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2-Path Geodetic Vertex Cover of Graphs

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Abstract. A vertex cover $S \subseteq V(G)$ is called a 2-*path geodetic vertex cover* of G if for every $v \in V(G) \setminus S$, there exist vertices $u, w \in S$ such that $d_G(u, w) = 2$ and $v \in I_G(u, w)$. The 2-*path geodetic vertex covering number* of G, denoted $\beta_{2pg}(G)$, is the minimum cardinality of a 2-path geodetic vertex covering of G.
In this paper, we show that given two positive integers a and b such that $2 \leq a \leq b$, there exists a connected graph G such that $\beta(G) = a$ and $\beta_{2pg} = b$. As a consequence, the difference between the 2-path geodetic vertex covering number and the classical vertex covering number of a graph can be made arbitrarily large. We characterize graphs with small and large values of the 2-path geodetic vertex covering number. Furthermore, we provide necessary and sufficient conditions for the 2-path geodetic vertex covers in certain graph operations. The exact values of 2-path geodetic vertex cover numbers of these graphs are also determined.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Geodetic, vertex cover, vertex cover number

1. Introduction

The concept of vertex covering in graphs has been extensively studied (see, for instance, [1], [2], [3], [4]). As noted by Angel and Amutha [5], this parameter has practical applications in network security. In particular, their study highlights that in computer networks, minimizing the vertex cover number provides an optimal strategy for network defense. Toregas et al. [6] further demonstrated that this concept is utilized in determining the optimal placement of emergency facilities within telecommunication networks.

Despite its significance, the vertex cover problem is classified as an NP-hard optimization problem. Specifically, Karp [7] established its NP-completeness by leveraging the known result that the clique problem is NP-complete. For cubic and planar graphs, the

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vertex covering problem is still NP-complete. For complete details, one may refer to [8] and [9].

The problem of determining bounds and exact values for the vertex cover number of specific classes of graphs has been extensively studied (see [10], [11]). In recent years, several variations of the vertex cover concepts have been introduced and investigated (see [5], [12], [13], [14], [15], [1], [16], [17], and [18]).

Motivated by the aforementioned studies, we introduce and initiate the study 2-path geodetic vertex cover of a graph. This new parameter naturally extends two existing concepts: 2-path geodetic set and vertex cover of a graph. For some related works on the concept of geodetic and 2-path closure absorbing set, readers may see [19], [20], [21], and [22].

2. Terminologies and Notations

Let $G = (V(G), E(G))$ be a simple undirected graph. The *open neighborhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ (the set consisting of all the neighbors of v), while its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V(G)$ is the set $N_G(S) = \cup_{v\in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = S \cup N_G(S)$. Any $v \in V(G)$ with $|N_G(v)| = 0$ is called an *isolated vertex.* Vertex v is a *leaf* or an *endvertex* if $|N_G(v)| = 1$. A vertex w of G is a *support vertex* if $wv \in E(G)$ for some leaf v in G. A vertex w is an *extreme vertex* in G if the induced subgraph $\langle N_G(v)\rangle$ of $N_G(v)$ is complete. The sets $I(G)$, $L(G)$, $S(G)$ and $Ext(G)$ denote the sets containing of all the isolated vertices, leaves, support vertices, and extreme vertices in G, respectively.

A subset A of $V(G)$ is an *independent* set if for every pair of distinct vertices in G do not form an edge. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the *independence number* of G. Any independent set with cardinality equal to $\alpha(G)$ is called an α-set in G.

A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. It is a 2-*dominating set* if $|N_G(v) \cap S| \geq 2$ for every $v \in V(G) \setminus S$. The domination number (resp. 2-domination number) of G, denoted $\gamma(G)$ (resp. $\gamma_2(G)$) is the minimum cardinality of a dominating (resp. 2-dominating) set in G. Any dominating set (2-dominating set) with cardinality $\gamma(G)$ (resp. $\gamma_2(G)$) is called a γ-set (resp. γ_2-set) in G.

A subset $S \subseteq V(G)$ is called a *geodetic set* of a graph G if for every vertex $v \in V(G)$, there exist vertices $u, w \in S$ such that $v \in I_G[u, w]$, where $I_G[u, w]$ denotes the set consisting of u, w, and all vertices that lie on some shortest path between u and w in G. This shortest path connecting u and w is called a u-w geodesic. A set $S \subseteq V(G)$ is a 2-*path geodetic* or 2-*path closure absorbing* set in G if for each $x \in V(G) \setminus S$, there exist $p, q \in S$ such that $x \in I_G(p, q)$ and $d_G(p, q) = 2$, where $I_G(p, q) = I_G[p, q] \setminus \{p, q\}$. The smallest cardinality among all 2-path geodetic sets in G, denoted $g_{2p}(G)$, is called the 2-*path geodetic number* of G.

A 2-path geodetic set S is 2-*path geodetic* 2-*dominating* if S is 2-dominating in G. The smallest cardinality of a 2-path geodetic 2-dominating set in G, denoted $\gamma_{2pg2}(G)$, is called the 2-*path geodetic* 2-*domination number* of G. Any 2-path geodetic 2-dominating set with cardinality $\gamma_{2pg2}(G)$ is referred to as a γ_{2pg2}-set.

A subset U of vertices of a graph G is called a *vertex cover* of G if for every $e = uv \in E(G)$, $u \in U$ or $v \in U$. The minimum cardinality of a vertex cover of G, denoted $\beta(G)$, is the *vertex cover number* of G. Any vertex cover of G with cardinality $\beta(G)$ is called a β-set.

A set $S \subseteq V(G)$ is called a 2-*path geodetic vertex cover* of G if it is both a vertex cover and a 2-path geodetic set in G. The smallest cardinality of a 2-path geodetic vertex cover of G, denoted $\beta_{2pg}(G)$, is called the 2-*path geodetic vertex cover number* of G. Any 2-path geodetic vertex cover of G with cardinality $\beta_{2pg}(G)$ is called a β_{2pg}-set.

Consider the graph G in Figure 1. Let $S = \{a, c, f\}$. Since every edge of G is incident to some vertex in S, it follows that S is a vertex cover of G. Moreover, $b, d \in I_G(a, c)$ and $e \in I_G(c, f)$. Thus, S is a 2-path geodetic vertex cover of G. Since there exists no 2-path geodetic vertex cover with cardinality less than 3, it follows that $\beta_{2pg}(G) = |S| = 3$.

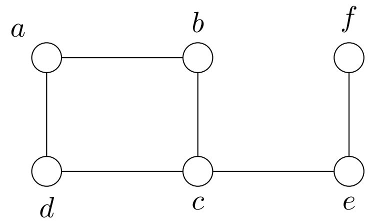

Figure 1: A graph G with $\beta_{2pg}(G) = 3$

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *shadow graph* $D_2(G)$ of graph G is constructed by taking two copies of G, say G_1 and G_2, and then joining each vertex $u \in V(G_1)$ to the neighbors of its corresponding vertex $u' \in V(G_2)$.

Readers are referred to [23] for other basic definitions that are not given here.

3. Results

Since $V(G)$ is a 2-path geodetic vertex cover of G, it follows that any graph G admits a 2-path geodetic vertex cover.

Remark 1. *A* 2-*path geodetic set need not be a vertex cover and a vertex cover need not be a* 2-*path geodetic set.*

To see this, consider graph G in Figure 2. Let $S_1 = \{v_1, v_3, w_1, w_3\}$ and $S_2 = \{v_2, w_2\}$. Clearly, S_1 and S_2 are 2-path geodetic set and vertex cover of G, respectively. Since $v_2w_2 \in E(G)$ and $v_2, w_2 \notin S_1$, it follows that S_1 is not a vertex cover of G. By definition, S_2 is not a 2-path geodetic set in G. It can easily be verified that $g_{2p}(G) = |S_1| = 4$ and

$\beta(G) = 2$.

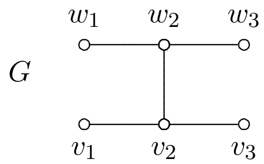

Figure 2: Graph G with $\beta(G) = 2$ and $g_{2p}(G) = 4$

Remark 2. *Let G be a graph and let S be a vertex cover of G. Then $V(G) \setminus S$ is an independent set. Moreover, $\alpha(G) + \beta(G) = |V(G)|$.*

Proposition 1. *Let G be any graph and let S be a 2-path geodetic set in G. Then each of the following holds:*

(i) $Ext(G) \cup I(G) \subseteq S$ and $|Ext(G)| + |I(G)| \leq g_{2p}(G)$.

(ii) S is a 2-dominating set and $\gamma_2(G) \leq g_{2p}(G)$.

Proof. Clearly, every 2-path geodetic set is a geodetic set. Now, since every geodetic set in a graph contains all the extreme and isolated vertices, it follows that $Ext(G) \cup I(G) \subseteq S$ and the given inequality in (i) holds.

Next, let $v \in V(G) \setminus S$. Since S is a 2-path geodetic set in G, there exist $p, q \in S$ such that $d_G(p, q) = 2$ and $v \in I_G(p, q)$. It follows that S is a 2-dominating set in G. Thus, $\gamma_2(G) \leq g_2(G)$. □

The next result is a direct consequence of Proposition 1(i).

Corollary 1. *Let n be a positive integer. Then $\beta_{2pg}(K_n) = \beta_{2pg}(\overline{K}_n) = n$.*

Theorem 1. *Let $G_1, G_2, .., G_k$, where $k \geq 1$, be the components of G. Then S is a 2-path geodetic vertex cover of G if and only if $S = \bigcup_{j \in [k]} S_j$, where S_j is a 2-path geodetic vertex cover of G_j for every $j \in [k] = \{1, 2, .., k\}$. Moreover,*

$$\beta_{2pg}(G) = \sum_{j \in [k]} \beta_{2pg}(G_j).$$

Proof. Suppose S is a 2-path geodetic vertex cover of G. Let $S_j = S \cap V(G_j)$ for each $j \in [k] = \{1, 2, .., k\}$. Then $S = \cup_{j \in [k]} S_j$. Since S is a 2-path geodetic set in G, $S_j \neq \emptyset$ for all $j \in [k]$. Next, let $j \in [k]$ and let $ab \in E(G_j)$. Since S is a vertex cover in G, it follows that $a \in S$ or $b \in S$. Hence, $a \in S_j$ or $b \in S_j$, showing that S_j is a vertex cover of G_j. Now let $v \in V(G_j) \setminus S$. Since S is a 2-path geodetic set in G, there exist $y, z \in S$ such that $d_G(y, z) = 2$ and $x \in I_G(y, z)$. This implies that there exist $y, z \in S_j$ such that

$d_{G_j}(y,z) = 2$ and $x \in I_{G_j}(y,z)$. Therefore S_j is a 2-path geodetic vertex cover in G_j. In particular, if S is a β_{2pg}-set in G, then

$$\beta_{2pg}(G) = |S| = \sum_{j\in[k]} |S_j| \geq \sum_{j\in[k]} \beta_{2pg}(G_j).$$

For the converse, suppose $S = \cup_{j\in[k]} S_j$, where S_j is a 2-path geodetic vertex cover of G_j for all $j \in [k] = \{1, 2, \ldots, k\}$. Let $pq \in E(G)$. Then there exists $j \in [k]$ such that $pq \in E(G_j)$. Since S_j is a vertex cover of G_j, we have $p \in S_j$ or $q \in S_j$. It follows that $p \in S$ or $q \in S$. Let $v \in V(G) \setminus S$, and let $t \in [k]$ such that $v \in V(G_t) \setminus S_t$. Since S_t is a 2-path geodetic set in G_t, there exist $u, w \in S_t$ such that $d_G(u,w) = 2$ and $v \in I_{G_t}(u,w) = I_G(u,w)$. Therefore S is a 2-path geodetic vertex cover of G. If each S_j is a β_{2pg}-set in G_j, then

$$\beta_{2pg}(G) \leq |S| = \sum_{j\in[k]} |S_j| = \sum_{j\in[k]} \beta_{2pg}(G_j).$$

This proves the assertion. □

Theorem 2. *Let G be a graph of order n. Then*

$$\max\{\beta(G), g_{2p}(G)\} \leq \beta_{2pg}(G) \leq n.$$

Moreover, the following statements hold.

(*i*) *$g_{2p}(G) = \beta_{2pg}(G)$ if and only if G has g_{2p}-set which is also a vertex cover of G.*

(*ii*) *$\beta(G) = \beta_{2pg}(G)$ if and only if G has β-set which is also a 2-path geodetic set in G.*

Proof. Clearly, $\beta_{2pg}(G) \leq n$. Since every 2-path geodetic vertex cover is both 2-path geodetic and a vertex cover, we have $\max\{\beta(G), g_{2p}(G)\} \leq \beta_{2pg}(G)$.

(*i*) Suppose that $g_{2p}(G) = \beta_{2pg}(G)$. Let S be a β_{2pg}-set in G. By assumption, S is a g_{2p}-set in G.

For the converse, suppose that G has a g_{2p}-set S which is also a vertex cover of G. Then S is a 2-path geodetic vertex cover of G. Hence, $\beta_{2pg}(G) \leq |S| = g_{2p}(G)$. By the first part, it follows that $\beta_{2pg}(G) = g_{2p}(G)$.

(*ii*) Suppose that $\beta(G) = \beta_{2pg}(G)$. Let S be a β_{2pg}-set in G. By assumption, S is a β-set in G.

Conversely, suppose that G has a β-set S which is also a 2-path geodetic set in G. Then S is a 2-path geodetic vertex cover of G. Hence, $\beta_{2pg}(G) \leq |S| = \beta(G)$. With the first part, this implies that $\beta_{2pg}(G) = \beta(G)$. □

Theorem 3. *Let G be a graph of order n. Then*

(*i*) *$\beta_{2pg}(G) = 1$ if and only if $G = K_1$*

(*ii*) *$\beta_{2pg}(G) = 2$ if and only if $G \in \{K_2, \overline{K}_2, K_{2,n-2}\}$*

(*iii*) $\beta_{2pg}(G) = 3$ *if and only if* $G \in \{K_3, \overline{K}_3, P_1 \cup P_2\}$ *or there exist distinct vertices* $x, y, z \in V(G)$ *such that* $V(G) \setminus \{x, y, z\}$ *is an independent set and one of the following conditions holds:*

(*a*) $\langle\{x, y, z\}\rangle = \overline{K}_3$ *and for every* $v \in V(G)\setminus\{x, y, z\}$*, it holds that* $|N_G(v) \cap \{x, y, z\}| \geq 2$ *or*

(*b*) $\langle\{x, y, z\}\rangle = \langle x\rangle \cup \langle\{y, z\}\rangle = P_1 \cup P_2$ *and for every* $v \in V(G) \setminus \{x, y, z\}$*, we have* $x, y \in N_G(v)$ *or* $x, z \in N_G(v)$*.*

(*c*) $\langle\{x, y, z\}\rangle = P_3 = [x, y, z] \neq G$ *and for every* $v \in V(G) \setminus \{x, y, z\}$*, it holds that* $x, z \in N_G(v)$ *and* $N_G(y) \setminus \{x, y, z\} \neq \varnothing$*.*

Proof. (*i*) Suppose that $\beta_{2pg}(G) = 1$, say $S = \{v\}$ is a β_{2pg}-set in G. Since S is a 2-path geodetic set, $V(G) = \{v\}$, i.e, $G = K_1$.

For the converse, suppose that $G = K_1$. By Proposition 1, $\beta_{2pg}(G) = 1$.

(*ii*) Suppoe $\beta_{2pg}(G) = 2$, say $S = \{u, v\}$ is a β_{2pg}-set. If $n = 2$, then $G \in \{K_2, \overline{K}_2\}$. Suppose $n \geq 3$. Let $x \in V(G) \setminus S$. Since S is a 2-path geodetic set, $x \in I_G(u, v)$ and $d_G(u, v) = 2$. Let $G_1 = \langle\{u, v\}\rangle$ and $G_2 = \langle V(G) \setminus S\rangle$. Then $G_1 = \overline{K}_2$ and, by Remark 2, $G_2 = \overline{K}_{n-2}$. Thus, $G = \overline{K}_2 + \overline{K}_{n-2} = K_{2,n-2}$.

The converse is clear.

(*iii*) Suppose that $\beta_{2pg}(G) = 3$. Let $S = \{x, y, z\}$ be a β_{2pg}-set in G. Then $V(G) \setminus S$ is an independent by Remark 2. Suppose $|V(G)| = 3$. Since $\beta_{2pg}(P_3) = 2$ and $\beta_{2pg}(K_3) = \beta_{2pg}(\overline{K}_3) = \beta_{2pg}(P_1 \cup P_2) = 3$, it follows that $G \in \{K_3, \overline{K}_3, P_1 \cup P_2\}$. Next, suppose that $|V(G)| \geq 4$. Then $V(G) \setminus \{x, y, z\} \neq \emptyset$. Clearly, $\langle S\rangle \neq K_3$ since any $v \in V(S) \setminus S$ cannot be in $I_G(S)$. Suppose now that $\langle S\rangle = \overline{K}_3$ and let $v \in V(G) \setminus S$. Since S is a 2-path geodetic set in G, it follows that $|N_G(v) \cap S| \geq 2$. This proves (*a*). Suppose now that $\langle S\rangle = \langle x\rangle \cup \langle\{y, z\}\rangle = P_1 \cup P_2$ and let $v \in V(G) \setminus S$. Since S is a 2-path geodetic set in G, it follows that $x, z \in N_G(v)$ or $x, y \in N_G(v)$. This proves (*b*). Finally, suppose that $\langle\{x, y, z\}\rangle = [x, y, z] = P_3 \neq G$. Let $v \in V(G)\setminus\{x, y, z\}$. Since $\{x, y, z\}$ is a 2-path geodetic set, $x, z \in N_G(v)$. Suppose $N_G(y) \setminus S = \varnothing$. Since $y \in I_G(x, z)$, it follows that $\{x, z\}$ is a 2-path geodetic vertex cover of G, a contradiction to the assumption that $\beta_{2pg}(G) = 3$. Thus, there exists $w \in N_G(y) \setminus S$, showing that (*c*) holds.

For the converse, suppose that $V(G) \setminus S$ is independent. Assume first that $G \in \{K_3, \overline{K}_3, P_1 \cup P_2\}$. Then $\beta_{2pg}(G) = 3$. Next, suppose that (*a*) holds. Let $S = \{x, y, z\}$ and let $vw \in E(G)$. Since $V(G) \setminus S$ is independent, it follows that $v \in S$ or $w \in S$. Hence S is a vertex cover of G. Let $v \in V(G) \setminus S$. Then, by (*a*), S is a 2-path geodetic set in G. Therefore, S is a 2-path geodetic vertex cover and $\beta_{2pg}(G) = |S| = 3$ by (*ii*). Suppose that (*b*) holds. Let $pq \in E(G)$. Again, since $V(G) \setminus S$ is an independent set, $p \in S$ or $q \in S$. Let $v \in V(G) \setminus S$. By (*b*), it follows that S is a 2-path geodetic set in G. Therefore, S is a geodetic vertex cover in G. Again, by (*ii*), it follows that $\beta_{2pg}(G) = |S| = 3$. Lastly, suppose that (*c*) holds. Since $V(G) \setminus S$ is independent, S is a vertex cover of G. Let $v \in V(G) \setminus S$. Since $x, z \in N_G(v)$, it follows that $v \in I_G(x, z)$. Hence, S is a 2-path geodetic set in G. Moreover, since $N_G(y) \setminus S \neq \varnothing$, $G \neq K_{2,n-2}$. Thus, by part (ii), we must have $\beta_{2pg}(G) = 3$. □

Theorem 4. *Let G be a graph of order n. Then $\beta_{2pg}(G) = n$ if and if G' is complete for every component G' of G.*

Proof. Suppose that $\beta_{2pg}(G) = n$. Suppose further that there exists a component G' of G which is not complete. Then there exist vertices $p, q \in V(G')$ such that $d_G(p,q) = 2$. Let $x \in N_G(p) \cap N_G(q)$. Then $D = V(G') \setminus \{x\}$ is a 2-path geodetic vertex cover of G'. This implies that $\beta_{2pg}(G') \leq |V(G')| - 1$. By Theorem 1, $\beta_{2pg}(G) \leq n-1$, a contradiction to the assumption. Therefore, every component of G is complete.

For the converse, suppose that every component G' of G is complete. By Corollary 1, $\beta_{2pg}(G') = |V(G')|$ for every component G' of G. Therefore, by Theorem1, we have $\beta_{2pg}(G) = n$. □

Theorem 5. *Let G be a graph of order n. Then $\beta_{2pg}(G) = n - 1$ if and only if all but a component H of G are complete and $\langle V(H) \setminus Ext(H) \rangle$ is complete.*

Proof. Let $G_1, G_2, \cdots, G_k$ be the components of G. Suppose $\beta_{2pg} = n-1$. By Theorem 1 and Theorem 4, there exists a component $H = G_t$ of G which is not complete. Hence, by Theorem 1 and the assumption, $\beta_{2pg}(H) = |V(H)| - 1$ and $\beta_{2pg}(G_j) = |V(G_j)|$ for every $j \in \{1, 2, \cdots, t-1, t+1, \cdots, k\}$. Let $H^* = \langle V(H) \setminus Ext(H) \rangle$. Since H is connected, it follows that H^* is connected. Suppose $H^* = \langle V(H) \setminus Ext(H) \rangle$ is not complete. Choose any $p, q \in V(H^*)$ such that $d_G(p,q) = d_H(p,q) = 2$. Since $p, q \notin Ext(H)$, each of them has non-adjacent neighbors. This implies that $S = V(H) \setminus \{p, q\}$ is a 2-path geodetic vertex covering of H, a contradiction. Thus, H^* is complete.

For the converse, suppose that G_j is complete for all $j \in \{1, 2, \cdots, t-1, t+1, \cdots, k\}$ and that $H = G_t$ is non-complete satisfying the property that $H^* = \langle V(H) \setminus Ext(H) \rangle$ is complete. Then $\beta_{2pg}(H) \leq |V(H)| - 1$ and $\beta_{2pg}(G_j) = |V(G_j)|$ for all $j \in \{1, 2, \cdots, t-1, t+1, \cdots, k\}$ by Theorem 4. Let D be a β_{2pg}-set in H. Then $Ext(H) \subseteq D$ by Proposition 1(i). If H^* is the trivial graph, say $H^* = \langle v \rangle$, then $D = Ext(H)$. Hence, $\beta_{2pg}(H) = |D| = |V(H)| - 1$. Suppose H^* is nontrivial. Since H^* is connected (it contains an edge) and D is a vertex cover , it follows that $D \cap V(H^*) \neq \varnothing$, i.e., $D \neq Ext(H)$. Suppose there exist distinct vertices $p, q \in V(H^*) \setminus D$. Then $pq \in E(H)$ because H^* is complete. This implies that D is not a vertex covering of H, a contradiction. Thus, $|D \cap V(H^*)| = |V(H^*)| - 1$. Therefore, $\beta_{2pg}(H) = |D| = |V(H)| - 1$. By Theorem 1, $\beta_{2pg}(G) = n - 1$. □

Theorem 6. *Let $G = K_{m_1, m_2, .., m_k}$, where $2 \leq m_1 \leq, .., \leq m_k$. Then $\beta_{2pg}(G) = \sum_{j=1}^{k-1} m_j$.*

Proof. Let $S_1, S_2, ...S_k$ be the partite sets of G. Clearly, $\bigcup\limits_{i=1}^{k-1} S_i$ is a 2-path geodetic vertex cover of G. It follows that $\beta_{2pg}(G) \leq \sum\limits_{j=1}^{k-1} m_j$. Next, let S be a β_{2pg}-set of G. Since

G is not complete, it follows that $S \neq V(G)$. Let $v \in V(G) \setminus S$ and let $r \in \{1, 2, \dots, k\}$ such that $v \in S_r$. Since $vw \in E(G)$ for all $w \in \bigcup_{i \neq r} S_i$ and S is a vertex cover of G, it follows that $\bigcup_{j \neq r} S_i \subseteq S$. Therefore,

$$\sum_{j=1}^{k-1} m_j \leq \sum_{j \neq r} m_j \leq |S| = \beta_{2pg}(G).$$

This proves the assertion. □

Observation 1. *Let n be a positive integer. Then*

(i) $\beta_{2pg}(P_n) = \lceil \frac{n+1}{2} \rceil$ *for all n and*

(ii) $\beta_{2pg}(C_n) = \lceil \frac{n}{2} \rceil$ *for all $n \geq 4$.*

We now give some realization results involving the parameters 2-path geodetic number, vertex cover number, and 2-path geodetic vertex cover number.

Theorem 7. *Given two positive integers a and b such that $3 \leq a \leq b$, there exists a connected graph G such that $g_{2p}(G) = a$ and $\beta_{2pg}(G) = b$.*

Proof. If $a = b$, then let $G = K_{1,a}$. Clearly, $g_{2p}(G) = \beta_{2pg}(G) = a$.

Suppose now that $a < b$. Consider the graph G in Figure 3 with complete subgraphs K_{a-1} and K_{b-a+1}, where $V(K_{a-1}) = \{v_1, \dots, v_{a-1}\}$ and $V(K_{b-a+1}) = \{x_1, \dots, x_{b-a+1}\}$. Let $D = \{v_1, \dots, v_{a-1}, v_a\}$, Then D is g_{2p}-set in G, implying that $g_{2p}(G) = a$. Let D_0 be a β_{2pg}-set in G. Then $Ext(G) = \{v_1, v_2, \dots v_{a-2}, v_a\} \subseteq D_0$ by Proposition 1(i). If $v_{a-1} \notin D_0$, then $V(K_{b-a+1}) \subseteq D_0$ because D_0 is a vertex cover of G. Hence, $D_0 = Ext(G) \cup V(K_{b-a+1})$. It follows that $|D_0| = (a-1) + (b-a+1) = b$. Suppose $v_{a-1} \in D_0$. Again, since D_0 is a vertex cover of G, $|V(K_{b-a+1}) \cap D_0| = b - a$.

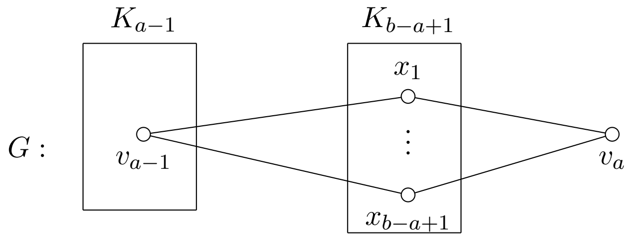

Figure 3: Graph G with $g_{2p}(G) = a < b = \beta_{2pg}(G)$

This implies that $D_0 = a + (b - a) = b$. Therefore, $\beta_{2pg}(G) = |D_0| = b$. □

The next result is direct consequence of Theorem 7.

Corollary 2. *Let n be a positive integer. Then there exists a connected graph G such that $\beta_{2pg}(G) - g_{2p}(G) = n$. In other words, the difference $\beta_{2pg}(G) - g_{2p}(G)$ can be increased arbitrarily.*

Theorem 8. *Given two positive integers a and b such that $2 \leq a \leq b$, there exists a connected graph G such that $\beta(G) = a$ and $\beta_{2pg}(G) = b$.*

Proof. If $a = b$, then consider the cycle $G = C_{2a}$. Clearly, $\beta(G) = \beta_{2pg}(G) = \lceil \frac{2a}{2} \rceil = a$.

Next, suppose $a < b$. Consider the graph G in Figure 4 with the complete graph K_a and the star $K_{1,b-a}$ as subgraphs, where $V(K_a) = \{v_1, \ldots, v_{a-1}, x\}$ and $V(K_{1,b-a}) = \{v_a, x_1, \ldots, x_{b-a}\}$. Let S_1 be a β-set in G. If $x \in S_1$, then $|\{v_1, \ldots, v_{a-1}\} \cap S_1| = a-2$ and $v_a \in S_1$ because S_1 is a β-set in G. If $x \notin S_1$, then $S_1 = \{v_1, \ldots, v_{a-1}, v_a\}$. In both cases, $|S_1| = a$. Hence, $\beta(G) = a$. Now, let S_2 be a β_{2pg}-set in G. Then $\{v_1, \ldots, v_{a-1}, x_1, \ldots, x_{b-a}\} \subseteq S_2$ by Proposition 1(i). Also, since S_2 is a β_{2pg}-set in G, we have $|\{x, v_a\} \cap S_2| = 1$. Thus, $\beta_{2pg}(G) = |S_2| = (a-1) + (b-a) + 1 = b$.

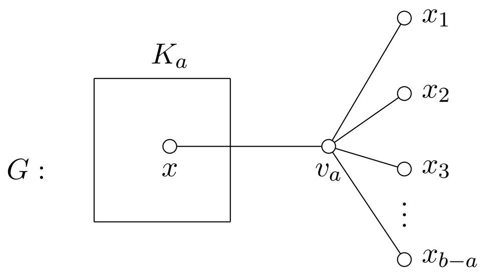

Figure 4: Graph G with $\beta(G) = a < b = \beta_{2pg}(G)$

Therefore, the assertion holds. □

The next result follows from Theorem 8.

Corollary 3. *Let n be a positive integer. Then there exists a connected graph G such that $\beta_{2pg}(G) - \beta(G) = n$. In other words, the difference $\beta_{2pg}(G) - \beta(G)$ can be made arbitrarily large.*

If what follows, we denote by G_1 and G_2 the copies of graph G in the definition of the shadow graph $D_2(G)$. Moreover, we denote by v' the vertex in G_2 corresponding to the vertex $v \in V(G_1)$.

Theorem 9. *Let G be a non-trivial connected graph. Then $S \subseteq V(D_2(G))$ is a vertex cover of $D_2(G)$ if and only if $S = S_{G_1} \cup S_{G_2}$ and satisfies the following conditions:*

(i) *$S = S_{G_1} \cup S_{G_2}$, where S_{G_1} and S_{G_2} are vertex covers of G_1 and G_2, respectively.*

(ii) *For each $v \in V(G_1) \setminus S_{G_1}$, it holds that $w \in S_{G_1}$ and $w' \in S_{G_2}$ for every $w \in N_{G_1}(v)$.*

(iii) *For each $p' \in V(G_2) \setminus S_{G_2}$, it holds that $q \in S_{G_1}$ and $q' \in S_{G_2}$ for every $q' \in N_{G_2}(p')$.*

Proof. Suppose S is a vertex cover of $D_2(G)$. Then $S_{G_1} = S \cap V(G_1)$ and $S_{G_2} = S \cap V(G_2)$ are vertex covers of G_1 and G_2, respectively, because S is a vertex cover of $D_2(G)$. This shows that (i) holds. Now let $v \in V(G_1) \setminus S_{G_1}$ and let $w \in N_{G_1}(v)$. Since S_{G_1} is vertex cover of G_1, it follows that $w \in S_{G_1}$. Also, since $vw' \in E(D_2(G))$ and S is a vertex cover of $D_2(G)$, we have $w' \in S_{G_2}$. This shows that (ii) holds. Similarly, (iii) holds.

For the converse, suppose that S has the given form and satisfies (i), (ii), and (iii). Let $xy \in E(D_2(G))$ and consider the following cases:

Case 1. $xy \in E(G_1) \cup E(G_2)$.

If $xy \in E(G_1)$, then $x \in S_{G_1}$ or $y \in S_{G_1}$ because S_{G_1} is a vertex cover of G_1. Similarly, $x \in S_{G_2}$ or $y \in S_{G_2}$ whenever $xy \in E(G_2)$.

Case 2. $x \in V(G_1)$ and $y \in V(G_2)$.

Let $y = z'$ where $z \in V(G_1)$. If $x \in S_{G_1}$, then we are done. So suppose $x \in V(G_1) \setminus S_{G_1}$. Since $z \in N_G(x)$, it follows from (ii) that $z' \in S_{G_2}$. Thus, $y \in S$.

Therefore, S is a vertex cover of $D_2(G)$. □

Theorem 10. *Let G be a non-trivial connected graph. Then $S \subseteq V(D_2(G))$ is a vertex cover of $D_2(G)$ if and only if it is a 2-path geodetic vertex cover of $D_2(G)$.*

Proof. Suppose S is a vertex cover of $D_2(G)$. Then $S = S_{G_1} \cup S_{G_2}$ and satisfies properties (i), (ii), and (iii) of Theorem 9. Now let $v \in D_2(G) \setminus S$. Suppose $v \in V(G_1) \setminus S_{G_1}$. Pick any $w \in V(G_1) \cap N_{G_1}(v)$. Then $w, w' \in S$ by (ii). Clearly, $d_{D_2(G)}(w, w') = 2$ and $v \in I_{D_2(G)}(w, w')$. If $v \in V(G_2) \setminus S_{G_2}$, say $v = z'$ where $z \in V(G_1)$, then we may choose any $x' \in V(G_2) \cap N_{G_2}(z')$. By (iii), $x, x' \in S$. Moreover, $d_{D_2(G)}(x, x') = 2$ and $v \in I_{D_2(G)}(x, x')$. Thus, S is a 2-path geodetic vertex cover of $D_2(G)$.

The converse is clear. □

Corollary 4. *Let G be a non-trivial connected graph. Then*

$$\beta_{2pg}(D_2(G)) = \beta(D_2(G)) = 2\beta(G).$$

Proof. Let S_1 be a β-set in G_1 and let $S_2 = \{v' \in V(G_2) : v \in S_1\}$. Then S_2 is a β-set in G_2. Moreover, $S = S_1 \cup S_2$ is a vertex cover of $D_2(G)$ by Theorem 9. Thus, by Theorem 10,

$$\beta_{2pg}(D_2(G)) = \beta(D_2(G)) \leq |S| = |S_1| + |S_2| = 2\beta(G).$$

Next, let S_0 be a β-set of $D_2(G)$. Then $S_0 = S_{G_1} \cup S_{G_2}$ where S_{G_1} and S_{G_2} are vertex covers of G_1 and G_2, respectively, by Theorem 9. By Theorem 10, we have

$$\beta_{2pg}(D_2(G)) = \beta(D_2(G)) = |S_0| = |S_{G_1}| + |S_{G_2}| \geq 2\beta(G).$$

This establishes the desired equality. □

Theorem 11. *Let G be a non-complete graph and let m be a positive integer. Then $S \subseteq V(K_m + G)$ is a 2-path geodetic vertex cover of $K_m + G$ if and only if $S = (V(K_m) \setminus$*

$\{v\}) \cup V(G)$ for some $v \in V(K_m)$ or $S = V(K_m) \cup S_G$, where S_G is a 2-path geodetic vertex cover of G.

Proof. Suppose S is a 2-path geodetic vertex cover of K_m+G. Suppose $V(K_m)\setminus S \neq \varnothing$, say $v \in V(K_m) \setminus S$. Since S is a vertex cover of $K_m + G$, it follows that $V(G) \subseteq S$ and $|V(K_m) \setminus S| = 1$. Hence, $S = (V(K_m) \setminus \{v\}) \cup V(G)$. Next, suppose that $V(K_m) \setminus S = \varnothing$. Let $S_G = S \cap V(G)$. Then $S = V(K_m) \cup S_G$. Since S is a vertex cover of $K_m + G$, it follows that S_G is vertex cover of G. Let $w \in V(G) \setminus S_G$. Since S is a 2-path geodetic set in K_m+G, there exist $y, z \in V(K_m+G)$ such that $w \in I_{K_m+G}(y, z)$ and $d_{K_m+G}(y, z) = 2$. This implies that $y, z \in V(G)$, $w \in I_G(y, z)$ and $d_G(y, z) = 2$. Hence, S_G is a 2-path geodetic vertex cover of G.

For the converse, suppose first $S = (V(K_m) \setminus \{v\}) \cup V(G)$ for some $v \in V(K_m)$. Then clearly, S is a 2-path geodetic vertex cover of K_m+G. Next, suppose that $S = V(K_m)\cup S_G$, where S_G is a 2-path geodetic vertex cover of G. Let $ab \in E(K_m+G)$. If $ab \notin E(G)$, then $a \in V(K_m)$ or $b \in V(K_m)$. Suppose $ab \in E(G)$. Since S_G is a vertex cover of G, $a \in S_G$ or $b \in S_G$. Hence, in both cases, $a \in S$ or $b \in S$. Let $w \in V(K_m + G) \setminus S$. Then $w \in V(G) \setminus S_G$. This implies that there exist $p, q \in S_G$ such that $d_{K_m+G}(p, q) = d_G(p, q) = 2$ and $w \in I_G(p, q)$. Since $I_G(p, q) = I_{K_m+G}(p, q)$, it follows that S is a 2-path geodetic vertex cover of $K_m + G$. □

The next results are consequence of Theorem 11.

Corollary 5. *Let G be a non-complete graph and let m be a positive integer. Then*

$$\beta_{2pg}(K_m + G) = min\{(m-1) + |V(G)|, \beta_{2pg}(G) + m\}.$$

Corollary 6. *Let G be a non-complete graph. Then*

$$\beta_{2pg}(K_1 + G) = min\{|V(G)|, \beta_{2pg}(G) + 1\}.$$

Moreover,

(i) $\beta_{2pg}(K_{1,n}) = \beta_{2pg}(K_1 + \overline{K}_n) = n$ *for all* $n \geq 2$;

(ii) $\beta_{2pg}(F_n) = \beta_{2pg}(K_1 + P_n) = \lceil \frac{n+1}{2} \rceil + 1$ *for all* $n \geq 3$;

(iii) $\beta_{2pg}(W_n) = \beta_{2pg}(K_1 + C_n) = \lceil \frac{n}{2} \rceil + 1$ *for all* $n \geq 4$; *and*

(iv) $\beta_{2pg}(K_1 + \bigcup_{j=1}^{k} K_{m_j}) = \sum_{j=1}^{k} m_j$ *for* $k \geq 2$.

Theorem 12. *Let G and H be non-complete graphs. Then $S \subseteq V(G + H)$ is a 2-path geodetic vertex cover of $G + H$ if and only if one of the following statements holds:*

(i) *$S = V(G) \cup S_H$ where S_H is a vertex cover of H.*

(ii) *$S = S_G \cup V(H)$ where S_G is a vertex cover of G.*

Proof. Suppose S is a 2-path geodetic vertex cover of $G+H$. Suppose $V(G)\setminus S\neq\emptyset$ and $V(H)\setminus S\neq\emptyset$. Let $x\in V(G)\setminus S$ and $y\in V(H)\setminus S$. Then $xy\in E(G+H)$ and $x,y\notin S$. This is not possible because S is a vertex cover of $G+H$. Hence, $V(G)\setminus S=\emptyset$ or $V(H)\setminus S=\emptyset$. Suppose $V(G)\setminus S=\emptyset$. Then $V(G)\subseteq S$. Let $S_H=S\cap V(H)$ and let $ab\in E(H)$. Since S is a vertex cover of $G+H$, it follows that $a\in S_H$ or $b\in S_H$. This implies that S_H is a vertex cover of H, showing that (i) holds. Similarly, (ii) holds if $V(H)\setminus S=\emptyset$.

For the converse, suppose that (i) holds. Let $st\in E(G+H)$. If $s\in V(G)$ or $t\in V(G)$, then st is incident to a vertex in S. Suppose $st\in E(H)$. Since S_H is a vertex cover of H, $s\in S_H$ or $t\in S_H$. It follows that S is a vertex cover of $G+H$. Let $z\in V(G+H)\setminus S$. Then $z\in V(H)\setminus S_H$. Choose any $p,q\in V(G)$ such that $d_G(p,q)\neq 1$. Then $p,q\in S$, $d_{G+H}(p,q)=2$, and $z\in I_{G+H}(p,q)$. Therefore, S is a 2-path geodetic vertex cover of $G+H$. We obtain the same conclusion if (ii) holds. □

The next result follows from Theorem 12.

Corollary 7. *Let G and H be non-complete graphs on m and n vertices, respectively. Then*

$$\beta_{2pg}(G+H)=min\{m+\beta(H),n+\beta(G)\}.$$

In particular, each of the following hold.

(i) $\beta_{2pg}(K_{m,n})=\min\{m,n\}$ *for* $m,n\geq 2$.

(ii) $\beta_{2pg}(P_m+P_n)=\min\{m+\lfloor\frac{n}{2}\rfloor,n+\lfloor\frac{m}{2}\rfloor\}$ *for* $m,n\geq 3$.

(iii) $\beta_{2pg}(C_m+C_n)=\min\{m+\lceil\frac{n}{2}\rceil,n+\lceil\frac{m}{2}\rceil\}$ *for* $m,n\geq 4$.

4. Conclusion

In this paper, the concept of 2-path geodetic vertex covering of a graph has been introduced and initially studied. It was shown that the difference $\beta_{2pg}(G)-\beta(G)$ can be increased arbitrarily. Graphs which attain small and large values of the parameter have been characterized. Also, 2-path geodetic vertex coverings in the shadow graph and the join of graphs have been characterized and, subsequently, values of the parameter for these graphs have been determined. This newly defined variant of vertex covering can also be investigated for other classes of graphs. Moreover, while the vertex cover problem is NP-complete, it remains to show whether or not the 2-path geodetic vertex covering problem is also NP-complete.

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