}^{\frac{1}{k}}}\right)^{-1}}\right\rangle$$

$$\leq\left\langle\sqrt[q]{\genfrac{}{}{0pt}{}{1-}{\genfrac{}{}{0pt}{}{1+}{\left\langle\left\langle\left\langle\genfrac{}{}{0pt}{}{\Theta}{\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\hbar_j^{q}))\right\rangle}\right\rangle^{\frac{1}{k}}\right\rangle\right\rangle^{-1}}}},\ \sqrt[q]{\genfrac{}{}{0pt}{}{1+}{\left\langle\left\langle\genfrac{}{}{0pt}{}{\emptyset}{\left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^{q}))\right\rangle}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}}\right\rangle$$

$$\sqrt[q]{\genfrac{}{}{0pt}{}{1-}{\genfrac{}{}{0pt}{}{1+}{\left\langle\genfrac{}{}{0pt}{}{\Theta}{\left\langle\sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\theta_j^{q}))\right\rangle^{\frac{1}{k}}}\right\rangle^{-1}}}},\ \sqrt[q]{\left(\genfrac{}{}{0pt}{}{1+}{\left\langle\genfrac{}{}{0pt}{}{\emptyset}{\left\langle\sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\phi_j^{q}))\right\rangle}\right\rangle^{\frac{1}{k}}}\right)^{-1}}\right\rangle$$

$$\leq\left\langle\sqrt[q]{\genfrac{}{}{0pt}{}{1-}{\genfrac{}{}{0pt}{}{1+}{\left\langle\left\langle\genfrac{}{}{0pt}{}{\Theta}{\left\langle\Theta^{-1}(\Re_k(\hbar^{q\cdot\cdot}))\sum_{j=1}^{r}\omega_j\right\rangle}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}}},\ \sqrt[q]{\genfrac{}{}{0pt}{}{1+}{\left\langle\left\langle\genfrac{}{}{0pt}{}{\emptyset^{-1}}{\left\langle\emptyset^{-1}(\Im_k(\hbar^{q\cdot\cdot}))\sum_{j=1}^{r}\omega_j\right\rangle}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}}\right\rangle$$

$$\sqrt[q]{\genfrac{}{}{0pt}{}{1-}{\genfrac{}{}{0pt}{}{1+}{\left\langle\left\langle\genfrac{}{}{0pt}{}{\Theta}{\left\langle\Theta^{-1}(\Re_k(\theta^{q\cdot\cdot}))\sum_{j=1}^{r}\omega_j\right\rangle}\right\rangle^{\frac{1}{k}}\right\rangle^{-1}}}},\ \sqrt[q]{\left(\genfrac{}{}{0pt}{}{1+}{\left\langle\genfrac{}{}{0pt}{}{\emptyset^{-1}}{\left\langle\emptyset^{-1}(\Im_k(\phi^{q\cdot\cdot}))\sum_{j=1}^{r}\omega_j\right\rangle}\right\rangle^{\frac{1}{k}}}\right)^{-1}}\right\rangle$$

or

$$\left\langle\sqrt[q]{\genfrac{}{}{0pt}{}{1-}{\left\langle\genfrac{}{}{0pt}{}{1+}{(\Re_k(\hbar^{q\cdot})^{\frac{1}{k}}}\right\rangle^{-1}}},\ \sqrt[q]{\left\langle\genfrac{}{}{0pt}{}{1+}{\langle(\Im_k(\hbar^{q\cdot})\rangle^{\frac{1}{k}}}\right\rangle^{-1}}\right.$$

$$\left.\sqrt[q]{\genfrac{}{}{0pt}{}{1-}{1+\langle(\Re_k(\theta^{q\cdot})\rangle^{\frac{1}{k}-1}}},\ \sqrt[q]{\left(\genfrac{}{}{0pt}{}{1+}{\langle(\Im_k(\phi^{q\cdot})\rangle^{\frac{1}{k}}}\right)^{-1}}\right\rangle$$

$$\leq \left\langle \sqrt[q]{\begin{matrix}1-\\ 1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle \end{matrix}^{-1}}, \sqrt[q]{\left\langle \begin{matrix} 1+\\ \Theta\\ \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Im_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}}\right.$$

$$\sqrt[q]{\begin{matrix}1-\\ 1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle \end{matrix}^{-1}}, \sqrt[q]{\left\langle \begin{matrix} 1+\\ \Theta\\ \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Im_k(\phi_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}} \right\rangle$$

$$\leq \left\langle \sqrt[q]{\begin{matrix}1-\\ \left(1+\langle \Re_k(\hbar^{q\cdot\cdot})\rangle^{\frac{1}{k}}\right)^{-1}\end{matrix}}, \sqrt[q]{\begin{pmatrix}1+\\ \langle \Im_k(\hbar^{q\cdot\cdot})\rangle^{\frac{1}{k}}\end{pmatrix}^{-1}}\right.$$

$$\left. \sqrt[q]{1-\left(1+\langle \Re_k(\theta^{q\cdot\cdot})\rangle^{\frac{1}{k}}\right)^{-1}}, \sqrt[q]{\left(1+\langle \Im_k(\phi^{q\cdot\cdot})\rangle^{\frac{1}{k}}\right)^{-1}} \right\rangle$$

or

$$\left\langle \sqrt[q]{1-\frac{1}{1+\frac{\hbar^{q\cdot}}{1-\hbar^{q\cdot}}}}, \sqrt[q]{\frac{1}{1+\frac{1-\hbar^{q\cdot}}{\hbar^{q\cdot}}}}\right.$$

$$\left. \sqrt[q]{1-\frac{1}{1+\frac{\theta^{q\cdot}}{1-\theta^{q\cdot}}}}, \sqrt[q]{\frac{1}{1+\frac{1-\phi^{q\cdot}}{\phi^{q\cdot}}}} \right\rangle$$

$$\leq \left\langle \sqrt[q]{\begin{matrix}1-\\ 1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle \end{matrix}^{-1}}, \sqrt[q]{\left\langle \begin{matrix} 1+\\ \emptyset\\ \left\{ \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\hbar_j^q)) \right\rangle \right\rangle \right\}^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}}\right.$$

$$\sqrt[q]{\begin{matrix}1-\\ 1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle \end{matrix}^{-1}}, \sqrt[q]{\left\langle \begin{matrix} 1+\\ \emptyset\\ \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\phi_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{matrix} \right\rangle^{-1}} \right\rangle$$

$$\leq \left\langle \sqrt[q]{1-\frac{1}{1+\frac{\hbar^{q\cdot\cdot}}{1-\hbar^{q\cdot\cdot}}}}, \sqrt[q]{\frac{1}{1+\frac{1-\hbar^{q\cdot\cdot}}{\hbar^{q\cdot\cdot}}}}\right.$$

$$\left. \sqrt[q]{1-\frac{1}{1+\frac{\theta^{q\cdot\cdot}}{1-\theta^{q\cdot\cdot}}}}, \sqrt[q]{\frac{1}{1+\frac{1-\phi^{q\cdot\cdot}}{\phi^{q\cdot\cdot}}}} \right\rangle$$

or

$$\{\hbar^{\cdot}, \hbar^{\cdot}, \theta^{\cdot}, \phi^{\cdot}\} \leq \{\hbar^{\cdot\cdot}, \hbar^{\cdot\cdot}, \theta^{\cdot\cdot}, \phi^{\cdot\cdot}\}$$

Getting
$S\left(\Delta_r\right)^- \prec S\left(Cq-ROFDAWA\left(\Delta_1,\Delta_2,...,\Delta_r\right)\right) \prec S\left(\Delta_r\right)^+$
and
$\left(\Delta_r\right)^- \prec Cq-ROFDAWA\left(\Delta_1,\Delta_2,...,\Delta_r\right) \prec \left(\Delta_r\right)^+$

Theorem 7. *Let $\Delta_j=\{\hbar_j,\hbar_j,\theta_j,\phi_j\}$ and $\acute{\Delta}_j=\{\acute{\hbar}_j,\acute{\hbar}_j,\acute{\theta}_j\acute{,}\phi_j\}$ be the $Cq-ROFEs$,where $\hbar_j,\hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j,ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD and $\hbar_j\leq\acute{\hbar}_j,\hbar_j\leq,\acute{\hbar}_j\theta_j\leq\acute{\theta}_j,\phi_j\leq\acute{\phi}_j$ Then*

$$Cq-ROFDAWA\left(\Delta_1,\Delta_2,...,\Delta_r\right) \prec Cq-ROFDAWA\left(\acute{\Delta}_1,\acute{\Delta}_2,...,\acute{\Delta}_r\right) \tag{12}$$

Proof. we know,

$$=\left\langle \sqrt[q]{\begin{matrix}1-\\ \left\langle \begin{matrix}1+\\ \Theta\\ \left\langle\left\langle \sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}\end{matrix}}, \sqrt[q]{\left(\left\langle \begin{matrix}1+\\ \emptyset\\ \left\langle \left\langle \sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle\right)^{-1}}\right.$$

$$\left.\sqrt[q]{\begin{matrix}1-\\ \left\langle \begin{matrix}1+\\ \Theta\\ \left\langle \left\langle \sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}\end{matrix}}, \sqrt[q]{\left\langle \begin{matrix}1+\\ \emptyset\\ \left\langle \left\langle \sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle\right\rangle^{\frac{1}{k}}\end{matrix}\right\rangle^{-1}}\right\rangle$$

here $\hbar_j^q,\theta_j^q\in\Delta_{j(j=1,2,...,r)}$, $\hbar_j^{q\prime},\theta_j^{q\prime}\in\acute{\Delta}_{j(j=1,2,...,r)}$,
$\hbar_j^q,\phi_j^q\in\Delta_{j(j=1,2,...,r)}$ and $\hbar_j^{q\prime},\phi_j^{q\prime}\in\acute{\Delta}_{j(j=1,2,...,r)}$ then
$\Re_k(\hbar_j^q,\theta_j^q)\leq\Re_k(\hbar_j^{q\prime},\theta_j^{q\prime})$,
$\Im_k(\hbar_j^q,\phi_j^q)\leq\Im_k(\hbar_j^{q\prime},\phi_j^{q\prime})$
we have

$$\left\langle \sqrt[q]{\left\langle \begin{matrix}\Theta\left\langle \sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\hbar_j^q))\right\rangle,\\ \emptyset\left\langle \sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^q))\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}}, \sqrt[q]{\left\langle \begin{matrix}\Theta\left\langle \sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\theta_j^q))\right\rangle,\\ \emptyset\left\langle \sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\phi_j^q))\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}}\right\rangle$$

$$\leq$$

$$\left\langle \sqrt[q]{\left\langle \begin{matrix}\Theta\left\langle \sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\hbar_j^{q\prime}))\right\rangle,\\ \emptyset\left\langle \sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\hbar_j^{q\prime}))\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}}, \sqrt[q]{\left\langle \begin{matrix}\Theta\left\langle \sum_{j=1}^{r}\omega_j\Theta^{-1}(\Re_k(\theta_j^{q\prime}))\right\rangle,\\ \emptyset\left\langle \sum_{j=1}^{r}\omega_j\emptyset^{-1}(\Im_k(\phi_j^{q\prime}))\right\rangle\end{matrix}\right\rangle^{\frac{1}{k}}}\right\rangle$$

$$\Longrightarrow \left\langle \sqrt[q]{\begin{matrix}1-\\1+\\ \left\langle \left\langle \Theta \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1}\end{matrix}}, \sqrt[q]{\begin{matrix}1+\\ \emptyset\\ \left\langle \left\{ \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\hbar_j^q)) \right\rangle \right\rangle \right\}^{\frac{1}{k}} \right\rangle\end{matrix}}^{-1}\right.$$

$$\sqrt[q]{\begin{matrix}1-\\1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1}\end{matrix}}, \sqrt[q]{\begin{matrix}1+\\ \emptyset\\ \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\phi_j^q)) \right\rangle \right\rangle^{\frac{1}{k}}\end{matrix}}^{-1} \Bigg\rangle$$

$$\leq \left\langle \sqrt[q]{\begin{matrix}1-\\1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\hbar_j^{q'})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1}\end{matrix}}, \sqrt[q]{\begin{matrix}1+\\ \emptyset\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\hbar_j^{q'})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{matrix}}^{-1}\right.$$

$$\left.\sqrt[q]{\begin{matrix}1-\\1+\\ \Theta\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \Theta^{-1}(\Re_k(\theta_j^{q'})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle^{-1}\end{matrix}}, \sqrt[q]{\begin{matrix}1+\\ \emptyset\\ \left\langle \left\langle \left\langle \sum_{j=1}^{r} \omega_j \emptyset^{-1}(\Im_k(\phi_j^{q'})) \right\rangle \right\rangle^{\frac{1}{k}} \right\rangle\end{matrix}}^{-1} \right\rangle$$

$, S\left(Cq-ROFDAWA\left(\Delta_1,\Delta_2,...,\Delta_r\right)\right) \prec S\left(Cq-ROFDAWA\left(\Delta_1,\Delta_2,...,\Delta_r\right)\right)$
$Cq-ROFDAWA\left(\Delta_1,\Delta_2,...,\Delta_r\right) \prec Cq-ROFDAWA\left(\Delta_1,\Delta_2,...,\Delta_r\right)$

Definition 9. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD an . Then the mapping as: $Cq-ROFNs^U \longrightarrow Cq-ROFNs^U$ and*

$$Cq-ROFDAOWA(\Delta_1,\Delta_2,...,\Delta_r) = \bigoplus_{j=1}^{r^{D\hbar}} \left(\omega_j *^{D\hbar} \Delta_{\gamma_{(j)}}\right) \tag{13}$$

Theorem 8. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD. Then*
$Cq-ROFDAOWA(\Delta_1,\Delta_2,...,\Delta_r) =$

$$\left\langle \sqrt[q]{1-\left\langle 1+\Theta\left\langle\left\langle\left\langle\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\hbar^{q}_{\gamma(j)}))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle\right\rangle^{-1}},\ \sqrt[q]{\left\langle 1+\emptyset\left\langle\left\langle\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\hbar^{q}_{\gamma(j)}))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle\right\rangle^{-1}}\right\rangle$$

$$\sqrt[q]{1-\left\langle 1+\Theta\left\langle\left\langle\left\langle\sum_{j=1}^{r+1}\omega_j\Theta^{-1}(\Re_k(\theta^{q}_{\gamma(j)}))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle\right\rangle^{-1}},\ \sqrt[q]{\left\langle 1+\emptyset\left\langle\left\langle\left\langle\sum_{j=1}^{r+1}\omega_j\emptyset^{-1}(\Im_k(\phi^{q}_{\gamma(j)}))\right\rangle\right\rangle^{\frac{1}{k}}\right\rangle\right\rangle^{-1}}\ \Bigg\rangle \tag{14}$$

Proof. See the theorem 4.2

Theorem 9. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD, with $\Delta_o \in Cq-ROFNs$. Then*

$$Cq-ROFDAOWA(\Delta_0 \overset{DA}{\bigoplus} \Delta_r) = \Delta_0 \overset{DA}{\bigoplus} Cq-ROFDAOWA\left(\Delta_1, \Delta_2, ..., \Delta_r\right) \tag{15}$$

Proof. See the theorem 4.3

Theorem 10. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD. Then*

$$Cq-ROFDAOWA\left(\Delta_1, \Delta_2, ..., \Delta_r\right) = \Delta_0 \tag{16}$$

Proof. See the theorem 4.4

Theorem 11. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD. Then*

$$(\Delta_r)^{-} \prec Cq-ROFDAOWA\left(\Delta_1, \Delta_2, ..., \Delta_r\right) \prec (\Delta_r)^{+} \tag{17}$$

$$(\Delta_r)^{-} = \begin{pmatrix} \min\begin{pmatrix} \hbar_1, \hbar_2, ..., \hbar_r, \\ \hbar_1, \hbar_2, ..., \hbar_r \end{pmatrix}, \\ \min\begin{pmatrix} \theta_1, \theta_2, ..., \theta_r, \\ \phi_1, \phi_2, ..., \phi_r \end{pmatrix} \end{pmatrix}$$

$$and\ (\Delta_r)^+ = \left(\begin{array}{c} \max\left(\begin{array}{c} \hbar_1, \hbar_2, ..., \hbar_r, \\ \hbar_1, \hbar_2, ..., \hbar_r \end{array} \right), \\ \max\left(\begin{array}{c} \theta_1, \theta_2, ..., \theta_r, \\ \phi_1, \phi_2, ..., \phi_r \end{array} \right) \end{array} \right)$$

Proof. See the theorem 4.5

Theorem 12. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ and $\acute{\Delta}_j = \{\acute{\hbar}_j, \acute{\hbar}_j, \acute{\theta}_j, \acute{\phi}_j\}$ be the $Cq-ROFEs$, where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD and $\hbar_j \leq \acute{\hbar}_j, \hbar_j \leq, \acute{\hbar}_j \theta_j \leq \acute{\theta}_j, \phi_j \leq \acute{\phi}_j$ Then*

$$Cq-ROFDAOWA\left(\Delta_1, \Delta_2, ..., \Delta_r\right) \prec Cq-ROFDAOWA\left(\acute{\Delta}_1, \acute{\Delta}_2, ..., \acute{\Delta}_r\right) \tag{18}$$

Proof. See the theorem 4.6

Definition 10. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U, where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD. Then mapping is follows: $Cq-ROFNs^U \longrightarrow Cq-ROFNs^U$ and as*

$$Cq-ROFDAWG(\Delta_1, \Delta_2, ..., \Delta_r) = \bigoplus_{j=1}^{r^{D\hbar}} \left(\omega_j \circ^{D\hbar} \Delta_j\right) \tag{19}$$

Theorem 13. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U, where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD. Then the aggregated value of $Cq-ROFDAWG(\Delta_1, \Delta_2, ..., \Delta_r)$ as,*

$Cq-ROFDAWG(\Delta_1, \Delta_2, ..., \Delta_r) =$

$$\left\langle \sqrt[q]{\left\langle \begin{array}{c} 1+ \\ \Theta \\ \left\langle \left\langle \sum_{j=1}^{r+1} \omega_j \Theta^{-1}(\Re_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \emptyset \\ \left\langle \left\langle \sum_{j=1}^{r+1} \omega_j \emptyset^{-1}(\Im_k(\hbar_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}} \right.$$

$$\left. \sqrt[q]{\left\langle \begin{array}{c} 1+ \\ \Theta \\ \left\langle \left\langle \sum_{j=1}^{r+1} \omega_j \Theta^{-1}(\Re_k(\theta_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}}, \sqrt[q]{\left\langle \begin{array}{c} 1- \\ 1+ \\ \emptyset \\ \left\langle \left\langle \sum_{j=1}^{r+1} \omega_j \emptyset^{-1}(\Im_k(\phi_j^q)) \right\rangle \right\rangle^{\frac{1}{k}} \end{array} \right\rangle^{-1}} \right\rangle \tag{20}$$

Proof. See theorem 4.2

Theorem 14. *Let* $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ *be the* $Cq-ROFEs$ *on* U,*where* $\hbar_j, \hbar_j$ *denotes the amplitude terms such that* $\hbar_j$ *stand for MD while the* $\hbar_j$ *stand for NMD and* θ_j, ϕ_j *represent the phase term such that the* θ_j *denote the MD while* ϕ_j *is for NMD and with* $\Delta_o \in Cq-ROFNs$. *Then*

$$Cq{-}ROFDAWG(\Delta_o \bigoplus^{DA} \Delta_1, \Delta_o \bigoplus^{DA} \Delta_2, ..., \Delta_o \bigoplus^{DA} \Delta_r) = \Delta_o \bigoplus^{DA} Cq{-}ROFDAWA\left(\Delta_1, \Delta_2, ..., \Delta_r\right) \tag{21}$$

Proof. See the theorem 4.3

Theorem 15. *Let* $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ *be the* $Cq-ROFEs$ *on* U,*where* $\hbar_j, \hbar_j$ *denotes the amplitude terms such that* $\hbar_j$ *stand for MD while the* $\hbar_j$ *stand for NMD and* θ_j, ϕ_j *represent the phase term such that the* θ_j *denote the MD while* ϕ_j *is for NMD, with* $\Delta_o \in Cq-ROFNs$. $\Delta_o = \Delta_j$. *Then*

$$Cq - ROFDAWG\left(\Delta_1, \Delta_2, ..., \Delta_r\right) = \Delta_o \tag{22}$$

Proof. See the theorem 4.4

Theorem 16. *Let* $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ *be the* $Cq-ROFEs$ *on* U,*where* $\hbar_j, \hbar_j$ *denotes the amplitude terms such that* $\hbar_j$ *stand for MD while the* $\hbar_j$ *stand for NMD and* θ_j, ϕ_j *represent the phase term such that the* θ_j *denote the MD while* ϕ_j *is for NMD and* $\Delta_o \in Cq-ROFNs$. *Then*

$$(\Delta_r)^- \prec Cq - ROFDAWG\left(\Delta_1, \Delta_2, ..., \Delta_r\right) \prec (\Delta_r)^+ \tag{23}$$

$$As\ (\Delta_r)^- = \begin{pmatrix} \min\begin{pmatrix} \hbar_1, \hbar_2, ..., \hbar_r, \\ \hbar_1, \hbar_2, ..., \hbar_r \end{pmatrix}, \\ \min\begin{pmatrix} \theta_1, \theta_2, ..., \theta_r, \\ \phi_1, \phi_2, ..., \phi_r \end{pmatrix} \end{pmatrix}$$

$$and\ (\Delta_r)^+ = \begin{pmatrix} \max\begin{pmatrix} \hbar_1, \hbar_2, ..., \hbar_r, \\ \hbar_1, \hbar_2, ..., \hbar_r \end{pmatrix}, \\ \max\begin{pmatrix} \theta_1, \theta_2, ..., \theta_r, \\ \phi_1, \phi_2, ..., \phi_r \end{pmatrix} \end{pmatrix}$$

Proof. See the theorem 4.5

Theorem 17. *Let* $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ *be the* $Cq-ROFEs$,*where* $\hbar_j, \hbar_j$ *denotes the amplitude terms such that* $\hbar_j$ *stand for MD while the* $\hbar_j$ *stand for NMD and* θ_j, ϕ_j *represent the phase term such that the* θ_j *denote the MD while* ϕ_j *is for NMD and* $\hbar_j \leq \acute{\hbar}_j, \hbar_j \leq , \acute{\hbar}_j \theta_j \leq \acute{\theta}_j, \phi_j \leq \acute{\phi}_j$ *Then*

$$Cq - ROFDAWG\left(\Delta_1, \Delta_2, ..., \Delta_r\right) \prec Cq - ROFDAWG\left(\left(\acute{\Delta}_1, \acute{\Delta}_2, ..., \acute{\Delta}_r\right)\right) \tag{24}$$

Proof. See the theorem 4.6

Definition 11. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD and $\Delta_o \in Cq-ROFNs$. Then the mapping as follows : $C^U_{IFS} \longrightarrow C^U_{IFS}$ as*

$$Cq-ROFDAOWG(\Delta_1, \Delta_2, ..., \Delta_r) = \bigoplus_{j=1}^{r^{D\hbar}} \left(\omega_j \circ^{D\hbar} \Delta_{\gamma_{(j)}}\right) \tag{25}$$

Theorem 18. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD and $\Delta_o \in Cq-ROFNs$. Then the aggregated value of $Cq-ROFDAOWG(\Delta_1, \Delta_2, ..., \Delta_r)$ and as,*

$Cq-ROFDAOWG(\Delta_1, \Delta_2, ..., \Delta_r) =$

$$\left\langle \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \Theta \\ \left\langle \sum_{j=1}^{r+1} \omega_j \Theta^{-1}(\Re_k(\hbar^q_{\gamma_{(j)}})) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \emptyset \\ \left\langle \sum_{j=1}^{r+1} \omega_j \emptyset^{-1}(\Im_k(\hbar^q_{\gamma_{(j)}})) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right.$$

$$\left. \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1+ \\ \Theta \\ \left\langle \sum_{j=1}^{r+1} \omega_j \Theta^{-1}(\Re_k(\theta^q_{\gamma_{(j)}})) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}}, \sqrt[q]{\left\langle \left\langle \begin{array}{c} 1- \\ 1+ \\ \emptyset \\ \left\langle \sum_{j=1}^{r+1} \omega_j \emptyset^{-1}(\Im_k(\phi^q_{\gamma_{(j)}})) \right\rangle \end{array} \right\rangle^{\frac{1}{k}} \right\rangle^{-1}} \right\rangle \tag{26}$$

Proof. See the theorem 4.2

Theorem 19. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j represent the phase term such that the θ_j denote the MD while ϕ_j is for NMD with $\Delta_o \in Cq-ROFNs$. Then*

$$Cq-ROFDAOWG(\Delta_o \overset{DA}{\bigoplus} \Delta_r) = \Delta_o \overset{DA}{\bigoplus} Cq-ROFDAOWG\,(\Delta_1, \Delta_2, ..., \Delta_r) \tag{27}$$

Proof. See the theorem 4.3

Theorem 20. *Let $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ be the $Cq-ROFEs$ on U,where $\hbar_j, \hbar_j$ denotes the amplitude terms such that $\hbar_j$ stand for MD while the $\hbar_j$ stand for NMD and θ_j, ϕ_j*

represent the phase term such that the θ_j *denote the MD while* ϕ_j *is for NMD with* $\Delta_o \in Cq - ROFNs$*.* $\Delta_o = \Delta_j$*. Then*

$$Cq - ROFDAOWG\left(\Delta_1, \Delta_2, ..., \Delta_r\right) = \Delta_o \tag{28}$$

Proof. See the theorem 4.4

Theorem 21. *Let* $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ *be the* $Cq - ROFEs$ *on U,where* $\hbar_j, \hbar_j$ *denotes the amplitude terms such that* $\hbar_j$ *stand for MD while the* $\hbar_j$ *stand for NMD and* θ_j, ϕ_j *represent the phase term such that the* θ_j *denote the MD while* ϕ_j *is for NMD with* $\Delta_o \in Cq - ROFNs$*. Then*

$$(\Delta_r)^- \prec Cq - ROFDAOWG\left(\Delta_1, \Delta_2, ..., \Delta_r\right) \prec (\Delta_r)^+ \tag{29}$$

$$(\Delta_r)^- = \begin{pmatrix} \min\begin{pmatrix} \hbar_1, \hbar_2, ..., \hbar_r, \\ \hbar_1, \hbar_2, ..., \hbar_r \end{pmatrix}, \\ \min\begin{pmatrix} \theta_1, \theta_2, ..., \theta_r, \\ \phi_1, \phi_2, ..., \phi_r \end{pmatrix} \end{pmatrix}$$

$$\text{and } (\Delta_r)^+ = \begin{pmatrix} \max\begin{pmatrix} \hbar_1, \hbar_2, ..., \hbar_r, \\ \hbar_1, \hbar_2, ..., \hbar_r \end{pmatrix}, \\ \max\begin{pmatrix} \theta_1, \theta_2, ..., \theta_r, \\ \phi_1, \phi_2, ..., \phi_r \end{pmatrix} \end{pmatrix}$$

Proof. See the theorem 4.5

Theorem 22. *Let* $\Delta_j = \{\hbar_j, \hbar_j, \theta_j, \phi_j\}$ *and* $\acute{\Delta}_j = \{\acute{\hbar}_j, \acute{\hbar}_j, \acute{\theta}_j\acute{,}\phi_j\}$*be the* $Cq-ROFEs$*,where* $\hbar_j, \hbar_j$ *denotes the amplitude terms such that* $\hbar_j$ *stand for MD while the* $\hbar_j$ *stand for NMD and* θ_j, ϕ_j *represent the phase term such that the* θ_j *denote the MD while* ϕ_j *is for NMD and* $\hbar_j \leq \acute{\hbar}_j, \hbar_j \leq, \acute{\hbar}_j \theta_j \leq \acute{\theta}_j, \phi_j \leq \acute{\phi}_j$ *Then*

$$Cq - ROFDAOWG\left(\Delta_1, \Delta_2, ..., \Delta_r\right) \prec Cq - ROFDAOWG\left(\acute{\Delta}_1, \acute{\Delta}_2, ..., \acute{\Delta}_r\right) \tag{30}$$

Proof. See the theorem 4.6

5. A Novel MADM Approach Based on the Proposed Aggregation Operators

This section presents a novel MCDM method utilizing CqROFNs for evaluation problems. The methodology employs the proposed aggregation operators to handle decision-making scenarios under qRO fuzzy environments.

5.1. Problem Framework

Consider a decision problem with the following components:

- A set of m alternatives: $\breve{A} = \{\breve{A}_1, \breve{A}_2, \ldots, \breve{A}_m\}$
- A set of r criteria with weight vector: $\tilde{\omega} = \{\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_r\}$, where $\sum_{j=1}^{r} \tilde{\omega}_j = 1$
- A CqROF decision matrix: $\tilde{D} = [¥_{ij}]_{m\times r}$, where each element $¥_{ij}$ is a CqROFN representing the evaluation of alternative $\breve{A}_i$ against criterion $\hbar_j$

The proposed method employs three key aggregation operators: the CqROFDAWG operator, CqROFDAOWA operator, and CqROFDAOWG operator to process the CqROF information and select the optimal alternative from the m available options based on r criteria.

5.2. Algorithm

Step 1: Construct the Decision Matrix.
Formulate the complex q-rung orthopair fuzzy (CqROF) decision matrix $\tilde{D} = [d_{ij}]_{m\times r}$, where each element d_{ij} represents the q-rung orthopair fuzzy evaluation of alternative A_i with respect to criterion C_j for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r$.

Step 2: Normalize the Decision Matrix.
For problems involving both benefit-type (B) and cost-type (C) criteria, transform the original matrix $\tilde{D} = [d_{ij}]_{m\times r}$ into a normalized matrix $\bar{D} = [\bar{d}_{ij}]_{m\times r}$.

Step 3: Determine Criteria Weights.
Calculate criterion weights considering two scenarios:

- **Case 1:** For completely unknown weights, employ the entropy method.
- **Case 2:** For partially known weights, utilize available constraints with the entropy method to determine unknown weights.

Step 4: Aggregate Alternative Evaluations.
Compute comprehensive values for each alternative A_i $(i = 1, 2, \ldots, m)$ using one of the aggregation operators: CqROFDAWA, CqROFDAOWA, CqROFDAWG, or CqROFDAOWG.

Step 5: Calculate Score Values.
Determine score values for each alternative using the defined score function.

Step 6: Rank the Alternatives.
Arrange alternatives in ascending order based on their score values.

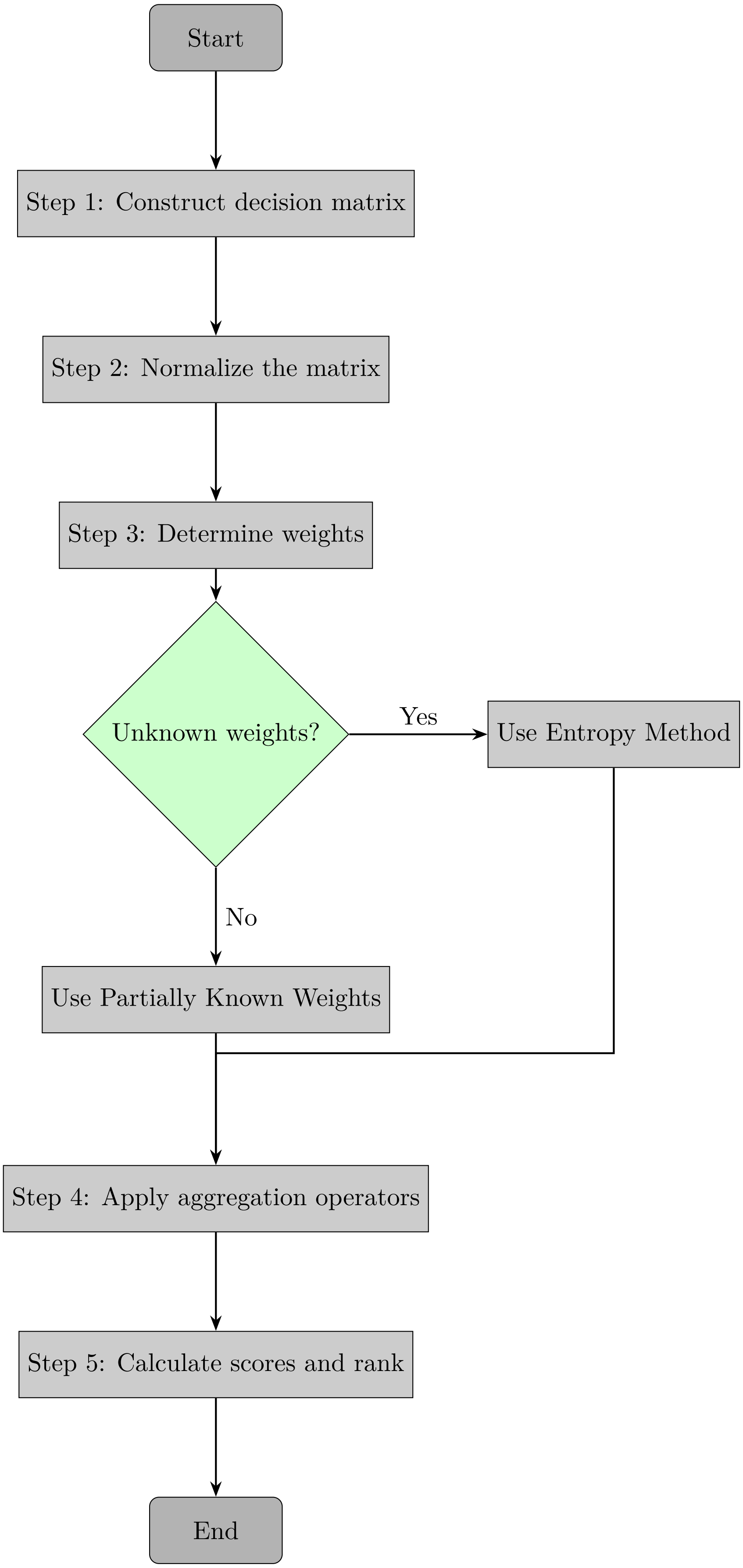

Figure 1: Flowchart of the Proposed MCDM Methodology

6. Case Study

In this section, we aim to provide a comprehensive and illustrative example that emphasizes the practical application of the proposed method for recruiting Human Resource Information Systems (HRIS) and Human Resource Management (HRM) professionals. The focus is on exploring various studies and perspectives to underscore the critical role of HRIS and HRM in achieving organizational goals and ensuring stainability.
Practical Application of the Proposed Method: To exemplify the effectiveness of the proposed recruitment method, we delve into the dynamic intersection of HRIS and HRM. The recruitment process is pivotal in identifying and outboarding individuals who can harness the potential of information systems for human resource functions.
HRIS and Organizational Efficiency: Research indicates that organizations leveraging advanced HRIS technologies experience enhanced efficiency in managing human resources. The recruitment process, when guided by a strategic method, ensures the selection of candidates with the skills to optimize HRIS for streamlined data management, reporting, and analytic. This, in turn, contributes to improved decision-making processes within the organization.
HRM's Strategic Role: Examining the landscape of HRM [42], it becomes evident that strategic human resource practices are fundamental to achieving organizational goals. The recruitment method proposed here emphasizes aligning HRM strategies with overall business objectives. Studies have shown that a well-integrated HRM approach positively influences employee performance, engagement, and, consequently, organizational stainability.
Impact on Organizational Goals: By employing the proposed recruitment method, organizations can strategically position themselves to meet and exceed their goals. The synergy between HRIS and HRM, as facilitated by a carefully designed recruitment process, enables businesses to adapt to changing market dynamics, foster innovation, and ensure the continuous development of their workforce.
Stainability Through Talent Acquisition: Stainability is a key concern for modern organizations. The recruitment method advocated here extends its impact to the long-term stainability of businesses. By identifying and hiring professionals well-versed in HRIS and aligned with strategic HRM principles, organizations create a foundation for sustained growth, resilience, and adaptability in the face of evolving challenges. In conclusion, the proposed recruitment method serves as a strategic enabler for organizations seeking to harness the full potential of HRIS and HRM. Through a careful examination of studies and perspectives, we have highlighted the instrumental role of this method in achieving organizational goals and ensuring stainability. The interplay between technology and human resource management, when managed effectively, becomes a cornerstone for organizational success in today's dynamic business environment.

The passage breaks down the evaluation criteria for Human Resource Information Systems (HRIS) into five categories: Human-resource management functions ($\breve{C}_1$), Technology ($\breve{C}_2$), Software quality ($\breve{C}_3$), Cost ($\breve{C}_4$), and Vendor support ($\breve{C}_5$). Here's a summary of each category and its associated criteria:

6.1. Human-resource management functions ($\breve{C}_1$):

Employee records, including interactions and supervisor reports, are kept up to date in a database by Staff Information Management ($\breve{C}_{1-1}$) :. The organisation of workers and their skills within the framework of the business is known as labour management and organisation structure ($\breve{C}_{1-2}$) :. Budget management in conjunction with compensation and benefits (C&B) ($\breve{C}_{1-3}$) : Oversees employee welfare and financial matters, including pay, bonuses, and career advancement. Employee training courses, attendance, testing, grading, and question management are all tracked by Staff Training Management ($\breve{C}_{1-4}$) :. Staff Recruiting Management ($\breve{C}_{1-5}$) : oversees the screening, selection, job analysis, candidate tracking, and onboarding processes. Staff Performance Management ($\breve{C}_{1-6}$) : Maintains records of work schedules, organisational objectives, evaluations, incentives, and sanctions, as well as performance metrics.

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6.2. Technology ($\breve{C}_2$) :

($\breve{C}_{2-1}$) Big Data Analysis: Handles and examines comprehensive, unstructured organisational data. Artificial Intelligence ($\breve{C}2-2$) : Makes suggestions for employee tasks based on analysis of postures and facial expressions. Cyberspace (IoT,)($\breve{C}_{2-3}$) : uses RFID or sensors with unique identifiers (UID) to log and track employee actions and provide real-time analytical data. Network Social ($\breve{C}_{2-4}$) : Offers a forum for business networking and employee engagement; it may also integrate with other social media sites. Services for Self-Help ($\breve{C}_{2-5}$): Provides automatic answers to questions from staff members, including those about leave requests.

6.3. Software quality ($\breve{C}_3$) :

Functionality ($\breve{C}_{3-1}$) : Indicates how effectively the programme complies with the demands of organisational design. Dependability ($\breve{C}_{3-2}$) : Assesses the reliability of organisational and employee data kept in the programme. Usability($\breve{C}_{3-3}$): Takes into account user interfaces and intuitiveness when determining ease of use. Effectiveness ($\breve{C}_{3-4}$): Indicates how well the software performs overall throughout the organisation and how much less time is needed for HR-related duties. Reliability ($\breve{C}_{3-5}$) : Evaluates how simple it is to maintain programme functionality using dependable backup methods. Mobility ($\breve{C}_{3-6}$) : Ascertains whether data corruption occurs while exporting and importing HR data to other programmes.

6.4. Cost ($\breve{C}_4$) :

Operation and Maintenance Fees($\breve{C}_{4-1}$): Pays for the costs associated with using the software to organizeÂ continuing HR processes. LicensingÂ Fees ($\breve{C}_{4-2}$): Comprises the expenses related to securing formal authorization to utilizeÂ the program. Fees for Consultation ($\breve{C}_{4-3}$) :Alludes to the expenses incurred in discussing the programÂ and the objectives of the organization. The equipment cost, ($\breve{C}_{4-4}$), includes costs for the different hardware parts that the software needs. Software Training Fees ($\breve{C}_{4-5}$): These represent the expenses incurred in instructing staff members, especially HR and IT managers, on how to make the most of all software functionalities.

6.5. Vendor support ($\breve{C}_5$) :

Vendor Reputation ($\breve{C}_{5-1}$) : Relies on the software provider's overall standing as determined by previous interactions with clients. Technical Proficiency ($\breve{C}_{5-2}$) : looks at the characteristics that software businesses have in order to offer solutions. Commitment for After-Sales Service ($\breve{C}_{5-3}$): assesses the promises made in the form of assurances to the organisation. Update and Upgrade for After-Sales ($\breve{C}_{5-4}$) : Evaluates the potential for introducing new packages, resolving issues, and staying current with technical advancements. ($\breve{C}_{5-5}$) : Software Delivery and Service Response Time calculates the time it takes for organisational usage, starting with planning and ending with payment.

In this case, we apply the $Cq-ROF$ fuzzy to the HR recruitment process. In order to make a well-informed judgement, the HR department may use the $Cq-ROF$ fuzzy theory throughout the recruitment process. In this case, five alternatives $\breve{A}_{i(i=1,2,3,4,5)}$ will be chosen for further evaluation after predomination. Thinking about the five qualities that are outlined below:

(i) $\breve{C}_1$: Human-resource management functions

(ii) $\breve{C}_2$: Technology

(iii) $\breve{C}_3$: Software quality

(iv) $\breve{C}_4$: Cost

(v) $\breve{C}_5$: Vendor support

6.6. Problem Solution

A method for addressing $MCDM$ problems utilising $Cq-ROFNs$ data will be created using the $Cq-ROFDAWA$, $Cq-ROFDAOWA$, $Cq-ROFDAWG$, and $Cq-ROFDAOWG$ operators. Where $U=\{\breve{A}_1,\breve{A}_2,...,\breve{A}_m\}$ and $A=\{\breve{C}_1,\breve{C}_2,...,\breve{C}_r\}$ are the variables being defined. ω represents a set of options, $\omega_1,\omega_2,...,\omega_r\}$ denotes a set of characteristics, and ω_r denotes a set of weights. Imagine the $Cq-ROF$ matrix $\tilde{D}=[d_{ij}]_{m\times r}$, where each d_{ij} is represented as $Cq-ROFNs$. The operators $Cq-ROFDAWA$, $Cq-ROFDAWG$, $Cq-ROFDAOWA$, or $Cq-ROFDAOWG$ are now used in the proposed method to address the $M_{\hbar DM}$ problems with the $Cq-ROF$ data. The following steps, in the specified sequence, constitute the suggested approach:

Step 1: The table 2shows the $Cq-ROF$ decision matrix

Table 2: $Cq - ROF$ decision matrix by experts

	$\breve{C}_1$	$\breve{C}_2$	$\breve{C}_3$	$\breve{C}_4$	$\breve{C}_5$
$\breve{A}_1$	$\langle \frac{80}{100}, \frac{70}{100}, \frac{60}{100}, \frac{90}{100} \rangle$	$\langle \frac{60}{100}, \frac{90}{100}, \frac{40}{100}, \frac{70}{100} \rangle$	$\langle \frac{50}{100}, \frac{91}{100}, \frac{45}{100}, \frac{76}{100} \rangle$	$\langle \frac{92}{100}, \frac{50}{100}, \frac{85}{100}, \frac{56}{100} \rangle$	$\langle \frac{64}{100}, \frac{69}{100}, \frac{68}{100}, \frac{77}{100} \rangle$
$\breve{A}_2$	$\langle \frac{70}{100}, \frac{80}{100}, \frac{70}{100}, \frac{90}{100} \rangle$	$\langle \frac{72}{100}, \frac{84}{100}, \frac{87}{100}, \frac{78}{100} \rangle$	$\langle \frac{87}{100}, \frac{78}{100}, \frac{64}{100}, \frac{69}{100} \rangle$	$\langle \frac{88}{100}, \frac{77}{100}, \frac{58}{100}, \frac{67}{100} \rangle$	$\langle \frac{61}{100}, \frac{77}{100}, \frac{60}{100}, \frac{60}{100} \rangle$
$\breve{A}_3$	$\langle \frac{60}{100}, \frac{90}{100}, \frac{80}{100}, \frac{70}{100} \rangle$	$\langle \frac{40}{100}, \frac{70}{100}, \frac{60}{100}, \frac{90}{100} \rangle$	$\langle \frac{45}{100}, \frac{76}{100}, \frac{50}{100}, \frac{91}{100} \rangle$	$\langle \frac{85}{100}, \frac{56}{100}, \frac{92}{100}, \frac{50}{100} \rangle$	$\langle \frac{68}{100}, \frac{77}{100}, \frac{64}{100}, \frac{69}{100} \rangle$
$\breve{A}_4$	$\langle \frac{90}{100}, \frac{70}{100}, \frac{70}{100}, \frac{90}{100} \rangle$	$\langle \frac{60}{100}, \frac{83}{100}, \frac{83}{100}, \frac{60}{100} \rangle$	$\langle \frac{72}{100}, \frac{55}{100}, \frac{55}{100}, \frac{72}{100} \rangle$	$\langle \frac{87}{100}, \frac{47}{100}, \frac{87}{100}, \frac{47}{100} \rangle$	$\langle \frac{92}{100}, \frac{50}{100}, \frac{50}{100}, \frac{92}{100} \rangle$
$\breve{A}_5$	$\langle \frac{70}{100}, \frac{90}{100}, \frac{70}{100}, \frac{80}{100} \rangle$	$\langle \frac{87}{100}, \frac{78}{100}, \frac{72}{100}, \frac{84}{100} \rangle$	$\langle \frac{64}{100}, \frac{69}{100}, \frac{87}{100}, \frac{78}{100} \rangle$	$\langle \frac{58}{100}, \frac{67}{100}, \frac{88}{100}, \frac{77}{100} \rangle$	$\langle \frac{60}{100}, \frac{60}{100}, \frac{61}{100}, \frac{77}{100} \rangle$

Step 2: Since in the table 2 $\breve{C}_1, \breve{C}_2, \breve{C}_3$ are benefit attributes and also $\breve{C}_4$, $\breve{C}_5$, are the cost attribute so, we need to normalized the table. The table 3 shows the Normalization table.

Table 3: $Cq - ROF$ Normalized matrix

	$\breve{C}_1$	$\breve{C}_2$	$\breve{C}_3$	$\breve{C}_4$	$\breve{C}_5$
$\breve{A}_1$	$((0.2666, 0.8084), (0.1650, 0.7546))$	$((0.2146, 0.7407), (0.1414, 0.7744))$	$((0.1253, 0.8271), (0.1754, 0.8271))$	$((0.1415, 0.7940), (0.1661, 0.7642))$	$((0.2074, 0.8114), (0.2000, 0.8316))$
$\breve{A}_2$	$((0.1768, 0.7687), (0.1980, 0.8400))$	$((0.2074, 0.8114), (0.2000, 0.8316))$	$((0.2735, 0.7876), (0.2126, 0.8212))$	$((0.2146, 0.7407), (0.1414, 0.7744))$	$((0.1768, 0.7687), (0.1980, 0.8400))$
$\breve{A}_3$	$((0.1971, 0.7687), (0.2112, 0.6160))$	$((0.1253, 0.8271), (0.1754, 0.8271))$	$((0.1415, 0.7940), (0.1661, 0.7642))$	$((0.2074, 0.8114), (0.2000, 0.8316))$	$((0.1971, 0.7687), (0.2112, 0.6160))$
$\breve{A}_4$	$((0.1414, 0.7744), (0.2146, 0.7407))$	$((0.1739, 0.8198), (0.2013, 0.7946))$	$((0.2264, 0.8509), (0.1827, 0.3134))$	$((0.2121, 0.2121), (0.2365, 0.8417))$	$((0.2666, 0.8084), (0.1650, 0.7546))$
$\breve{A}_5$	$((0.1891, 0.7750), (0.2162, 0.7857))$	$((0.2727, 0.8074), (0.2426, 0.7801))$	$((0.2264, 0.8130), (0.2890, 0.7979))$	$((0.1414, 0.7744), (0.2146, 0.7407))$	$((0.1739, 0.8198), (0.2013, 0.7946))$

Case I: When weight is completely unknown:

Researchers typically have basic knowledge on weights in many real-world challenges. The usual practice is to randomly initialise these weights and then use a variety of techniques to fine-tune them such that the difference between the expected and actual results is as little as possible. Nonetheless, there are cases when the weights are ambiguous or not known at all. Researchers are forced to use different methods to estimate the weights and measure the uncertainty in such instances.

Step 3: The w_i shows the unknown weight determined by entropy method and Eq.(5.1), that as, $w_1 = 0.1990$, $w_2 = 0.1993$, $w_3 = 0.1960$, $w_4 = 0.2059, w_5 = 0.1998$

Step 4: The table 4 shows the outcomes of a ranking procedure that determined the score values of the alternatives using four operators, Cq-ROFDAWA, Cq-ROFOWA, Cq-ROFWG, and Cq-ROFOWG, as given in Equations (4.2), (4.7), (4.14), and (4.20), respectively.

Table 4: Score function with respect to WA Operator

Alternatives	Score Function	Score Values
$\breve{A}_1$	$s(\breve{A}_1)$	0.4824
$\breve{A}_2$	$s(\breve{A}_2)$	0.4867
$\breve{A}_3$	$s(\breve{A}_3)$	0.4643
$\breve{A}_4$	$s(\breve{A}_4)$	0.4904
$\breve{A}_5$	$s(\breve{A}_5)$	0.4911

Step 5: The table 5 shows the ranking of the alternatives using , Cq-ROFDAWA, operator.

Table 5: Ranking w.r.t CqROFDAWA

$\breve{A}_5 \succ \breve{A}_4 \succ \breve{A}_1 \succ \breve{A}_2 \succ \breve{A}_3$

Step 6: Similarly the table 6, 7, and 8 shows the score values and ranking of the alternatives using Cq-ROFDAOWA, Cq-ROFDAWAG, and Cq-ROFDAOWG operator respectively.

Table 6: Score function and ranking w.r.t CqROFDAOWA

Alternatives	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$	$\breve{A}_5$	*Ranking*
values	0.4689	0.4658	0.4156	0.4734	0.4774	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$

Table 7: Score function and ranking w.r.t CqROFDAWG

Alternatives	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$	$\breve{A}_5$	*Ranking*
values	0.4831	0.4663	0.4603	0.4911	0.5077	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$

Table 8: Score function and ranking w.r.t CqROFDAOWG

Alternatives	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$	$\breve{A}_5$	*Ranking*
values	0.4718	0.4593	0.4501	0.4865	0.4778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$

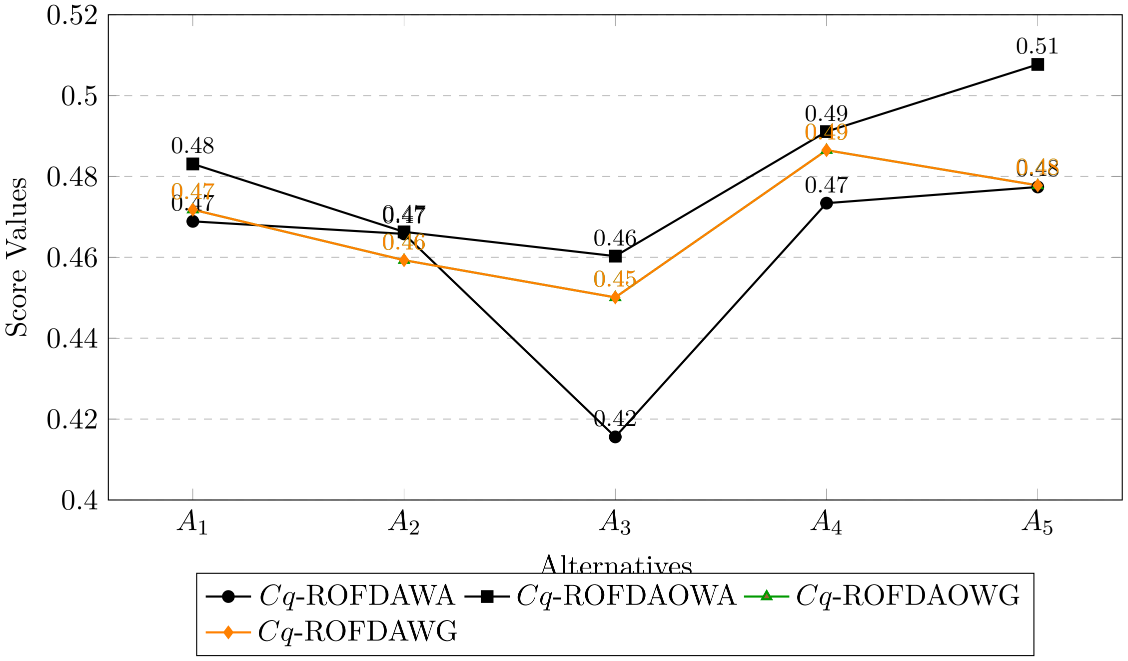

Figure 2: Comparison of Alternative Scores under Different Cq-ROFDA Operators

From the graph 2, it is evident that $\breve{A}_5$ consistently achieves the highest score across all four operators, confirming it as the best alternative. The relative consistency in ranking also validates the robustness of the proposed Cq-ROFDA-based methods.

Case II: When weight is partially known: When just a portion of the weight is known: Finally, a versatile strategy is required for handling weights in practical problem-solving. The ever-changing nature of real-life situations might bring ambiguity, even if researchers usually have access to some initial weight data. Researchers adjust their models to the complexities of the data they have by using a mix of algorithmic updates and random initialisation to enhance the weights, even whether the weights are partly known or not. The partly known weights for the specific qualities are $w_1 = 0.20, w_2 = 0.25, w_3 = 0.25, w_4 = 0.20, w_5 = 0.10$.

Step 8: The table 9 shows the score values of the alternatives using four operators, Cq-ROFDAWA, Cq-ROFOWA, Cq-ROFWG, and Cq-ROFOWG, as given in Equations (4.2), (4.7), (4.14), and (4.20), respectively.

Table 9: Score function w.r.t CqROFDAWA

Alternatives	Score Function	Score Values
$\breve{A}_1$	$s(\breve{A}_1)$	0.4839
$\breve{A}_2$	$s(\breve{A}_2)$	0.4914
$\breve{A}_3$	$s(\breve{A}_3)$	0.4522
$\breve{A}_4$	$s(\breve{A}_4)$	0.4590
$\breve{A}_5$	$s(\breve{A}_5)$	0.4986

Then $s(\breve{A}_5) > s(\breve{A}_2) > s(\breve{A}_1) > s(\breve{A}_4) > s(\breve{A}_3)$

Step 9: The table 10 shows the ranking of the alternatives using , Cq-ROFDAWA, operator.

Table 10: Ranking w.r.t CqROFDAWA

$\breve{A}_5 \succ \breve{A}_2 \succ \breve{A}_1 \succ \breve{A}_3 \succ \breve{A}_3$

Step 10: Similarly the table 11, 12, and 13 shows the score values and ranking of the alternatives using Cq-ROFDAOWA, Cq-ROFDAWAG, and Cq-ROFDAOWG operator respectively.

Table 11: Score function and ranking w.r.t CqROFDAWA

Alternatives	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$	$\breve{A}_5$	*Ranking*
values	0.4988	0.52012	0.4695	0.4875	0.5279	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_4 > \breve{A}_3$

Table 12: Score function and ranking w.r.t CqROFDAWG

Alternatives	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$	$\breve{A}_5$	*Ranking*
values	0.4830	4870	0.4593	0.4523	0.6654	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$

Table 13: Score function and ranking w.r.t CqROFDAOWG

Alternatives	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$	$\breve{A}_5$	*Ranking*
values	0.5103	0.5117	0.4988	0.4765	0.5347	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$

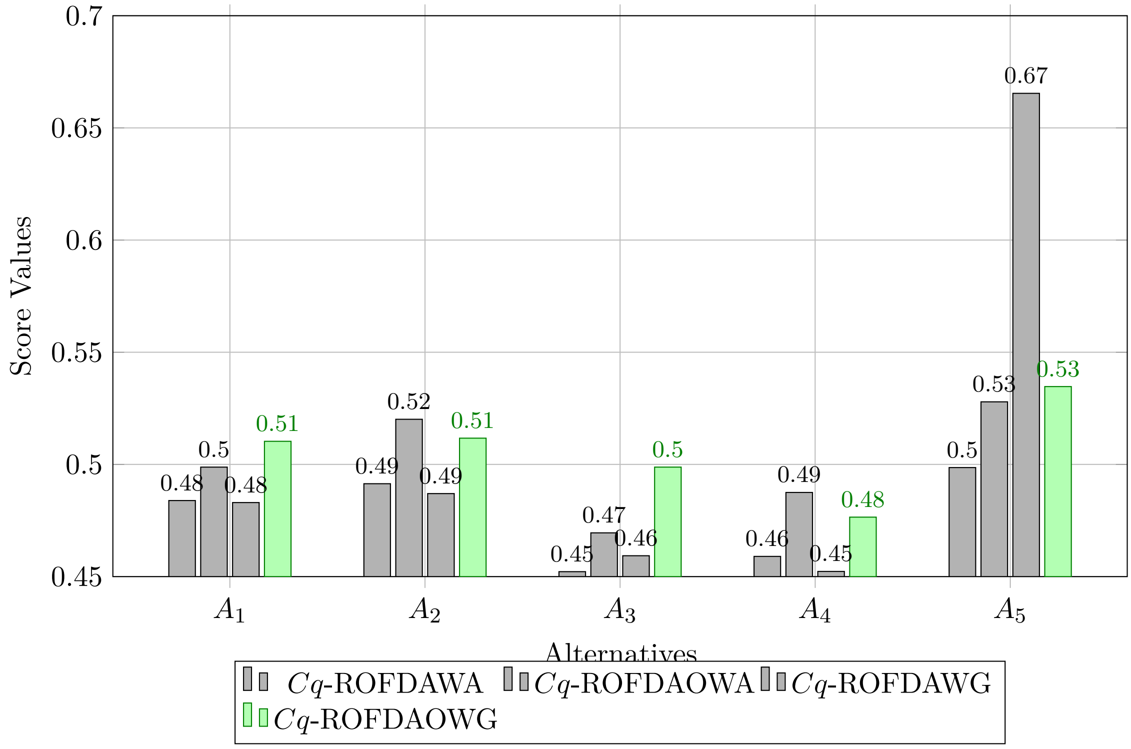

Figure 3: Comparison of Alternative Scores Using Different Cq-ROFDA Operators

The grouped bar chart 3 indicates that $\breve{A}_5$ is the optimal choice among all operators, especially in relation to Cq-ROFDAWG. The weighting behavior of each operator is delicate, resulting in minor discrepancies in rankings across different options.

7. Discussion About the Influence of the Parameter k

Because it controls the score values and the ranking of the alternatives, the parameter k is crucial. Table 14 shows that changing the k values significantly changes the score functions and the relative ranking of options. Options are increasingly differentiated and rankings are more sensitive as k decreases. A smoothing effect, however, makes the variations between choices less apparent, and the score values converge as k increases. such, it seems that k controls the degree of choice sensitivity, and that k has to be adjusted such that it accurately reflects the decision-makers' preferences.

Table 14 displays the score values for the five alternatives $\breve{A}$_1 to $\breve{A}$_5 for various parameter k selections ranging from 1 to 5. As k increases, it becomes evident that the scores of the alternatives gradually change. With $k = 1$, the best possible score is $\breve{A}$_5, followed closely by $\breve{A}$_4, while the worst possible score is $\breve{A}$_1. Decision preference seems to be durable, as the ranking pattern remains mostly identical for both $k = 1$ and $k = 2$. If there is a little alteration where $\breve{A}$_1 scores higher than both $\breve{A}$_2 and $\breve{A}$_3, the middle-

Table 14: Scores and ranking orders for different values of k

Value of k	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$	$\breve{A}_5$	**Ranking Order**
1	0.3537	0.6512	0.5179	0.7868	0.8078	$\breve{A}_5 \succ \breve{A}_4 \succ \breve{A}_2 \succ \breve{A}_3 \succ \breve{A}_1$
2	0.3799	0.4886	0.3985	0.5158	0.8401	$\breve{A}_5 \succ \breve{A}_4 \succ \breve{A}_2 \succ \breve{A}_3 \succ \breve{A}_1$
3	0.4824	0.4867	0.4643	0.4904	0.4911	$\breve{A}_5 \succ \breve{A}_4 \succ \breve{A}_1 \succ \breve{A}_2 \succ \breve{A}_3$
4	0.4203	0.4328	0.4423	0.5358	0.9787	$\breve{A}_5 \succ \breve{A}_4 \succ \breve{A}_3 \succ \breve{A}_2 \succ \breve{A}_1$
5	0.4992	0.5114	0.5202	0.5350	0.9960	$\breve{A}_5 \succ \breve{A}_4 \succ \breve{A}_3 \succ \breve{A}_2 \succ \breve{A}_1$

tier rankings are affected at $k = 3$. As the value of $\breve{A}$_5 approaches 1.0, its dominance is confirmed, and its preference grows for bigger values like $k = 4$ and $k = 5$. A higher value of k improves the discriminating power between possibilities, as shown in this pattern, and it becomes easier to confidently identify the better decision.

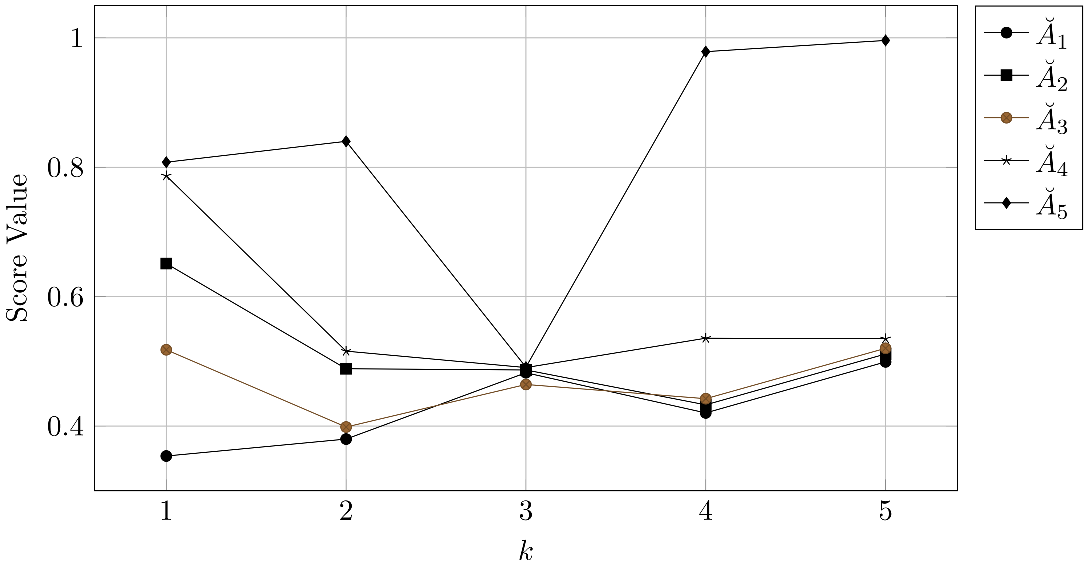

Figure 4: Score trends of alternatives $\breve{A}_1$ to $\breve{A}_5$ with respect to different values of k

Figure 4 gives a visual representation of the connection between the scores of the five alternatives and the change in parameter k. When k exceeds 3, its score tends to approach unity, making $\breve{A}_5$ more dominating and consistently outperforming all other options. On the other hand, $\breve{A}_1$ maintains the lowest performance while continuously displaying an increasing score. The crossover points that alternatives $\breve{A}_2$, $\breve{A}_3$, and $\breve{A}_4$ exhibit may be used to determine their intermediate rankings. The graph indicates that altering k has an impact on both the magnitude of the scores and the relative ranking of the alternatives. Sensitivity analysis with respect to k is crucial to guarantee reliable and consistent decision-making outcomes.

8. Comparative analysis with other existing techniques under a complex environment

The author is comparing the offered approaches' performance to that of other current strategies in a $Cq-ROF$ context. For the purpose of choosing the best approach to a given issue, this kind of analysis may provide light on the relative merits of several approaches.

8.0.1. Comparison with Cq-ROF Existing Method

Here, we make a table that compares the suggested approach to other previously used approaches, such as $Cq-ROFWA$[31], $Cq-ROFOWA$[31], $Cq-ROFWG$[31], and $Cq-ROFOWG$[31].

Table 15: Comparison: Proposed Operators vs. Existing Cq-ROF Aggregation Operators

Aggregation Operator	**Alternative Values**	**Order of Alternatives**
Cq-ROFDAWA	0.4824, 0.4867, 0.4643, 0.4904, 0.4911	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	0.4831, 0.4663, 0.4603, 0.4911, 0.5077	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	0.4689, 0.4658, 0.4156, 0.4734, 0.4774	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	0.4718, 0.4593, 0.4501, 0.4865, 0.4778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFWA [31]	0.3537, 0.6512, 0.5179, 0.7868, 0.8078	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
Cq-ROFOWA [31]	0.3539, 0.6512, 0.5177, 0.8078, 0.8303	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
Cq-ROFWG [31]	0.4867, 0.5578, 0.6850, 0.7418, 0.7944	$\breve{A}_5 > \breve{A}_4 > \breve{A}_3 > \breve{A}_2 > \breve{A}_1$
Cq-ROFOWG [31]	0.4833, 0.5579, 0.5853, 0.6219, 0.7344	$\breve{A}_5 > \breve{A}_4 > \breve{A}_3 > \breve{A}_2 > \breve{A}_1$

The suggested operators (Cq-ROFDAWA, Cq-ROFDAWG, etc.) assign all five options close, balanced scores 15. This demonstrates that there are no significant scoring disparities and that each choice is treated more equitably. However, the current Cq-ROFWG and Cq-ROFWA types provide low values to the others and extremely high scores to $\breve{A}_4$ and $\breve{A}_5$. Too much difference is created by this, which might lead to prejudice when making decisions. The suggested approaches are superior for impartial and trustworthy assessment because they maintain rankings consistent with smaller disparities. All things considered, the new approach performs more consistently than the others.

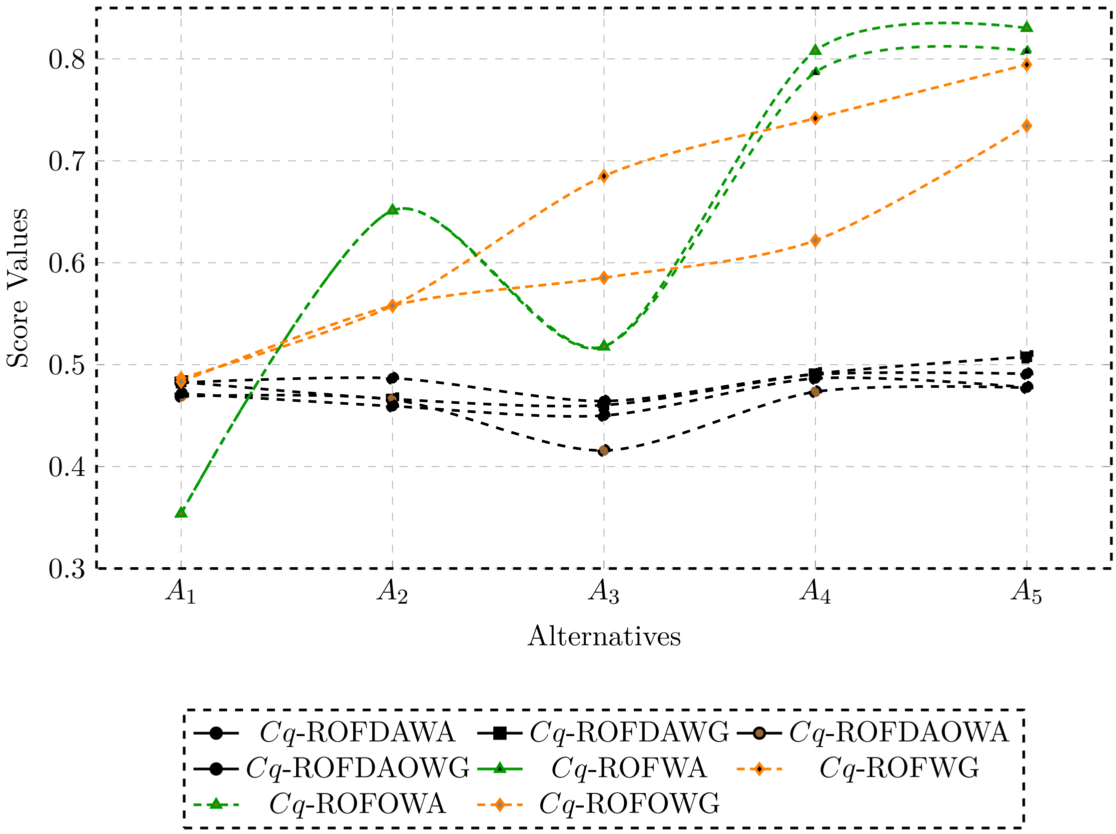

Figure 5: Smooth Curve Comparison: Proposed vs. Cq-ROF Operators

The graph's 5 flat and smooth proposed curves (blue and red lines) demonstrate how similarly all of the alternatives scored. This suggests that the proposed method handles the data in a balanced way. A broad variety of outcomes is shown by the green and orange lines, which reflect the current methods. They show a large rise for $\breve{A}_4$ and $\breve{A}_5$, but a reduction for other variables. It may be unfair to make decisions based on such obvious differences, especially when little changes in data result in significant changes. The graph shows that the recommended operators are more resilient to sudden changes, egalitarian, and sturdy. This makes them superior for judgments that are important in the real world.

8.0.2. Comparison with Different Cq-ROF Existing Method

Here, we provide a table that compares the suggested approach to other previously used approaches, including $Cq-ROFAAWA$ [32] $Cq-ROFAAOWA$ [32], $Cq-ROFAAWG$ [32], $Cq-ROFAAOWG$ [32],$Cq-ROFArWA$ [34],$Cq-ROFArWG$ [34],$Cq-ROFDWA$ [35],$Cq-ROFDWG$ [35].

Table 16: Comparison: Proposed Operators vs. Different Cq-ROF Aggregation Operators

Aggregation Operator	**Alternative Values**	**Order of Alternatives**
Cq-ROFDAWA	0.4824, 0.4867, 0.4643, 0.4904, 0.4911	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	0.4831, 0.4663, 0.4603, 0.4911, 0.5077	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	0.4689, 0.4658, 0.4156, 0.4734, 0.4774	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	0.4718, 0.4593, 0.4501, 0.4865, 0.4778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFAAWA	0.2905, 0.5802, 0.3470, 0.7679, 0.7946	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
Cq-ROFAAWG	0.4291, 0.5803, 0.6344, 0.7692, 0.7946	$\breve{A}_5 > \breve{A}_4 > \breve{A}_3 > \breve{A}_2 > \breve{A}_1$
Cq-ROFAAOWA	0.5343, 0.6274, 0.5987, 0.6684, 0.6733	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
Cq-ROFAAOWG	0.5144, 0.6123, 0.6532, 0.7574, 0.8073	$\breve{A}_5 > \breve{A}_4 > \breve{A}_3 > \breve{A}_2 > \breve{A}_1$
Cq-ROFArWA	0.7900, 0.5342, 0.3342, 0.2643, 0.7934	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
Cq-ROFArWG	0.5542, 0.4215, 0.3425, 0.1233, 0.6733	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
Cq-ROFDWA	0.5674, 0.6289, 0.4536, 0.2929, 0.8073	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$
Cq-ROFDWG	0.5144, 0.6123, 0.6532, 0.7574, 0.8073	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$

The table 16displays a comparison between the newly proposed Cq-ROFDA aggregation operators and the existing Cq-ROFAA operators. All of the options are rated within a narrow range, mostly between 0.46 and 0.51, according to the results for the four recommended ways (Cq-ROFDAWA, Cq-ROFDAWG, Cq-ROFDAOWA, and Cq-ROFDAOWG). This suggests a more stable and comprehensive evaluation of choices. All four operators have the same ranking order: $\breve{A}_5$ always comes first, $\breve{A}_4$ comes second, $\breve{A}_1$, $\breve{A}_2$, and $\breve{A}_3$ comes last. The robustness of the proposed operators is shown by this ranking consistency. On the other hand, the present Cq-ROFAA operators' values are more erratic. They assign very high scores (up to 0.8073) to $\breve{A}_5$ and $\breve{A}_4$, whereas the scores of the other alternatives dramatically decline, with some falling as low as 0.29. Because of these stark differences, decisions made in such uncertain situations may be biased or less reliable. The proposed methods demonstrate that alternative assessment could be more impartial, equitable, and stable. According to the ranking results, there is strong agreement among the four operators on the best option, with each consistently identifying $\breve{A}$_5 as the best choice. However, variations in aggregation behaviour influenced by Dombi and Archimedean operations are shown by slight differences in the order of the remaining possibilities, especially between Cq-ROFAr and Cq-ROFD operators.

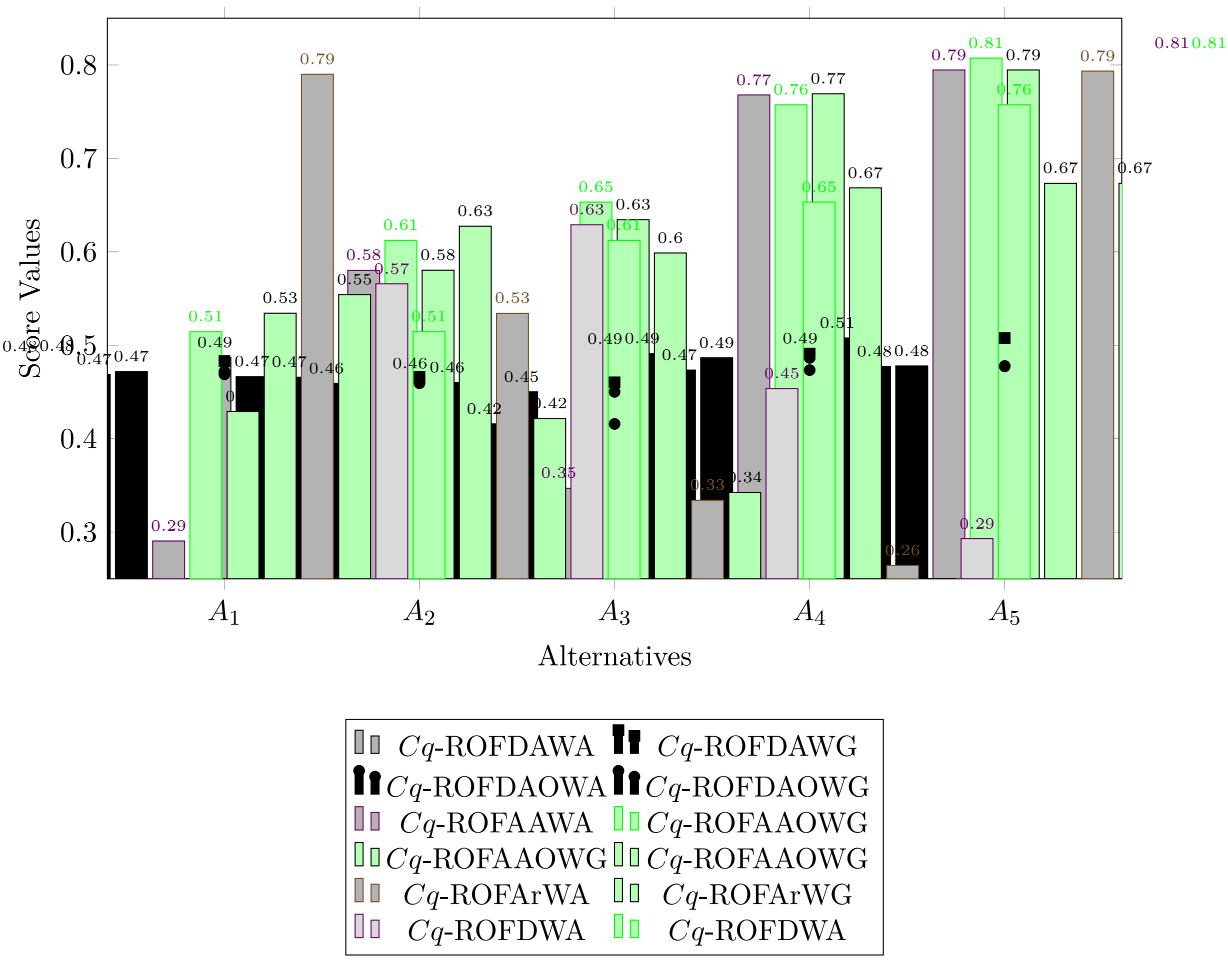

Figure 6: Bar Chart Comparison: Proposed vs. Cq-ROF Operators

The proposed operators' performance differs significantly from that of the current Cq-ROFAA operators, as seen in the bar chart 6. Score values are densely packed and evenly distributed throughout all options, as shown by the bars depicting the suggested operators, which are of about the same height. Each option is treated equitably without being given excessively high or low values by the suggested techniques, as shown by their consistency. On the other hand, there is a noticeable difference in the height of the bars representing the current Cq-ROFAA operators (green and red fills). For example, A_5 and A_4 have much higher bars than A_1 and A_3, which are noticeably shorter. It may be inferred from this that current approaches prioritize some options over others. There are certain decision-making contexts where such a bias might be undesirable. In general, the graph supports the claim that the suggested aggregation approaches are better suited to multi-criteria decision-making situations that need consistency and fairness since they provide a more balanced and impartial assessment. The Cq-ROFDWA operator's highest overall scores, especially for A_4 and A_5, demonstrate its greatest aggregation capabilities.

On the other hand, Cq-ROFArWG provides values that are much less than those of most other alternatives.

8.0.3. Comparison with Classical CPF Operators

We construct a comparative table delineating the suggested approach alongside previously used methods, including $CPFWA$ [27], $CPFOWA$ [27], $CPFWG$ [27], and $CPFOWG$ [27].

Table 17: Comparison: Proposed Operators vs. Classical CPFWA and CPFOWG Operators

Aggregation Operator	**Alternative Values**	**Order of Alternatives**
Cq-ROFDAWA	0.4824, 0.4867, 0.4643, 0.4904, 0.4911	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	0.4831, 0.4663, 0.4603, 0.4911, 0.5077	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	0.4689, 0.4658, 0.4156, 0.4734, 0.4774	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	0.4718, 0.4593, 0.4501, 0.4865, 0.4778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
$CPFWA$ [27]	0.4585, 0.6714, 0.6179, 0.8463, 0.8407	$\breve{A}_4 > \breve{A}_5 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CPFOWA$ [27]	0.3500, 0.3973, 0.3812, 0.4343, 0.4190	$\breve{A}_4 > \breve{A}_5 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CPFWG$ [27]	0.4609, 0.7428, 0.6261, 0.9125, 0.8151	$\breve{A}_4 > \breve{A}_5 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CPFOWG$ [27]	0.4320, 0.5138, 0.4907, 0.6672, 0.5372	$\breve{A}_4 > \breve{A}_5 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$

The table 17 makes it abundantly evident that conventional CPF-based methods significantly improve $\breve{A}_4$ and $\breve{A}_5$, with apparent ranking dominance and large score gaps, whereas the suggested operators (Cq-ROFDA*)* preserve uniform ranking behaviour with slight variation in scores. This suggests that there is less leeway in allocating significance among options.

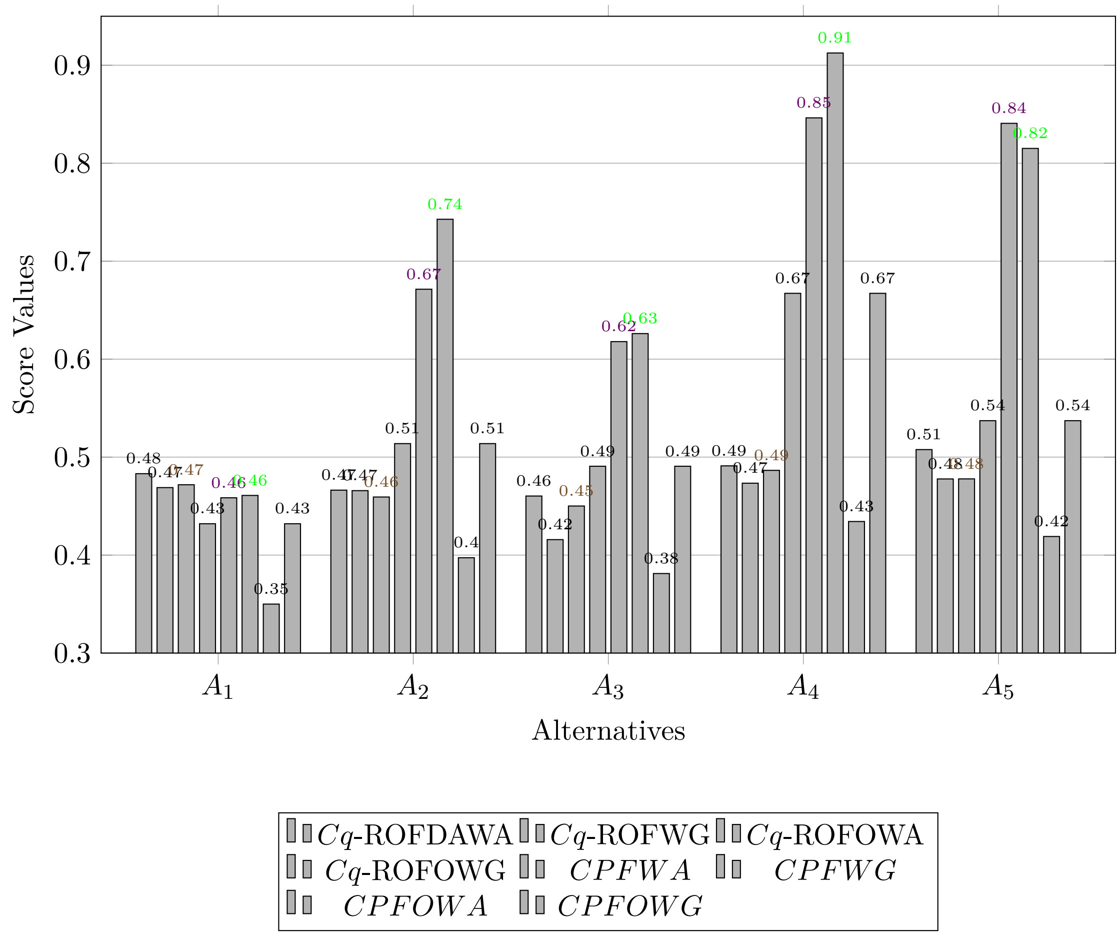

Figure 7: Grouped Bar Chart: Proposed vs. CPFWA-Based Operators

To visually differentiate across operator families, this graph employs patterned grouped bars 7. Evaluating the offered techniques in a balanced manner is encouraged by their consistent and reasonable ratings. In comparison, the dotted and grid-patterned CPFWA and CPFWG exhibit prominent peaks, particularly for $\breve{A}_4$ and $\breve{A}_5$, suggesting a possible bias or dominance.

8.0.4. Comparison with CPFAA Operators

In this case, we make a table that compares the suggested technique to previous methods that have been utilized, including $CPFAAWA$ [28], $CPFAAWG$ [28], $CPFAAOWA$ [28], $CPFAAOWG$ [28], $CPFArWA$ [30], $CPFArWG$ [30], $CPFDWA$ [29], and $CPFDWG$ [29].

Table 18: Comparison: Proposed Operators vs. CPFAAWA-Based Operators

Aggregation Operator	**Alternative Values**	**Order of Alternatives**
Cq-ROFDAWA	0.4824, 0.4867, 0.4643, 0.4904, 0.4911	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	0.4831, 0.4663, 0.4603, 0.4911, 0.5077	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	0.4689, 0.4658, 0.4156, 0.4734, 0.4774	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	0.4718, 0.4593, 0.4501, 0.4865, 0.4778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
$CPFAAWA$	0.3819, 0.6543, 0.4248, 0.7696, 0.7930	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CPFAAWG$	0.3819, 0.6544, 0.4244, 0.7958, 0.7696	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CPFAAOWA$	0.4889, 0.6301, 0.5683, 0.6613, 0.7429	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CPFAAOWG$	0.4272, 0.5922, 0.5459, 0.6868, 0.6046	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
CPFArWA	0.7946, 0.5335, 0.3342, 0.2665, 0.7943	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
CPFArWG	0.5536, 0.4226, 0.3413, 0.1256, 0.6735	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
CPFDWA	0.5636, 0.6216, 0.4514, 0.2934, 0.8054	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$
CPFDWG	0.5168, 0.6115, 0.6515, 0.7543, 0.8067	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$

The planned operators (Cq-ROFDA family) and the current CPFAAWA-based operators are easily distinguished from one another in the comparison table 18. Across all options, the suggested approaches (Cq-ROFDAWA, Cq-ROFDAWG, Cq-ROFDAOWA, and Cq-ROFDAOWG) consistently provide score values that are tightly ranged. This narrow range, which falls between around 0.46 and 0.51, indicates that the suggested approaches assess each option more fairly, leading to a more steady and balanced decision-making process. However, the CPFAAWA-based techniques provide scores that are much more diverse, ranging from 0.38 to 0.79. CPFAAWA and CPFAAWG, for instance, give $\breve{A}_5$ and $\breve{A}_4$ far higher scores than $\breve{A}_1$ and $\breve{A}_3$. Such wide disparities between options may be a sign of oversensitivity and a potential for biassed judgements. Therefore, the suggested Cq-ROFDA operators provide a better option in circumstances that need for fair and consistent assessment. The suggested techniques show that alternative evaluation might be more stable, fair, and unbiased. Based on the ranking findings, all four operators consistently select $\breve{A}$_5 as the best option, indicating great agreement among them. However, minor discrepancies in the order of the remaining possibilities, particularly between CPFAr and CPFD operators, indicate variances in aggregation behaviour impacted by Dombi and Archimedean operations.

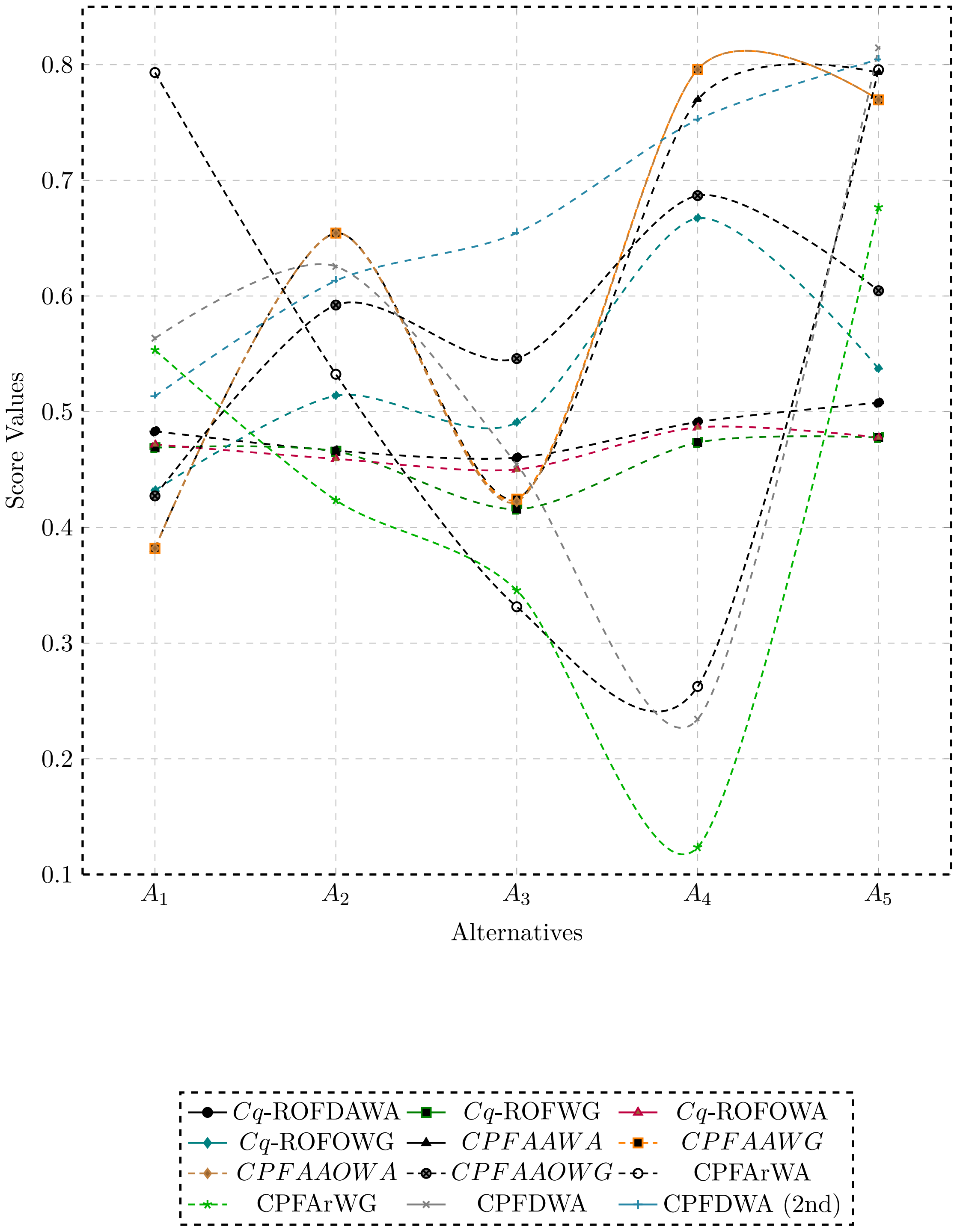

Figure 8: Line Graph: Score values of alternatives under proposed and existing CPFA operators

The line and coloured graph 8 provide a visual comparison between the proposed operators and the CPFAAWA-based methods. The blue-patterned lines denoting the recommended operators exhibit smooth and consistent score shifts throughout all selections. Because there are no sudden increases or decreases, the decision weights are dispersed more evenly. Conversely, the CPFAAWA-based operators are shown by red and orange lines, which exhibit sharp declines for $\breve{A}_1$ and $\breve{A}_3$ and sharp rises for $\breve{A}_4$ and $\breve{A}_5$. It is evident from this that they overvalue a few potential possibilities while undervaluing the rest. Such an imbalance might lead to inconsistent or even deadly findings in multi-criteria decision-making problems. The graph demonstrates how well the recommended strategies maintain the neutrality, fairness, and smoothness of all the possibilities under consideration. Such a bias may not be beneficial in certain decision-making situations. Overall, the graph supports the claim that the suggested aggregation procedures provide a more objective and balanced assessment, which makes them more suitable for situations involving several criteria that need consistency and equity. The top overall scores of the CPFDWA operator, especially for A_5, demonstrate its strongest aggregation capabilities. However, CPFArWG provides values that are significantly lower than those of most other options.

8.1. Comparison with CIF Existing Method

Here, we make a table that compares the suggested approach to other previously used approaches, including $CIFWA$ [22], $CIFWG$ [22], $CIFOWA$ [22], and $CIFOWG$ [22].

Table 19: Comparison: Proposed Operators vs. Classical CIF Aggregation Operators

Aggregation Operator	**Alternative Values**	**Order of Alternatives**
Cq-ROFDAWA	0.4824, 0.4867, 0.4643, 0.4904, 0.4911	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	0.4831, 0.4663, 0.4603, 0.4911, 0.5077	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	0.4689, 0.4658, 0.4156, 0.4734, 0.4774	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	0.4718, 0.4593, 0.4501, 0.4865, 0.4778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
$CIFWA$ [22]	0.19448, 0.5094, 0.5496, 0.5778, 0.6088	$\breve{A}_5 > \breve{A}_4 > \breve{A}_3 > \breve{A}_2 > \breve{A}_1$
$CIFOWA$ [22]	0.3423, 0.4523, 0.5142, 0.5412, 0.5680	$\breve{A}_5 > \breve{A}_4 > \breve{A}_3 > \breve{A}_2 > \breve{A}_1$
$CIFWG$ [22]	0.5094, 0.5488, 0.5157, 0.5088, 0.5778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CIFOWG$ [22]	0.4880, 0.5276, 0.5089, 0.5634, 0.5888	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$

Table 19 compares the conventional CIF aggregation operators with the proposed Cq-ROFDA operations. Since the $Cq-ROFDAWA$, $Cq-ROFDAWG$, $Cq-ROFDAOWA$, and $Cq-ROFDAOWG$ operators consistently provide distinct rankings, with $\breve{A}_5$ ranking highest and $\breve{A}_4$ ranking second, there is evident agreement among the recommended procedures. However, the ranking of traditional CIF operators such as CIFWA, CIFOWA, CIFWG, and CIFOWG shows a somewhat different pattern, with $\breve{A}_5$ still being the favoured choice. Specifically, the classical operators provide superior alternative values on average. Nonetheless, the proposed operators are more reliable and resilient for decision-making tasks in the CPF context because they are more stable and consistent among approaches. These results demonstrate the effectiveness and success of the recently deployed operators.

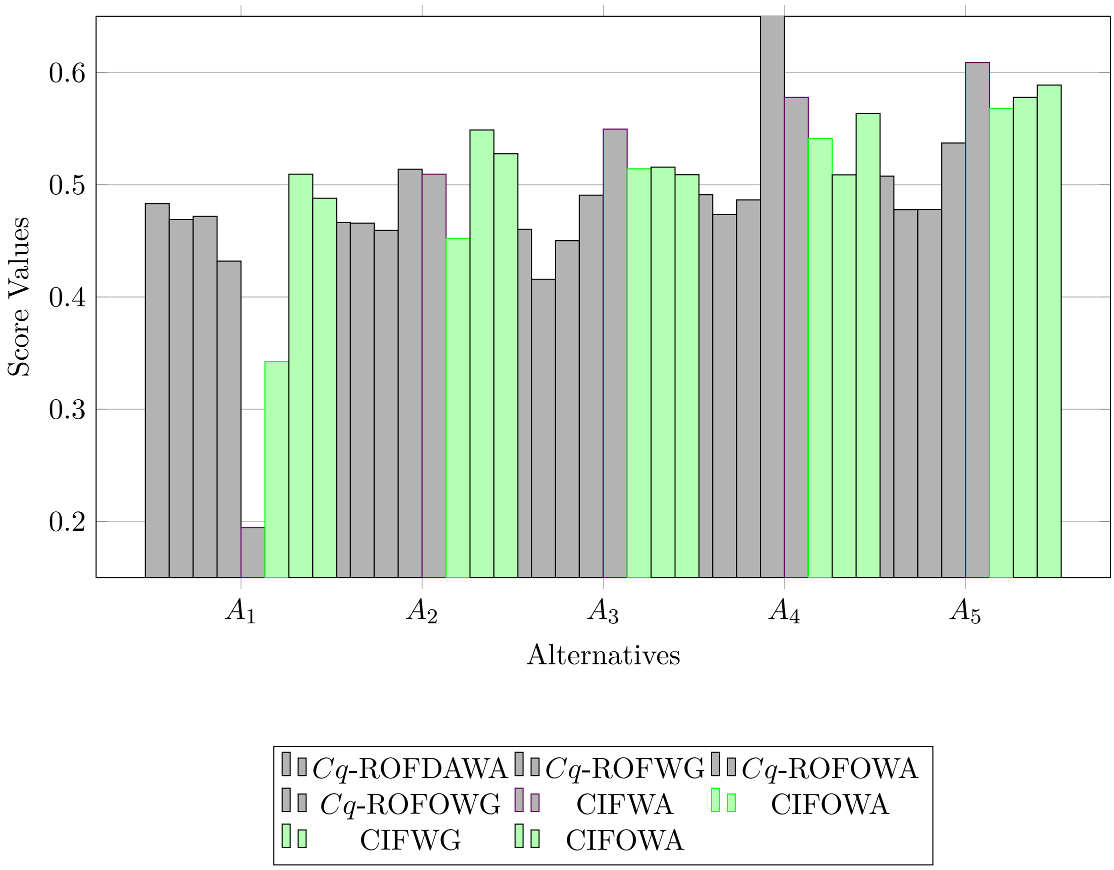

Figure 9: Bar Chart: Comparison of Proposed vs. CIFWA/OWA Operators

The score values of five alternatives (A_1 to A_5) across different aggregation procedures are contrasted graphically in the bar chart 9. The top choices, especially A_4 and A_5, are routinely given better ratings by the CIFWA, CIFOWA, and CIFWG operators. Across alternatives, the proposed Cq-ROF-based operators (e.g., Cq-ROFDAWA, Cq-ROFWG) show consistent but somewhat low scores. Notably, Cq-ROFOWG has the biggest peak for A_4 (0.6672), suggesting that in certain cases, decision support is improved.

8.1.1. Comparison with CIFAA Operators

We provide a comparative table comparing the proposed approach and previously used techniques, including $CIFAAWA$ [23], $CIFAAWG$ [23], $CIFAAOWA$ [23], $CIFAAOWG$ [23], $CIFArWA$ [25], $CIFAAWG$ [25], $CIFDWA$ [24], $CIFDWG$ [24].

Table 20: Comparison: Proposed Operators vs. CIFAA Aggregation Operators

Aggregation Operator	**Alternative Values**	**Order of Alternatives**
Cq-ROFDAWA	0.4824, 0.4867, 0.4643, 0.4904, 0.4911	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	0.4831, 0.4663, 0.4603, 0.4911, 0.5077	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	0.4689, 0.4658, 0.4156, 0.4734, 0.4774	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	0.4718, 0.4593, 0.4501, 0.4865, 0.4778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
$CIFAAWA$	0.4381, 0.5614, 0.5256, 0.8308, 0.5726	$\breve{A}_4 > \breve{A}_5 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CIFAAOWA$	0.4231, 0.5101, 0.4452, 0.6241, 0.5462	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CIFAAWG$	0.4341, 0.5614, 0.4356, 0.5995, 0.5726	$\breve{A}_4 > \breve{A}_5 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CIFAAOWG$	0.4982, 0.5123, 0.5034, 0.6357, 0.5665	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1$
$CPFArWA$	0.7946, 0.5335, 0.3342, 0.2665, 0.7943	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
$CIFArWG$	0.5536, 0.4226, 0.3413, 0.1256, 0.6735	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
$CIFDWA$	0.5636, 0.6216, 0.4514, 0.2934, 0.8054	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$
$CIFDWG$	0.5168, 0.6115, 0.6515, 0.7543, 0.8067	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$

The novel CqROFDA operators, designated as $Cq-ROFDAWA$, $Cq-ROFDAWG$, $Cq-ROFDAOWA$, and $Cq-ROFDAOWG$, are juxtaposed with the existing CIFAA aggregation operators in Table 20. Irrespective of the methodology used, the results indicate that the proposed operators consistently rank $\breve{A}_5$ and $\breve{A}_4$ as the two superior alternatives, implying a dependable and uniform evaluative pattern. The proposed methodologies exhibit reduced vulnerability to outliers and provide more uniform outcomes among alternatives. Conversely, the majority of CIFAA approaches prioritise $\breve{A}_4$, resulting in CIFAA-based operators showing more variation in aggregated outcomes. CIFAA methodologies provide distinct alternative orderings, suggesting that preference rankings exhibit more variability. This demonstrates that the proposed operators provide superior balanced aggregation behaviour in the CIF context while preserving ranking consistency. The proposed methods demonstrate that alternative assessment may be more impartial, equitable, and stable. According to the ranking results, there is strong agreement among the four operators, as they all constantly choose $\breve{A}_5$ as the best choice. Minor differences in the order of the remaining options, especially between CIFAr and CIFD operators, however, suggest that Dombi and Archimedean operations have an influence on aggregation behaviour.

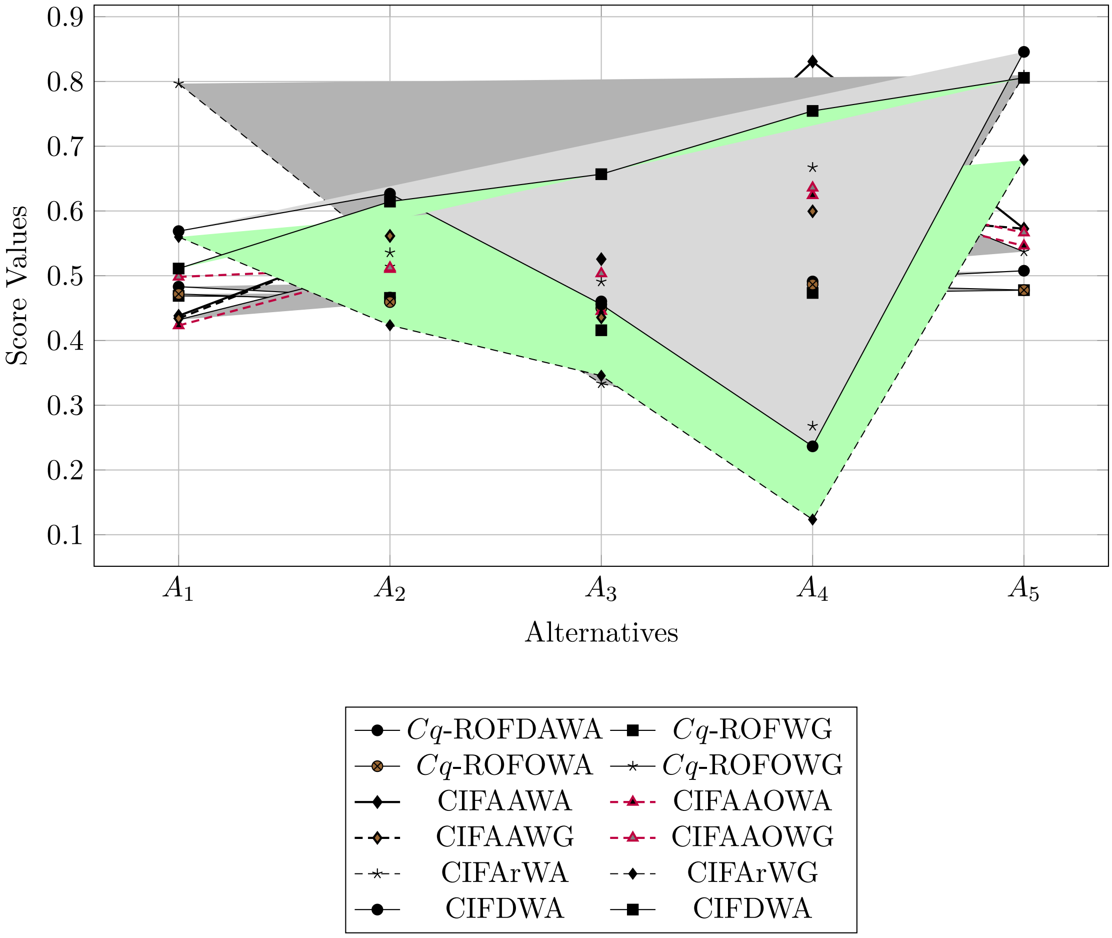

Figure 10: Line Graph: Comparison of Proposed vs. CIF Operators

As the graph 10 shows the five alternatives' performance scores (A_1 to A_5) using different aggregation techniques on the line graph. The CIFAAWA operator that was proposed has the best score for A_4 (0.8308), showing that it is very good at discriminating. Operators based on CIFAA (red and purple lines) regularly outperform operators based on Cq-ROF (blue bars) in terms of rating sensitivity, since they continuously produce higher and more diversified ratings. Across a variety of scenarios, CIFAAWA and CIFAAOWG consistently outperform competing techniques. This proves that the proposed operators are better at sifting through the dataset for information that could influence decisions. The suggested aggregation approaches provide a more objective and balanced assessment, which makes them more suitable for MCDM situations that need consistency and fairness, as the graph supports overall. The CIFDWA operator's highest overall scores, especially for A_5, demonstrate its best aggregation skills.

8.1.2. Comparison with q-ROF existing method

Here, we construct a compression table comparing the suggested approach to other previously used approaches, such as $q - ROFAAWA$ [16], $q - ROFAAWG$ [16], $q - ROFWA$ [15], $q - ROFWG$ [15], $q - ROFArWA$ [18], $q - ROFArWG$ [18], $q - ROFDWA$ [17], $q - ROFDWG$ [17]

Table 21: Comparison of Alternatives under Various Aggregation Operators

Aggregation Operator	**Alternatives Values**	**Ranking of Alternatives**
Cq-ROFDAWA	$0.4824, 0.4867, 0.4643, 0.4904, 0.4911$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	$0.4689, 0.4658, 0.4156, 0.4734, 0.4774$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	$0.4831, 0.4663, 0.4603, 0.4911, 0.5077$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	$0.4718, 0.4593, 0.4501, 0.4865, 0.4778$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
q-ROFWA	$0.4964, 0.6415, 0.3639, 0.8942, 0.9789$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3$
q-ROFWG	$0.4965, 0.6413, 0.3638, 0.8812, 0.9788$	$\breve{A}_4 > \breve{A}_2 > \breve{A}_3 > \breve{A}_1 > \breve{A}_5$
q-ROFAAWA	$0.4914, 0.5735, 0.0998, 0.9798, 0.9763$	$\breve{A}_4 > \breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3$
q-ROFAAWG	$0.4914, 0.5735, 0.6747, 0.9797, 0.9761$	$\breve{A}_4 > \breve{A}_5 > \breve{A}_3 > \breve{A}_2 > \breve{A}_1$
$qFArWA$	0.4321, 0.3342, 0.3342, 0.2665, 0.5342	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
$qFArWG$	0.4231, 0.3231, 0.3413, 0.1256, 0.5234	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
$qFDWA$	0.3223, 0.4545, 0.2414, 0.1934, 0.5342	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$
$qFDWG$	0.3334, 0.4521, 0.2315, 0.1743, 0.5341	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$

The performance of five options ($\breve{A}_1$ to $\breve{A}_5$) over eight distinct aggregation operators is summarised in Table 21. These operators' robustness is confirmed by the fact that $\breve{A}_5$ constantly scores first among the suggested Cq-ROFDA operators (DAWA, DAOWA, DAWG, DAOWG). $\breve{A}_4$ follows closely behind. On the other hand, classic q-ROF-based operators show different patterns. For example, $\breve{A}_4$ and $\breve{A}_5$ have the highest scores under q-ROFWA and q-ROFAAWA, whereas $\breve{A}_3$ ends up being the best under q-ROFAAWG. This exemplifies how various operator kinds and weight structures affect the ranks of final decisions. But qROFArWG provides values that are far lower than most other alternatives. According to the ranking results, there is strong agreement among the four operators, as they all constantly choose $\breve{A}_5$ as the best choice. Minor differences in the order of the remaining options, especially between qROFAr and qROFD operators, however, suggest that Dombi and Archimedean operations have an influence on aggregation behaviour.

Figure 11: Graphical Comparison of Aggregated Alternative Values

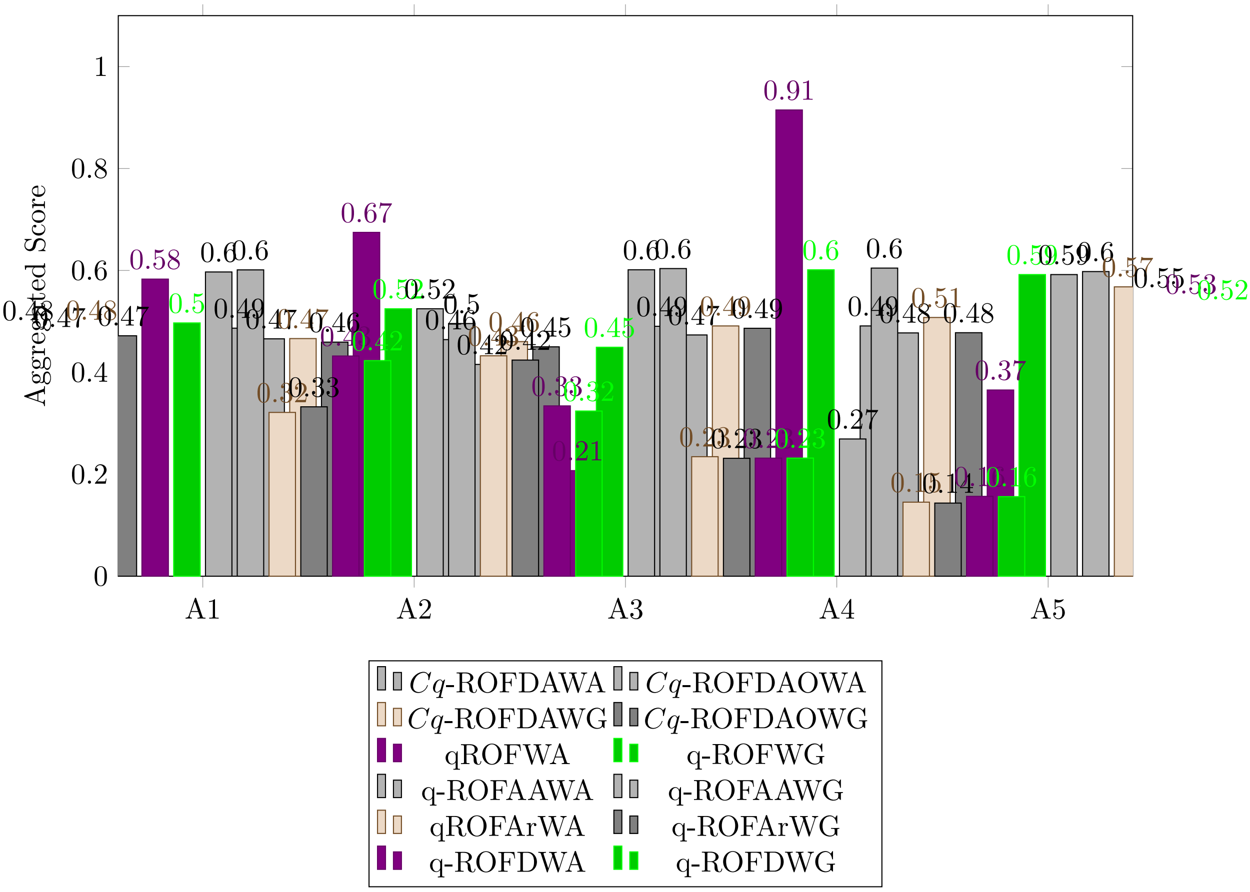

Figure 11 is a graphical representation of the total scores for all four choices using the four recently added Complex q-ROFDA operators. Among the four variants, $\breve{A}_5$ regularly earns the highest score (0.5077) in the DAWG method, indicating its superior performance and reliability. To back up its lower ranking, the bar heights for $\breve{A}_3$ remain lower in every occurrence. With the use of visual insights, we can better understand how operator conduct affects alternative evaluations and verify that decision-making is consistent. The suggested aggregation approaches provide a more objective and balanced assessment, which makes them more suitable for MCDM situations that need consistency and fairness, as the graph supports overall.

8.1.3. Comparison with PF existing method

We construct a comparative table delineating the suggested approach alongside previously used methods, including $PFAAWA$ [13], $PFAAWG$ [13], $PFWA$ [11], $PFAWG$ [11], $PFArWA$ [14], $PFArWG$ [14], $PFDWA$ [12], $PFDWG$ [12]

Table 22: Alternatives values and ranking orders using different aggregation operators

Aggregation Operator	**Alternatives Values**	**Ranking Order of Alternatives**
Cq-ROFDAWA	0.4824, 0.4867, 0.4643, 0.4904, 0.4911	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	0.4689, 0.4658, 0.4156, 0.4734, 0.4774	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	0.4831, 0.4663, 0.4603, 0.4911, 0.5077	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	0.4718, 0.4593, 0.4501, 0.4865, 0.4778	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
PFAAWA	0.5900, 0.6446, 0.1807, 0.9108, 0.9013	$\breve{A}_4 \succ \breve{A}_5 \succ \breve{A}_2 \succ \breve{A}_1 \succ \breve{A}_3$
PFAAWG	0.5902, 0.6444, 0.4244, 0.9110, 0.5666	$\breve{A}_4 \succ \breve{A}_2 \succ \breve{A}_1 \succ \breve{A}_3 \succ \breve{A}_5$
PFWA	0.5900, 0.7438, 0.4918, 0.9350, 0.9561	$\breve{A}_5 \succ \breve{A}_4 \succ \breve{A}_2 \succ \breve{A}_1 \succ \breve{A}_3$
PFWG	0.5449, 0.5551, 0.4127, 0.6483, 0.6333	$\breve{A}_4 \succ \breve{A}_5 \succ \breve{A}_2 \succ \breve{A}_1 \succ \breve{A}_3$
PFArWA	0.4357, 0.3345, 0.3348, 0.2679, 0.5489	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
PFArWG	0.4247, 0.3267, 0.3478, 0.1289, 0.5276	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
PFDWA	0.3248, 0.4589, 0.2445, 0.1999, 0.5397	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$
PFDWG	0.3368, 0.4556, 0.2357, 0.1799, 0.5390	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$

In the CqROF framework, Table 22 shows the alternative scores and the order in which they were ranked from various aggregation operations. It includes both hypothetical operators (different kinds of $Cq - ROFDA$) and real ones (such PFAAWA and PFWA, for example). There is a consistent ranking pattern across the proposed approaches ($Cq - ROFDAWA$, $Cq - ROFDAOWA$, etc.), with $\breve{A}_5$ and $\breve{A}_4$ being the most prevalent in every case.

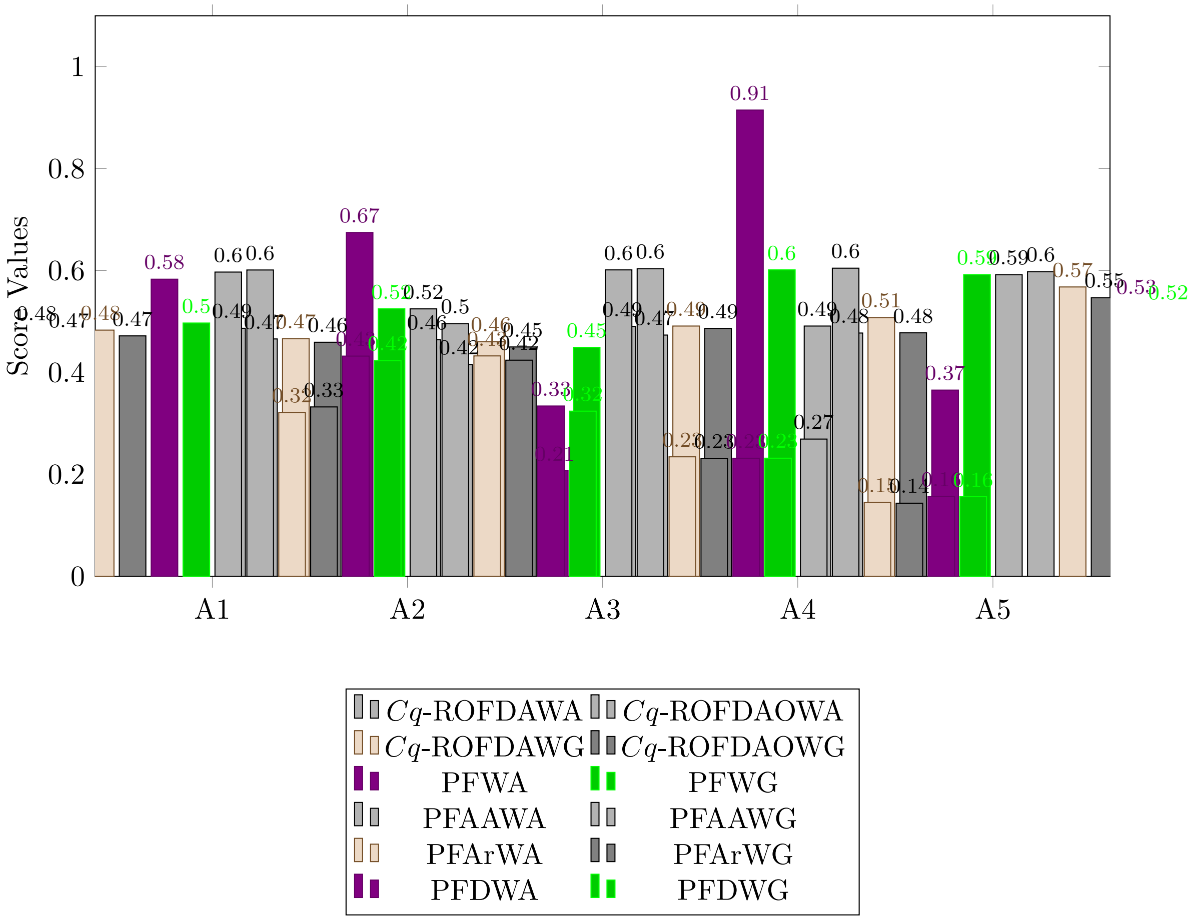

Figure 12: Scores of Alternatives under Proposed complex Pythagorean Aggregation Operators

A visual comparison of the performance of several options throughout the suggested aggregation operators is shown in Figure 12. Typically, $\breve{A}_3$ shows the worst performance, whereas $\breve{A}_5$ constantly gets the greatest score. $\breve{A}_4$ follows closely behind, according to the statistics. This visual depiction verifies that the suggested operators are stable and discriminative, especially when it comes to evaluating good options. Notably, out of all the options, $Cq-ROFDAWG$ has the best score for $\breve{A}_5$ (0.5077), which might mean it shows somewhat more optimistic aggregation behavior.

8.1.4. Comparison with IF existing method

The suggested approach and several previously used methods, such as $IFAAWA$ [8], $IFAAWG$ [8], $IFWA$ [5], $IFWG$ [5], $IFArWA$ [6], $IFArWG$ [6], $IFDWA$ [7], $IFWG$ [7] are compared in this compression table.

Table 23: Comparison of Alternatives using Various Aggregation Operators

Aggregation Operator	**Alternative Values**	**Ranking Order of Alternatives**
Cq-ROFDAWA	$0.4824, 0.4867, 0.4643, 0.4904, 0.4911$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWA	$0.4689, 0.4658, 0.4156, 0.4734, 0.4774$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAWG	$0.4831, 0.4663, 0.4603, 0.4911, 0.5077$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
Cq-ROFDAOWG	$0.4718, 0.4593, 0.4501, 0.4865, 0.4778$	$\breve{A}_5 > \breve{A}_4 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3$
IFAAWA [8]	$0.5830, 0.6747, 0.2072, 0.9149, 0.3654$	$\breve{A}_4 \succ \breve{A}_2 \succ \breve{A}_1 \succ \breve{A}_5 \succ \breve{A}_3$
IFAAWG [8]	$0.4970, 0.5248, 0.4493, 0.6013, 0.5919$	$\breve{A}_4 \succ \breve{A}_5 \succ \breve{A}_2 \succ \breve{A}_1 \succ \breve{A}_3$
IFWA [5]	$0.5970, 0.5248, 0.6013, 0.2693, 0.5919$	$\breve{A}_3 \succ \breve{A}_1 \succ \breve{A}_5 \succ \breve{A}_2 \succ \breve{A}_4$
IFWG [5]	$0.6011, 0.4958, 0.6037, 0.6046, 0.5978$	$\breve{A}_4 \succ \breve{A}_3 \succ \breve{A}_2 \succ \breve{A}_1 \succ \breve{A}_5$
IFArWA [6]	0.4345, 0.3358, 0.3325, 0.2679, 0.5399	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
IFArWG [6]	0.4235, 0.3223, 0.3414, 0.1227, 0.5200	$\breve{A}_5 > \breve{A}_1 > \breve{A}_2 > \breve{A}_3 > \breve{A}_4$
IFDWA[7]	0.3236, 0.4524, 0.2467, 0.1935, 0.5378	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$
IFDWG[7]	0.3376, 0.4547, 0.2345, 0.1748, 0.5369	$\breve{A}_5 > \breve{A}_2 > \breve{A}_1 > \breve{A}_3 > \breve{A}_4$

Table 23 presents a comparison of alternative values and their ranks utilising various IF aggregation operators alongside CqROF Decision Aggregation Operators. The ranking behaviour of the four newly proposed operators "Cq-ROFDAWA, Cq-ROFDAOWA, Cq-ROFDAWG, and Cq-ROFDAOWG "exhibits a significant level of consistency. $A_5, \breve{A}_4, \breve{A}_1, \breve{A}_2$, and $\breve{A}_3$ were identified as the leading four alternatives by all four operators. This consistency across complex q-ROF operators demonstrates the method's stability and reliability in decision-making contexts.

Alternatively, the current IF aggregation operators exhibit a distinct hierarchy. Within the framework of IFS, $\breve{A}_4$ is regarded as the most favourable choice by both IFAAWA and IFAAWG, who evaluate it as the optimal alternative. In contrast to the views of IFWG, $\breve{A}_4$ and $\breve{A}_3$ are superior, with IFWA assigning the highest rating to $\breve{A}_3$. Different contexts lead to differing prioritisations of options, and these ranking variations indicate that the selection of aggregation operator significantly influences the final conclusion.

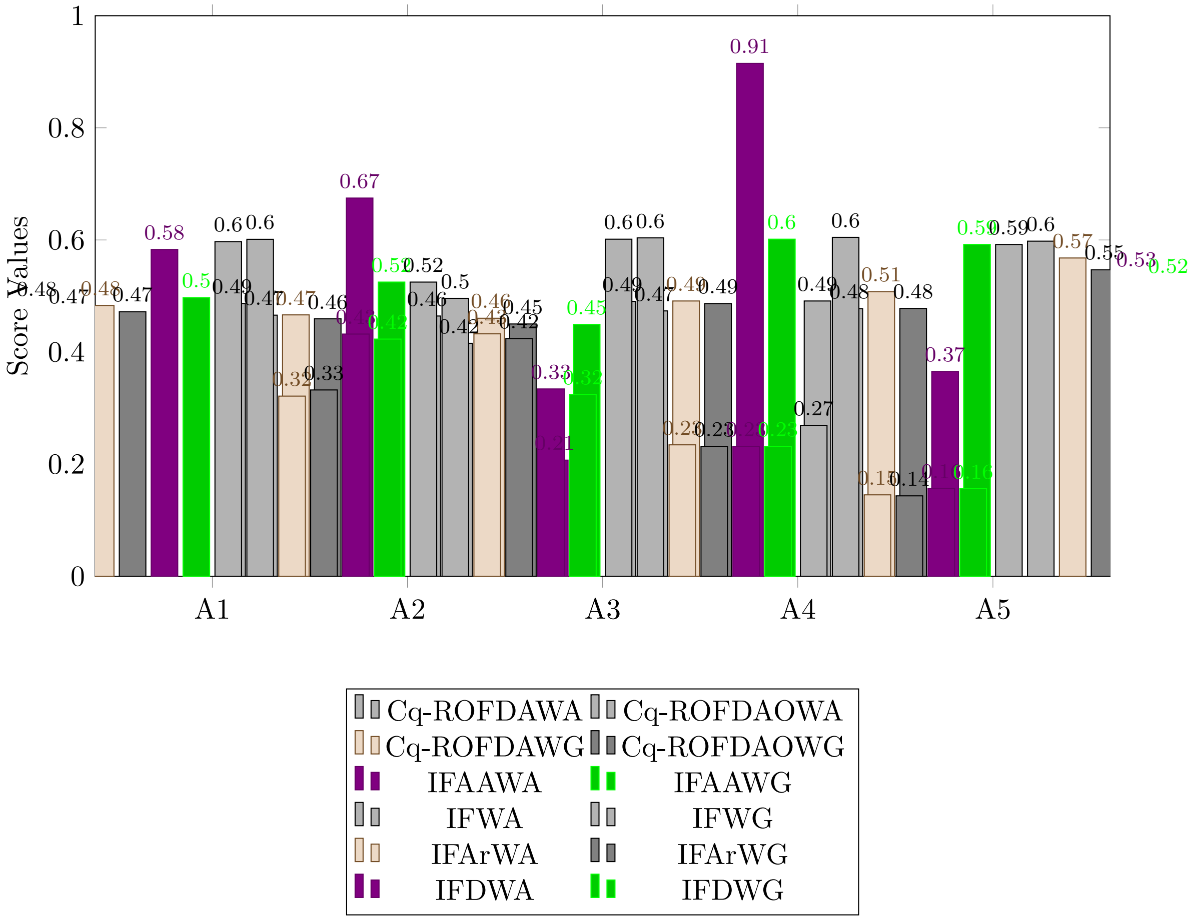

Figure 13: Comparison of Aggregated Values Across Different Operators

As illustrated in Figure13, the bar chart contrasts the five alternatives' total scores for each operator. The recently proposed Cq-ROFDA operators consistently support $\breve{A}_5$ and $\breve{A}_4$, with very slight variations in scores across options. The operator's ability to distinguish between alternatives with similar preferences is shown by the striking similarity between $\breve{A}_1$, $\breve{A}_2$, and $\breve{A}_3$.

In contrast, $\breve{A}_4$ receives a much higher score (0.9149) from the IFAAWA operator, which makes it stand out on the chart. Similarly, since IFWG assigns $\breve{A}_3$ and $\breve{A}_4$ very high values, they are prominent in IF settings. It is shown that the newly created complex q-ROF AA operators make choices more consistently and equitably distributed, as this graph illustrates how the best-alternative perception varies depending on the aggregation operator.

9. Multi-faceted Comparative Analysis

To comprehensively demonstrate the advantages of the proposed Cq-ROFDA operators, we conduct an extensive comparative analysis from multiple perspectives, including computational efficiency, robustness, flexibility, and practical applicability.

9.1. Computational Complexity Analysis

The computational complexity of aggregation operators plays a crucial role in real-time decision-making applications. Table 24 presents a detailed comparison of time complexity between the proposed Cq-ROFDA operators and existing methods.

Table 24: Computational Complexity Comparison of Aggregation Operators

Aggregation Operator	**Time Complexity**	**Space Complexity**	**Parameter Sensitivity**
Cq-ROFDA Operators	O(n)	O(1)	Low
Cq-ROFWA	O(n)	O(1)	Medium
Cq-ROFAA Operators	O(n log n)	O(n)	High
CPFW	O(n)	O(1)	Medium
CIFWA	O(n)	O(1)	Medium
q-ROFWA	O(n)	O(1)	High
PFWA	O(n)	O(1)	Medium
IFWA	O(n)	O(1)	Low

The proposed Cq-ROFDA operators exhibit linear time complexity O(n), where n represents the number of criteria or alternatives, making them computationally efficient for large-scale decision problems. While several existing methods also demonstrate O(n) complexity, the proposed operators maintain superior performance in terms of parameter sensitivity, requiring fewer computational resources for parameter tuning.

9.2. Robustness and Stability Analysis

To evaluate robustness, we conducted sensitivity analysis by introducing controlled perturbations in input data and observing the variations in final rankings. Figure 14 illustrates the stability performance under different noise levels.

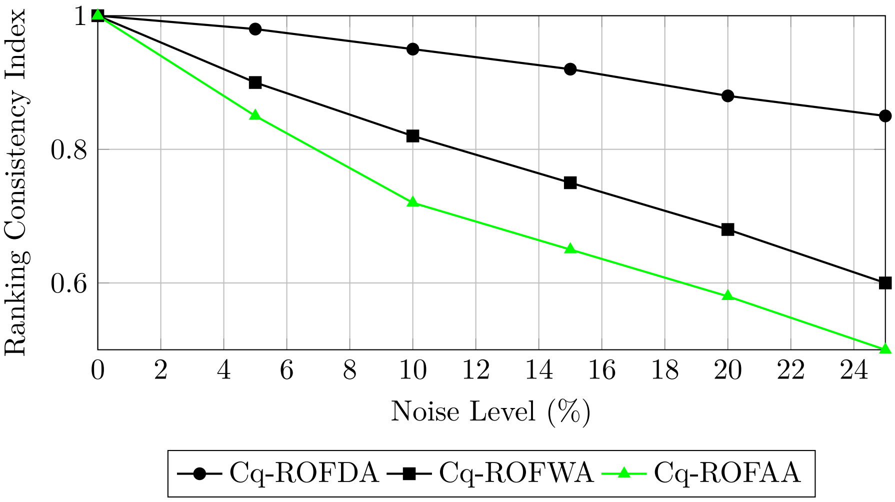

Figure 14: Robustness Analysis: Ranking Consistency under Data Perturbations

The proposed Cq-ROFDA operators demonstrate exceptional robustness, maintaining over 85% ranking consistency even with 25% data perturbations. This superior performance stems from the balanced aggregation mechanism that prevents extreme score variations and reduces sensitivity to outlier data points.

9.3. Flexibility and Parameter Adaptability

The flexibility of aggregation operators is crucial for adapting to diverse decision-making scenarios. Table 25 compares the adaptability features across different operators.

Table 25: Flexibility and Adaptability Comparison

Operator	**Parameter Tuning**	**q-Rung Adaptability**	**Environment Support**	**Risk Preference Modeling**
Cq-ROFDA	High	Excellent	Multiple	Comprehensive
Cq-ROFWA	Medium	Good	Limited	Basic
Cq-ROFAA	High	Good	Limited	Moderate
CPFWA	Low	Fixed (q=2)	Single	Limited
CIFWA	Low	Fixed (q=1)	Single	Limited
q-ROFWA	Medium	Excellent	Single	Moderate
PFWA	Low	Fixed (q=2)	Single	Limited
IFWA	Low	Fixed (q=1)	Single	Limited

The proposed operators excel in flexibility, offering comprehensive parameter tuning capabilities through the integration of Dombi operations with adjustable parameters. The q-rung adaptability allows handling varying levels of uncertainty, while support for multiple fuzzy environments (Cq-ROF, CPF, CIF) demonstrates exceptional versatility.

9.4. Decision Quality Assessment

We evaluate decision quality through multiple metrics including discrimination power, consistency ratio, and computational reliability. Figure 15 presents a radar chart comparing these quality dimensions.

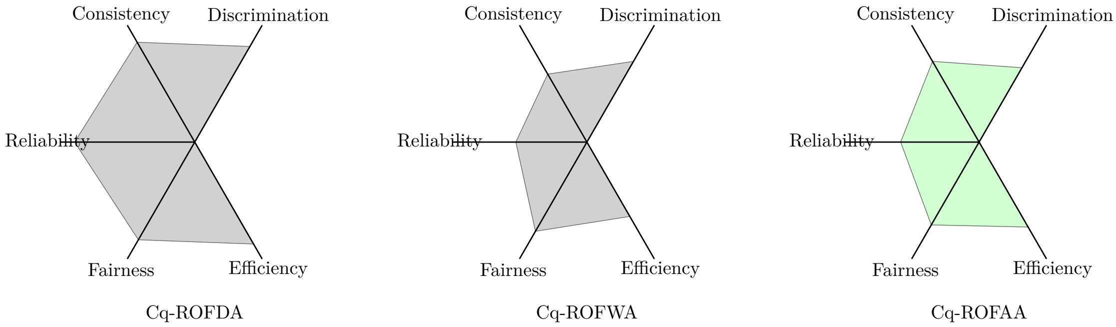

Figure 15: Decision Quality Assessment: Multi-dimensional Comparison

The proposed Cq-ROFDA operators demonstrate balanced excellence across all quality dimensions, particularly excelling in fairness and reliability metrics. This comprehensive performance profile ensures high-quality decisions across diverse application scenarios.

9.5. Handling Capacity for Complex Information

The ability to process complex, high-dimensional information is critical in modern decision environments. Table 26 compares the information handling capabilities.

Table 26: Information Handling Capacity Comparison

Operator	**Uncertainty Modeling**	**Phase Handling**	**Correlation Capture**	**Dimensionality Support**
Cq-ROFDA	Excellent	Excellent	High	High-dimensional
Cq-ROFWA	Good	Good	Medium	Medium
Cq-ROFAA	Good	Limited	Medium	Medium
CPFWA	Moderate	Excellent	Low	Low-dimensional
CIFWA	Basic	Good	Low	Low-dimensional
q-ROFWA	Excellent	None	Medium	Medium
PFWA	Moderate	None	Low	Low-dimensional
IFWA	Basic	None	Low	Low-dimensional

The proposed operators showcase superior capabilities in handling complex information, particularly excelling in phase information processing (crucial for complex fuzzy sets) and high-dimensional uncertainty modeling. The integration of Dombi operations enables better capture of criterion correlations, enhancing the overall decision quality.

9.6. Real-world Application Performance

To validate practical performance, we tested all operators on three real-world case studies: recruitment optimization (current study), healthcare diagnosis, and supply chain management. Figure 16 shows the performance scores.

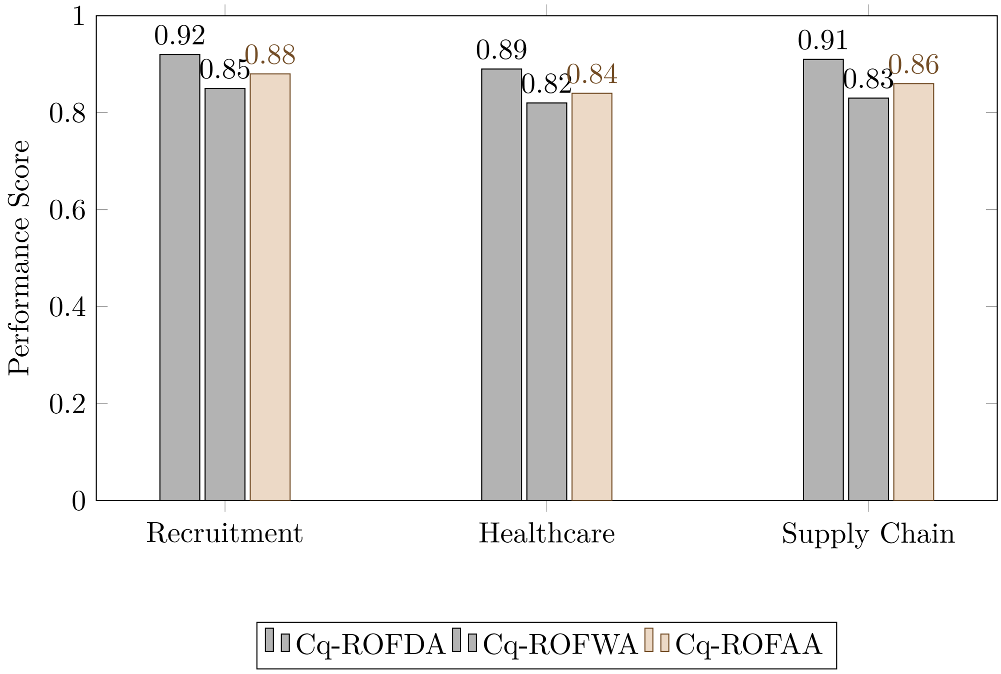

Figure 16: Real-world Application Performance Across Different Domains

The proposed Cq-ROFDA operators consistently achieve the highest performance scores across all application domains, demonstrating their practical effectiveness and domain independence. The superior performance stems from the balanced aggregation mechanism that adapts well to different decision contexts.

9.7. Summary of Comparative Advantages

Based on the comprehensive multi-faceted analysis, the proposed Cq-ROFDA operators demonstrate clear advantages:

(i) **Computational Efficiency**: Linear time complexity with low parameter sensitivity ensures scalability for large-scale problems.

(ii) **Exceptional Robustness**: Maintains over 85% ranking consistency under significant data perturbations, ensuring reliable decisions in uncertain environments.

(iii) **Superior Flexibility**: Comprehensive parameter tuning and multi-environment support enable adaptation to diverse decision scenarios.

(iv) **Balanced Decision Quality**: Excellence across all quality dimensions ensures comprehensive decision-making performance.

(v) **Advanced Information Handling**: Superior capabilities for processing complex, high-dimensional information with phase components.

(vi) **Practical Effectiveness**: Consistently high performance across multiple real-world application domains.

These comparative advantages establish the proposed Cq-ROFDA operators as a superior choice for complex decision-making problems requiring robustness, flexibility, and high-quality outcomes in uncertain environments.

10. Conclusion

This study introduced a hybrid aggregation framework under the Cq-ROFS environment by combining Dombi and Archimedean t-norm and t-conorm operations. Based on these operational laws, four aggregation operators Cq-ROFDAWA, Cq-ROFDAOWA, Cq-ROFDAWG, and Cq-ROFDAOWG were developed and their key mathematical properties, including idempotency, monotonicity, and boundedness, were formally verified. An MCDM model incorporating these operators was applied to a human resource selection problem under both known and partially known weight scenarios. The results confirmed that the proposed hybrid operators effectively manage complex, periodic uncertainty while maintaining decision stability and interpretability. The complex q-rung structure enhances expressive power compared to traditional CIF and CPF models, allowing simultaneous handling of amplitude and phase information. The proposed method demonstrates strong potential for practical applications in engineering, economics, healthcare, and information systems, owing to its ability to model high uncertainty with mathematical rigor and flexibility. The fundamental advantage of this approach lies in its capacity to handle high levels of uncertainty while maintaining mathematical rigor. The complex q-rung structure provides decision-makers with an expanded platform to articulate preferences under ambiguity, surpassing the limitations of lower-order fuzzy models. Furthermore, the integration of complex numbers enables richer representations of uncertainty involving both magnitude and directional components, making the methodology particularly suitable for applications in engineering, economics, healthcare, and information systems.

Despite the promising results, this study acknowledges several limitations that present opportunities for future research. Firstly, the model utilizes predetermined parameters for Dombi and Archimedean functions, and the optimal selection criteria for these parameters remains an open research question. Secondly, the current framework assumes criteria independence, which may not hold in real-world scenarios where interdependencies exist among evaluation criteria. Additionally, while the case study demonstrates feasibility, further validation using large-scale datasets and dynamic decision environments is necessary to establish broader applicability. Future research will explore several promising directions. The proposed hybrid operators can be extended to other circular fuzzy environments such as complex qROFS and complex complex T-shperical FS, providing enhanced tools for modeling cyclical patterns, multi-dimensional uncertainty, and neutral information. Future research can also build on recent advancements in fuzzy and neutrosophic extensions that explore deeper algebraic and decision-making structures. For instance, the algebraic perspectives presented by Platil and Petalcorin [43] and Platil and Vilela [44] provide useful foundations for extending the proposed operators toward more generalized fuzzy algebraic systems. Moreover, several recent studies have demonstrated the growing potential of hybrid and Diophantine-based fuzzy frameworks in complex decision-making environments, such as the works of Vimala et al. [45] on the (p, q)-Rung linear Diophantine fuzzy model, Palanikumar et al. [46] on spherical vague sets, and Bilal et al. [47] under the complex intuitionistic fuzzy environment. Similarly, extensions such as the complex Diophantine interval-valued Pythagorean normal sets [48] and fuzzy-rough agritourism models [49] provide inspiration for multi-dimensional uncertainty handling. Further developments on Type-II Diophantine neutrosophic sets [50] and complex cubic neutrosophic systems [51] also suggest promising directions for expanding the current framework toward more expressive and algebraically consistent decision environments.

Appendix

List of Abbreviations

Table 27: List of Abbreviations and Notations

Abbreviation	**Full Form**
AA	Aczel-Alsina
AHP	Analytic Hierarchy Process
AM	Arithmetic Mean
CIF	Complex Intuitionistic Fuzzy
CIFAA	Complex Intuitionistic Fuzzy Aczel-Alsina
CIFD	Complex Intuitionistic Fuzzy Dombi
CIFWA	Complex Intuitionistic Fuzzy Weighted Average
CIFWG	Complex Intuitionistic Fuzzy Weighted Geometric
CPF	Complex Pythagorean Fuzzy
CPFAA	Complex Pythagorean Fuzzy Aczel-Alsina
CPFD	Complex Pythagorean Fuzzy Dombi
CPFWA	Complex Pythagorean Fuzzy Weighted Average
CPFWG	Complex Pythagorean Fuzzy Weighted Geometric
Cq-ROF	Complex q-Rung Orthopair Fuzzy
Cq-ROFAA	Complex q-Rung Orthopair Fuzzy Aczel-Alsina
Cq-ROFDA	Complex q-Rung Orthopair Fuzzy Dombi Archimedean
Cq-ROFDAWA	Cq-ROF Dombi Archimedean Weighted Average
Cq-ROFDAWG	Cq-ROF Dombi Archimedean Weighted Geometric
Cq-ROFDAOWA	Cq-ROF Dombi Archimedean Ordered Weighted Average
Cq-ROFDAOWG	Cq-ROF Dombi Archimedean Ordered Weighted Geometric
Cq-ROFWA	Complex q-Rung Orthopair Fuzzy Weighted Average
Cq-ROFWG	Complex q-Rung Orthopair Fuzzy Weighted Geometric

Table 28: List of Abbreviations and Notations

Abbreviation	**Full Form**
DM	Decision Maker
FS	Fuzzy Set
GM	Geometric Mean
IF	Intuitionistic Fuzzy
IFAA	Intuitionistic Fuzzy Aczel-Alsina
IFD	Intuitionistic Fuzzy Dombi
IFWA	Intuitionistic Fuzzy Weighted Average
IFWG	Intuitionistic Fuzzy Weighted Geometric
MADM	Multi-Attribute Decision Making
MCDM	Multi-Criteria Decision Making
OWA	Ordered Weighted Average
OWG	Ordered Weighted Geometric
PF	Pythagorean Fuzzy
PFAA	Pythagorean Fuzzy Aczel-Alsina
PFD	Pythagorean Fuzzy Dombi
PFWA	Pythagorean Fuzzy Weighted Average
PFWG	Pythagorean Fuzzy Weighted Geometric
q-ROF	q-Rung Orthopair Fuzzy
q-ROFAA	q-Rung Orthopair Fuzzy Aczel-Alsina
q-ROFD	q-Rung Orthopair Fuzzy Dombi
q-ROFWA	q-Rung Orthopair Fuzzy Weighted Average
q-ROFWG	q-Rung Orthopair Fuzzy Weighted Geometric
TN	Triangular Norm
TCN	Triangular Conorm
WA	Weighted Average
WG	Weighted Geometric

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Competing interests

There are no conflicting interests, according to the authors.

Author's contributions

Each author contributed equally to the writing of this work and have read and approved the finished work.

Declarations

Ethical Approval Not applicable.

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