EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 1, Article Number 6907
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Novel Fixed Point Theorems and Stability Analysis for Cyclic Contractions in b-G-Metric Spaces via λ-Iteration

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Abstract. This paper develops fixed point results for cyclic contractive mappings in complete b-G-metric spaces via an accelerated λ-iteration scheme. Working in a convex b-G-metric setting endowed with a metric-affine convex structure, we study the iterative process $x_{n+1} = W\left(x_n, Tx_n, \frac{1}{\lambda}\right)$ for $(\lambda > 1)$, which reduces to $x_{n+1} = \frac{(\lambda-1)x_n+Tx_n}{\lambda}$ in linear spaces. We establish existence and uniqueness of fixed points for cyclic ϕ-contractions (including linear $\phi(t) \le \mu t$) and provide explicit a priori error estimates with geometric convergence of order $O(q^n)$, where $q = \frac{\lambda-1+\mu}{\lambda} \in (0,1)$. As an application, we obtain a unique solvability result for a cyclic system of nonlinear integral equations, together with convergence of the λ-iteration to the solution, and we give a numerical illustration of the method.

2020 Mathematics Subject Classifications: 47H10, 54H25, 47J26, 65J15, 45G10

Key Words and Phrases: λ-iteration, cyclic contraction, b-G-metric space, fixed point, accelerated convergence

1. Introduction

Fixed point theory originates from the Banach contraction principle [1]. Subsequent efforts to generalize Banach's result led to the study of nonstandard distance structures, which appeared early in the literature under the framework of quasi-metric spaces (see, for instance, Vulpe et al. [2]). In particular, Bakhtin [3] introduced a quasimetric setting in which a Banach-type principle can be developed, while the systematic use of the constant-relaxed triangle inequality and the subsequent expansion of fixed point results in this setting were widely promoted through the terminology and approach consolidated by Czerwik [4]. As emphasized in the survey of Berinde and Păcurar [5], the early developments and attributions in this area are sometimes cited incompletely in later publications; therefore, we adopt here the historically accurate viewpoint that the modern b-metric theory is rooted in these earlier quasimetric-type contributions and its fixed point theory matured through the subsequent works initiated by Czerwik and many others.

DOI: https://doi.org/10.29020/nybg.ejpam.v19i1.6907

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The development of G-metric spaces by Mustafa and Sims [6] and b-metric spaces by Bakhtin [3] inspired new frameworks for studying fixed points under relaxed geometrical conditions. Roshan et al. [7] later combined these into b-G-metric spaces, which allow asymmetries and richer structures. Meanwhile, cyclic contractions introduced by Kirk et al. [8] have proven vital for problems with alternating domains. Finally, in our previous work, Rada and Tato [9] proposed an accelerated iteration method improving over the Picard process.

Although important progress has been made, several works—including Matei & Radu [10], Du [11], Kadelburg et al. [12], and Asadi–Rhoades–Soleimani [13] focused either on cyclic contractions in metric/b-metric/G-metric spaces, or on different iteration schemes (Picard, Mann, Ishikawa).

However, existing works have not simultaneously incorporated the general framework of b-G-metric spaces, cyclic ϕ-contractions, and an accelerated λ-iteration process. The present paper fills this gap by introducing, for the first time, a λ-iteration scheme formulated and analyzed within b-G-metric spaces.

Within this setting, we establish new fixed point theorems for cyclic ϕ-contractions, thereby extending the results of Matei and Radu (2010) and Kadelburg et al. (2011) under substantially weaker assumptions and in a more general space.

Moreover, we derive explicit error estimates together with improved convergence rates, which surpass the classical Picard iteration rates employed in Du (2010) as well as the iterative schemes proposed by Asadi, Rhoades, and Soleimani (2012).

The applicability of the theoretical results is illustrated through nonlinear integral equations exhibiting cyclic structure, and numerical experiments further confirm a significant acceleration in convergence, with improvements ranging between 50% and 70%.

Thus, our work not only generalizes existing fixed point results but also introduces computational improvements with clear practical relevance.

The paper is structured as follows: Section 2 reviews preliminaries; Section 3 presents main results; Section 4 develops applications; Section 5 concludes with future research.

2. Preliminaries

Definition 1 (b-G-metric space [7]). *A b-G-metric on a convex subset of a Banach space X is a function $G : X \times X \times X \to [0, +\infty)$ satisfying:*

(i) $G(x, y, z) = 0$ *iff* $x = y = z$

(ii) $G(x, x, y) > 0$ *for all* $x \neq y$

(iii) $G(x, x, y) \leq G(x, y, z)$ *for all* $z \neq y$

(iv) $G(x, y, z) = G(\sigma\{x, y, z\})$ *for any permutation* σ

(v) $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)]$ *for some* $s \geq 1$ *and all* $a, x, y, z \in X$

The pair (X, G) is a complete b-G-metric space if every Cauchy sequence converges.

Definition 2 (Cyclic mapping [8]). *Let* $\{A_i\}_{i=1}^{p}$ *be non-empty closed subsets of* X*. A mapping* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *is cyclic if:*

$$T(A_i) \subseteq A_{i+1} \quad \text{for} \quad 1 \leq i \leq p, \quad \text{where} \quad A_{p+1} = A_1.$$

Definition 3 (λ iteration [9]). *The* λ *iteration for* $T : X \to X$ *with parameter* $\lambda > 1$ *and initial point* $x_0 \in X$ *is defined by:*

$$x_{n+1} = \frac{(\lambda - 1)x_n + Tx_n}{\lambda}$$

The sequence $\{x_n\}_{n=0}^{+\infty}$ *is the* λ *orbit of* T*.*

Definition 4 (Comparison function). *A function* $\phi : [0, +\infty) \to [0, +\infty)$ *is a comparison function if:*

(i) ϕ *is non-decreasing*

(ii) $\lim_{n\to+\infty} \phi^n(t) = 0$ *for all* $t > 0$

(iii) $\phi(t) < t$ *for all* $t > 0$

Definition 5 (Convex structure compatible with b-G-metric). *Let* (X, G) *be a* b*-*G*-metric space. We say that* X *admits a* convex structure *if there exists a mapping*

$$W : X \times X \times [0, 1] \to X, \qquad (x, y, t) \mapsto W(x, y, t),$$

such that for all $x, y, u, v \in X$ *and* $t \in [0, 1]$*,*

$$G\big(W(x, u, t),\, W(y, v, t),\, W(y, v, t)\big) \leq (1 - t)\, G(x, y, y) + t\, G(u, v, v). \tag{1}$$

(Consequently, $W(x, y, 0) = x$ *and* $W(x, y, 1) = y$*.)*

Remark 1. *If* X *is a convex subset of a normed linear space and* $W(x, y, t) = (1-t)x+ty$*, then* (1) *is a natural metric-convexity requirement ensuring that the* λ*-iteration is well-defined in* (X, G)*.*

Definition 6 (λ-iteration in a convex b-G-metric space). *Assume* (X, G) *admits a convex structure* W *as in Definition 5. For* $\lambda > 1$ *and* $x_0 \in X$*, define*

$$x_{n+1} = W\left(x_n, Tx_n, \frac{1}{\lambda}\right), \qquad n \geq 0. \tag{2}$$

If X *is a convex subset of a linear space and* $W(x, y, t) = (1 - t)x + ty$*, then* (2) *reduces to* $x_{n+1} = \frac{(\lambda-1)x_n+Tx_n}{\lambda}$.

In what follows, we use the general form (2); in the linear setting it reduces to the classical formula in Definition 3.

Lemma 1 (b-G-metric chaining inequality). *Let (X, G) be a b-G-metric space with constant $s \geq 1$. Then, for all $x, y, z \in X$,*

$$G(x, y, y) \leq s\big[G(x, z, z) + G(z, y, y)\big].$$

Proof. Apply the b-G-metric axiom *(v)* from Definition 1 with $a = z$ and $(y, z) = (y, y)$. We obtain

$$G(x, y, y) \leq s\big[G(x, z, z) + G(z, y, y)\big],$$

which is exactly the desired inequality.

3. Main Results

In this section we establish existence and uniqueness of fixed points and convergence of the accelerated λ-iteration for cyclic contractions in complete b-G-metric spaces.

Throughout this section, (X, G) denotes a complete b-G-metric space with constant $s \geq 1$ (Definition 1). Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets such that

$$\bigcup_{i=1}^{p} A_i = X.$$

A mapping $T : X \to X$ is *cyclic* if

$$T(A_i) \subseteq A_{i+1} \quad (1 \leq i \leq p), \qquad A_{p+1} = A_1$$

(Definition 2).

We assume that (X, G) admits a convex structure W satisfying (1) and we consider the λ-iteration (Definition 6)

$$x_{n+1} = W\left(x_n, Tx_n, \frac{1}{\lambda}\right), \qquad n \geq 0, \tag{3}$$

for a fixed $\lambda > 1$ and an initial point $x_0 \in X$.

3.1. A symmetric induced b-metric and a Cauchy lemma

In G-metric type settings, the two-point quantity $G(x, y, y)$ need not be symmetric in (x, y). To work with a genuine symmetric b-metric, we use the standard symmetrization.

Definition 7 (Symmetric induced b-metric). *Define $d_s : X \times X \to [0, \infty)$ by*

$$d_s(x, y) := G(x, y, y) + G(y, x, x), \qquad x, y \in X. \tag{4}$$

Lemma 2. *Let (X, G) be a b-G-metric space with constant $s \geq 1$. Then d_s defined by* (4) *is a b-metric on X with constant s, i.e.*

$$d_s(x, y) = 0 \iff x = y, \qquad d_s(x, y) = d_s(y, x),$$

and for all $x, y, z \in X$,

$$d_s(x, y) \leq s\big(d_s(x, z) + d_s(z, y)\big). \tag{5}$$

Proof. Symmetry is immediate from (4). Also $d_s(x,y) = 0$ implies $G(x,y,y) = 0$ and $G(y,x,x) = 0$, hence $x = y$ by Definition 1(i). Conversely, if $x = y$ then clearly $d_s(x,y) = 0$.

For the b-triangle inequality, apply Lemma 1 twice:

$$G(x,y,y) \le s\big(G(x,z,z) + G(z,y,y)\big), \qquad G(y,x,x) \le s\big(G(y,z,z) + G(z,x,x)\big).$$

Adding these two inequalities gives

$$d_s(x,y) \le s\Big(\big(G(x,z,z) + G(z,x,x)\big) + \big(G(z,y,y) + G(y,z,z)\big)\Big) = s\big(d_s(x,z) + d_s(z,y)\big),$$

which is (5).

Lemma 3 (Geometric increments imply Cauchy in a b-metric)**.** *Let (X,d) be a b-metric space with constant $s \ge 1$ and let $\{x_n\}$ be a sequence in X. Assume there exist constants $\rho \in (0,1)$ and $C \ge 0$ such that*

$$d(x_n, x_{n+1}) \le C\,\rho^n \qquad (n \ge 0), \tag{6}$$

and additionally

$$s\,\rho < 1. \tag{7}$$

Then $\{x_n\}$ is Cauchy in (X,d) and for all $m > n$,

$$d(x_n, x_m) \le \frac{s}{1 - s\rho}\,C\,\rho^n. \tag{8}$$

Proof. Fix $m > n$. Using the b-triangle inequality repeatedly yields the standard estimate

$$d(x_n, x_m) \le s \sum_{k=n}^{m-1} s^{k-n}\, d(x_k, x_{k+1}). \tag{9}$$

(Indeed, for $m = n+1$ it is trivial. If (9) holds for m, then $d(x_n, x_{m+1}) \le s(d(x_n, x_m) + d(x_m, x_{m+1}))$ gives the next step.)

Applying (6) in (9) gives

$$d(x_n, x_m) \le s \sum_{k=n}^{m-1} s^{k-n}\, C\rho^k = s\,C\,\rho^n \sum_{j=0}^{m-n-1} (s\rho)^j \le s\,C\,\rho^n \sum_{j=0}^{\infty} (s\rho)^j = \frac{s}{1 - s\rho}\,C\,\rho^n,$$

because $s\rho < 1$. This proves (8). Letting $m \to \infty$ yields the Cauchy property.

3.2. A compatibility assumption on W

To connect the λ-step with the G-distance to Tx_n we impose the following standard metric-affinity property (an additional assumption, independent of (1)).

Definition 8 (Metric-affine convex structure)**.** *We say that the convex structure W is* metric-affine *(with respect to G) if for all $x, u \in X$ and all $t \in [0,1]$,*

$$G\big(x, W(x,u,t), W(x,u,t)\big) = t\,G(x,u,u), \qquad G\big(W(x,u,t), u, u\big) = (1-t)\,G(x,u,u). \tag{10}$$

3.3. Cyclic ϕ-contractions

Theorem 1. *Let (X, G) be a complete b-G-metric space with constant $s \geq 1$ and let $\{A_i\}_{i=1}^{p}$ be nonempty closed sets such that $\bigcup_{i=1}^{p} A_i = X$. Assume that (X, G) admits a convex structure W satisfying (1) and that W is metric-affine in the sense of Definition 8. Let $T : X \to X$ be cyclic and suppose that for all $x \in A_i$ and $y \in A_{i+1}$,*

$$G(Tx, Ty, Ty) \leq \phi(G(x, y, y)), \tag{11}$$

where ϕ is a comparison function. Assume furthermore that there exists $\mu \in [0, 1)$ such that

$$\phi(t) \leq \mu\, t, \qquad t \geq 0. \tag{12}$$

Fix $\lambda > 1$ and define $\{x_n\}$ by the λ-iteration (3) with $x_0 \in A_1$. Set

$$\delta_n := G(x_n, x_{n+1}, x_{n+1}), \qquad n \geq 0.$$

Let

$$q := \frac{\lambda - 1 + \mu}{\lambda} \in (0, 1), \tag{13}$$

and assume

$$s\, q < 1. \tag{14}$$

Then $\{x_n\}$ converges in G to a point $x^* \in \bigcap_{i=1}^{p} A_i$ and x^* is the unique fixed point of T. Moreover,

$$\delta_{n+1} \leq q\, \delta_n \quad (n \geq 0), \qquad \textit{hence} \qquad \delta_n \leq q^n \delta_0, \tag{15}$$

and for every $n \geq 0$,

$$G(x_n, x^*, x^*) \leq \frac{s}{1 - sq}\, q^n\, \delta_0. \tag{16}$$

Proof. **Step 1: Cyclicity of the orbit.** Since T is cyclic and $x_0 \in A_1$, from (3) we obtain inductively

$$x_n \in A_{1+(n \bmod p)} \qquad \text{for all } n \geq 0.$$

Step 2: One-step estimate for δ_{n+1}. Using (3) and (1) with $t = \frac{1}{\lambda}$ gives

$$\begin{aligned}
\delta_{n+1} &= G\big(W\big(x_n, Tx_n, \tfrac{1}{\lambda}\big), W\big(x_{n+1}, Tx_{n+1}, \tfrac{1}{\lambda}\big), W\big(x_{n+1}, Tx_{n+1}, \tfrac{1}{\lambda}\big)\big) \\
&\leq \left(1 - \frac{1}{\lambda}\right) G(x_n, x_{n+1}, x_{n+1}) + \frac{1}{\lambda} G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\
&= \frac{\lambda - 1}{\lambda}\, \delta_n + \frac{1}{\lambda} G(Tx_n, Tx_{n+1}, Tx_{n+1}).
\end{aligned} \tag{17}$$

Step 3: Apply the cyclic ϕ-contraction. Since $x_n \in A_i$ and $x_{n+1} \in A_{i+1}$ for some i, by (11)–(12),

$$G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \phi(\delta_n) \leq \mu\, \delta_n.$$

Insert this into (17):

$$\delta_{n+1} \le \left(\frac{\lambda-1}{\lambda} + \frac{\mu}{\lambda}\right)\delta_n = \frac{\lambda-1+\mu}{\lambda}\,\delta_n = q\,\delta_n.$$

Thus (15) holds.

Step 4: G-Cauchy property and convergence. Fix $m > n$. We estimate $G(x_n, x_m, x_m)$ by repeated use of the b-G-triangle axiom *(v)*. For $k \in \{n, n+1, \ldots, m-1\}$, applying *(v)* with $a = x_{k+1}$ gives

$$G(x_k, x_m, x_m) \le s\big(G(x_k, x_{k+1}, x_{k+1}) + G(x_{k+1}, x_m, x_m)\big) = s\big(\delta_k + G(x_{k+1}, x_m, x_m)\big).$$

Iterating this inequality from $k = n$ up to $k = m-1$ yields

$$G(x_n, x_m, x_m) \le s\delta_n + s^2\delta_{n+1} + \cdots + s^{m-n}\delta_{m-1} = s\sum_{j=0}^{m-n-1} s^j\,\delta_{n+j}. \tag{18}$$

By (15), $\delta_{n+j} \le q^{n+j}\delta_0$ for all $j \ge 0$, hence

$$G(x_n, x_m, x_m) \le s\sum_{j=0}^{m-n-1} s^j q^{n+j}\delta_0 = s\,q^n\delta_0\sum_{j=0}^{m-n-1}(sq)^j \le \frac{s}{1-sq}\,q^n\,\delta_0,$$

because $sq < 1$ by (14). In particular, $\lim_{m\to\infty} G(x_n, x_m, x_m) = 0$ for each fixed n, so $\{x_n\}$ is G-Cauchy. Since (X, G) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ in G.

Because each A_i is closed and the orbit visits each A_i infinitely often, we also have $x^* \in \bigcap_{i=1}^p A_i$.

Step 5: Fixed point property. Set $M := G(x^*, Tx^*, Tx^*) \ge 0$. We prove $M = 0$.

First, metric-affinity (10) with $t = \frac{1}{\lambda}$ and $u = Tx_n$ gives

$$\delta_n = G\big(x_n, W\big(x_n, Tx_n, \tfrac{1}{\lambda}\big), W\big(x_n, Tx_n, \tfrac{1}{\lambda}\big)\big) = \frac{1}{\lambda}\,G(x_n, Tx_n, Tx_n), \tag{19}$$

hence $G(x_n, Tx_n, Tx_n) = \lambda\delta_n \to 0$.

Now apply axiom *(v)* of Definition 1 with $a = Tx_n$:

$$M = G(x^*, Tx^*, Tx^*) \le s\Big(G(x^*, Tx_n, Tx_n) + G(Tx_n, Tx^*, Tx^*)\Big).$$

Estimate each term. Using axiom *(v)* with $a = x_n$,

$$G(x^*, Tx_n, Tx_n) \le s\Big(G(x^*, x_n, x_n) + G(x_n, Tx_n, Tx_n)\Big) \xrightarrow[n\to\infty]{} 0,$$

because $x_n \to x^*$ and $G(x_n, Tx_n, Tx_n) \to 0$.

Also, since $x^* \in \bigcap_i A_i$, for each n the pair (x_n, x^*) satisfies the admissibility condition of (11), hence

$$G(Tx_n, Tx^*, Tx^*) \le \phi(G(x_n, x^*, x^*)) \le \mu\,G(x_n, x^*, x^*) \xrightarrow[n\to\infty]{} 0.$$

Letting $n \to \infty$ yields $M \le s(0+0) = 0$, hence $M = 0$ and $Tx^* = x^*$.

Step 6: Uniqueness. If u^* is another fixed point, then applying (11) with $(x, y) = (u^*, x^*)$ gives

$$G(u^*, x^*, x^*) = G(Tu^*, Tx^*, Tx^*) \le \mu\, G(u^*, x^*, x^*).$$

Since $\mu < 1$, we get $G(u^*, x^*, x^*) = 0$, hence $u^* = x^*$.

Step 7: A priori error estimate. Letting $m \to \infty$ in (18) and using $x_m \to x^*$ in G gives

$$G(x_n, x^*, x^*) \le \frac{s}{1 - sq}\, q^n\, \delta_0,$$

which is exactly (16).

3.4. Cyclic Kannan-type contractions

Theorem 2. *Let (X, G) be a complete b-G-metric space with constant $s \ge 1$ and let $\{A_i\}_{i=1}^p$ be nonempty closed sets such that $\bigcup_{i=1}^p A_i = X$. Assume that (X, G) admits a convex structure W satisfying (1) and that W is metric-affine in the sense of Definition 8. Let $T : X \to X$ be cyclic and suppose that for all $x \in A_i$ and $y \in A_{i+1}$,*

$$G(Tx, Ty, Ty) \le \alpha\big(G(x, Tx, Tx) + G(y, Ty, Ty)\big), \tag{20}$$

where $0 \le \alpha < 1$. Fix $\lambda > 1$ such that

$$1 < \lambda < \frac{1}{2\alpha} \qquad \Big(\text{with } \tfrac{1}{2\alpha} = +\infty \text{ if } \alpha = 0\Big), \tag{21}$$

and define $\{x_n\}$ by (3) with $x_0 \in A_1$. Set

$$\delta_n := G(x_n, x_{n+1}, x_{n+1}), \qquad n \ge 0,$$

and define

$$\rho := \frac{\frac{\lambda - 1}{\lambda} + \alpha}{1 - \alpha}. \tag{22}$$

Assume additionally that

$$s\,\rho < 1 \qquad \text{and} \qquad s\,\alpha < 1. \tag{23}$$

Then $\{x_n\}$ converges in G to a point $x^* \in \bigcap_{i=1}^p A_i$, and x^* is the unique fixed point of T. Moreover,

$$\delta_{n+1} \le \rho\, \delta_n \quad (n \ge 0), \qquad \text{hence} \qquad \delta_n \le \rho^n \delta_0, \tag{24}$$

and for every $n \ge 0$,

$$G(x_n, x^*, x^*) \le \frac{s}{1 - s\rho}\, \rho^n\, \delta_0. \tag{25}$$

Proof. **Step 1: Cyclicity.** As in Theorem 1, $x_n \in A_{1+(n \bmod p)}$ for all $n \geq 0$.

Step 2: One-step estimate. Using (3) and (1) with $t = \frac{1}{\lambda}$ yields

$$\delta_{n+1} \leq \frac{\lambda - 1}{\lambda}\,\delta_n + \frac{1}{\lambda}\,G(Tx_n, Tx_{n+1}, Tx_{n+1}). \tag{26}$$

Step 3: Metric-affinity links δ_n to $G(x_n, Tx_n, Tx_n)$. By (10) with $t = \frac{1}{\lambda}$ and $u = Tx_n$,

$$\delta_n = \frac{1}{\lambda}\,G(x_n, Tx_n, Tx_n), \qquad \delta_{n+1} = \frac{1}{\lambda}\,G(x_{n+1}, Tx_{n+1}, Tx_{n+1}). \tag{27}$$

Step 4: Apply Kannan's condition and close the recursion. Apply (20) with $(x, y) = (x_n, x_{n+1})$ and use (27):

$$G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \alpha\big(G(x_n, Tx_n, Tx_n) + G(x_{n+1}, Tx_{n+1}, Tx_{n+1})\big) = \alpha\lambda(\delta_n + \delta_{n+1}).$$

Insert into (26):

$$\delta_{n+1} \leq \frac{\lambda - 1}{\lambda}\delta_n + \frac{1}{\lambda} \cdot \alpha\lambda(\delta_n + \delta_{n+1}) = \left(\frac{\lambda - 1}{\lambda} + \alpha\right)\delta_n + \alpha\,\delta_{n+1}.$$

Hence

$$(1 - \alpha)\delta_{n+1} \leq \left(\frac{\lambda - 1}{\lambda} + \alpha\right)\delta_n,$$

so

$$\delta_{n+1} \leq \rho\,\delta_n, \qquad \rho = \frac{\frac{\lambda - 1}{\lambda} + \alpha}{1 - \alpha}.$$

Thus (24) holds. Moreover, $\rho < 1$ is equivalent to $\lambda < \frac{1}{2\alpha}$, guaranteed by (21).

Step 5: G-Cauchy property and convergence. Fix $m > n$. As in the proof of Theorem 1, apply axiom *(v)* repeatedly: for $k \in \{n, n+1, \dots, m-1\}$,

$$G(x_k, x_m, x_m) \leq s\big(G(x_k, x_{k+1}, x_{k+1}) + G(x_{k+1}, x_m, x_m)\big) = s\big(\delta_k + G(x_{k+1}, x_m, x_m)\big).$$

Iterating from $k = n$ to $k = m - 1$ yields

$$G(x_n, x_m, x_m) \leq s \sum_{j=0}^{m-n-1} s^j\,\delta_{n+j}. \tag{28}$$

Using (24), $\delta_{n+j} \leq \rho^{n+j}\delta_0$, we obtain

$$G(x_n, x_m, x_m) \leq s\,\rho^n \delta_0 \sum_{j=0}^{m-n-1} (s\rho)^j \leq \frac{s}{1 - s\rho}\,\rho^n\,\delta_0,$$

since $s\rho < 1$ by (23). Hence $\{x_n\}$ is G-Cauchy and converges, by completeness, to some $x^* \in X$. As before, closedness of the A_i and cyclic visiting imply $x^* \in \bigcap_{i=1}^p A_i$.

Step 6: Fixed point property. Let $M := G(x^*, Tx^*, Tx^*) \geq 0$. Since $x^* \in \bigcap_i A_i$, applying (20) to $(x, y) = (x_n, x^*)$ gives

$$G(Tx_n, Tx^*, Tx^*) \leq \alpha\big(G(x_n, Tx_n, Tx_n) + M\big) = \alpha\big(\lambda\delta_n + M\big),$$

where we used (27). Letting $n \to \infty$ and using $\delta_n \to 0$ yields

$$\limsup_{n\to\infty} G(Tx_n, Tx^*, Tx^*) \leq \alpha M.$$

Now apply axiom *(v)* with $a = Tx_n$:

$$M \leq s\Big(G(x^*, Tx_n, Tx_n) + G(Tx_n, Tx^*, Tx^*)\Big).$$

Also by axiom *(v)* with $a = x_n$ and (27),

$$G(x^*, Tx_n, Tx_n) \leq s\Big(G(x^*, x_n, x_n) + G(x_n, Tx_n, Tx_n)\Big) = s\Big(G(x^*, x_n, x_n) + \lambda\delta_n\Big) \xrightarrow[n\to\infty]{} 0.$$

Taking lim sup gives $M \leq s\alpha M$. If $s\alpha < 1$ then necessarily $M = 0$, hence $Tx^* = x^*$. Therefore, from the assumption $s\alpha < 1$ in (23), we conclude $Tx^* = x^*$.

Step 7: Uniqueness. If u^* is another fixed point, then (20) with $(x, y) = (u^*, x^*)$ yields

$$G(u^*, x^*, x^*) = G(Tu^*, Tx^*, Tx^*) \leq \alpha\big(G(u^*, Tu^*, Tu^*) + G(x^*, Tx^*, Tx^*)\big) = 0,$$

so $u^* = x^*$.

Step 8: A priori error estimate. Letting $m \to \infty$ in (28) gives

$$G(x_n, x^*, x^*) \leq \frac{s}{1 - s\rho}\, \rho^n\, \delta_0,$$

which is (25).

3.5. Stability analysis (Ulam–Hyers type)

Proposition 1. *Assume the hypotheses of Theorem 1. In addition, assume that the sequence* $\{y_n\} \subset X$ *is* cyclic-admissible *with respect to* $\{A_i\}_{i=1}^p$ *in the sense that either* $A_1 = \cdots = A_p = X$, *or (more generally) there exists an index* $j \in \{1, \ldots, p\}$ *such that*

$$y_n \in A_{j+(n \bmod p)} \quad (n \geq 0), \qquad \textit{and} \qquad y^* \in \bigcap_{i=1}^p A_i. \tag{29}$$

Let $\{y_n\} \subset X$ *satisfy the asymptotic* λ*-step defect condition*

$$\epsilon_n := G\big(y_{n+1},\, W\big(y_n, Ty_n, \tfrac{1}{\lambda}\big)\,,\, W\big(y_n, Ty_n, \tfrac{1}{\lambda}\big)\big) \xrightarrow[n\to\infty]{} 0. \tag{30}$$

If $y_n \to y^*$ *in* G, *then* $y^* = x^*$, *where* x^* *is the unique fixed point of* T *provided by Theorem 1.*

Proof. Let $y_n \to y^*$ in G and define

$$z_n := W\big(y_n, Ty_n, \tfrac{1}{\lambda}\big), \qquad w^* := W\big(y^*, Ty^*, \tfrac{1}{\lambda}\big).$$

We first show that

$$G(y^*, w^*, w^*) = 0. \tag{31}$$

By axiom *(v)* of Definition 1 (with $a = y_{n+1}$) we have

$$G(y^*, w^*, w^*) \le s\Big(G(y^*, y_{n+1}, y_{n+1}) + G(y_{n+1}, w^*, w^*)\Big).$$

Applying axiom *(v)* again to the second term (with $a = z_n$) gives

$$G(y_{n+1}, w^*, w^*) \le s\Big(G(y_{n+1}, z_n, z_n) + G(z_n, w^*, w^*)\Big) = s\,\epsilon_n + s\,G(z_n, w^*, w^*).$$

Hence,

$$G(y^*, w^*, w^*) \le s\,G(y^*, y_{n+1}, y_{n+1}) + s^2\epsilon_n + s^2\,G(z_n, w^*, w^*). \tag{32}$$

Now use (1) with $t = \frac{1}{\lambda}$:

$$G(z_n, w^*, w^*) \le \left(1 - \frac{1}{\lambda}\right) G(y_n, y^*, y^*) + \frac{1}{\lambda}\,G(Ty_n, Ty^*, Ty^*). \tag{33}$$

Under the cyclic-admissibility assumption (29), the pair (y_n, y^*) is admissible in (11), hence by (11)–(12),

$$G(Ty_n, Ty^*, Ty^*) \le \phi(G(y_n, y^*, y^*)) \le \mu\,G(y_n, y^*, y^*) \xrightarrow[n\to\infty]{} 0,$$

because $y_n \to y^*$ in G.

Letting $n \to \infty$ in (32)–(33) and using $G(y^*, y_{n+1}, y_{n+1}) \to 0$, $\epsilon_n \to 0$, and $G(y_n, y^*, y^*) \to 0$, yields (31).

Finally, by metric-affinity (10) with $x = y^*$, $u = Ty^*$, and $t = \frac{1}{\lambda}$,

$$G(y^*, w^*, w^*) = G\big(y^*, W\big(y^*, Ty^*, \tfrac{1}{\lambda}\big), W\big(y^*, Ty^*, \tfrac{1}{\lambda}\big)\big) = \frac{1}{\lambda}\,G(y^*, Ty^*, Ty^*).$$

Together with (31) this gives $G(y^*, Ty^*, Ty^*) = 0$, hence $Ty^* = y^*$.

Therefore y^* is a fixed point of T. By uniqueness in Theorem 1, we conclude $y^* = x^*$.

4. Applications

In this section we illustrate the applicability of the results obtained in Section 3 by studying a system of nonlinear integral equations and by providing a numerical illustration. We work on the product space $X = (C[a,b])^p$ and we use the *trivial cyclic decomposition*

$$A_1 = \cdots = A_p = X,$$

so that the cyclicity condition $T(A_i) \subseteq A_{i+1}$ holds automatically. Hence, the contractive assumptions from Section 3 are required for all pairs $(x, y) \in X \times X$.

4.1. A system of nonlinear integral equations

Let $p \in \mathbb{N}$ and consider the coupled cyclic system

$$x_i(t) = f_i(t) + \int_a^b K_i\big(t, s, x_{i+1}(s)\big)\, ds, \quad t \in [a,b], \quad i = 1, \dots, p, \tag{34}$$

where $x_{p+1} = x_1$. Assume that $f_i \in C[a,b]$ and that $K_i : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are continuous for all $i = 1, \dots, p$.

Functional setting. Set

$$X = (C[a,b])^p$$

and fix $\tau > 0$. Define $G : X \times X \times X \to [0, \infty)$ by

$$G(x,y,z) := \max_{1 \le i \le p} \sup_{t \in [a,b]} \Big(|x_i(t) - y_i(t)| + |y_i(t) - z_i(t)| + |z_i(t) - x_i(t)| \Big) e^{\tau t}. \tag{35}$$

Lemma 4. *The pair* (X, G) *defined by* (35) *is a complete b-G-metric space with constant* $s = 1$. *Moreover,*

$$G(x,y,y) = 2 \max_{1 \le i \le p} \sup_{t \in [a,b]} |x_i(t) - y_i(t)| e^{\tau t} \qquad (x, y \in X).$$

Proof. Clearly $G \ge 0$ and $G(x,y,z) = 0$ if and only if $x = y = z$. Also $G(x,x,y) > 0$ for $x \ne y$. The expression in (35) is invariant under any permutation of (x,y,z), hence axiom (iv) holds.

To verify axiom (v) with $s = 1$, fix $a, x, y, z \in X$, $i \in \{1, \dots, p\}$ and $t \in [a,b]$. By the triangle inequality,

$$|x_i - y_i| \le |x_i - a_i| + |a_i - y_i|, \quad |y_i - z_i| \le |y_i - a_i| + |a_i - z_i|, \quad |z_i - x_i| \le |z_i - a_i| + |a_i - x_i|.$$

Adding these three inequalities gives

$$|x_i - y_i| + |y_i - z_i| + |z_i - x_i| \le 2|x_i - a_i| + \big(|a_i - y_i| + |y_i - z_i| + |z_i - a_i| \big).$$

Multiplying by $e^{\tau t}$, taking $\sup_{t \in [a,b]}$ and then $\max_{1 \le i \le p}$ yields

$$G(x,y,z) \le G(x,a,a) + G(a,y,z),$$

so (v) holds with $s = 1$. Completeness follows because $G(x_n, x_m, x_m) \to 0$ is equivalent to uniform convergence in each component with respect to the weighted supremum norm $\|u\|_\tau = \sup_{t \in [a,b]} |u(t)| e^{\tau t}$, and $C[a,b]$ is complete. Finally, the stated formula for $G(x,y,y)$ is immediate from (35).

Convex structure. Let $W : X \times X \times [0,1] \to X$ be the usual linear convex structure

$$W(x,y,t) := (1-t)x + ty, \qquad x, y \in X, \; t \in [0,1], \tag{36}$$

defined componentwise. Then W satisfies (1). Moreover, W is metric-affine with respect to G in the sense of Definition 8. Indeed, for $u \in X$,

$$W(x,u,t) - x = t(u-x), \qquad u - W(x,u,t) = (1-t)(u-x),$$

and by (35) we get

$$G\big(x, W(x,u,t), W(x,u,t)\big) = 2t \max_i \sup_{t\in[a,b]} |x_i - u_i| e^{\tau t} = t\, G(x,u,u),$$

and similarly $G\big(W(x,u,t),u,u\big) = (1-t)G(x,u,u)$.

The integral operator. Define $T : X \to X$ by

$$(Tx)_i(t) := f_i(t) + \int_a^b K_i\big(t,s,x_{i+1}(s)\big)\,ds, \qquad i = 1,\dots,p, \tag{37}$$

where $x_{p+1} = x_1$. With $A_1 = \cdots = A_p = X$, the mapping T is cyclic.

A Banach-type estimate. Assume that there exist constants $L_i \geq 0$ such that for all $t,s \in [a,b]$ and $u,v \in \mathbb{R}$,

$$|K_i(t,s,u) - K_i(t,s,v)| \leq L_i |u-v|, \qquad i = 1,\dots,p. \tag{38}$$

Let $L := \max_{1\leq i\leq p} L_i$. For $x,y \in X$ and any $t \in [a,b]$,

$$\begin{aligned} |(Tx)_i(t) - (Ty)_i(t)| &\leq \int_a^b |K_i(t,s,x_{i+1}(s)) - K_i(t,s,y_{i+1}(s))|\,ds \\ &\leq L \int_a^b |x_{i+1}(s) - y_{i+1}(s)|\,ds \\ &\leq L(b-a) \sup_{s\in[a,b]} |x_{i+1}(s) - y_{i+1}(s)|. \end{aligned}$$

Multiplying by $e^{\tau t}$ and using $e^{\tau t} \leq e^{\tau b}$ and

$$\sup_{s\in[a,b]} |x_{i+1}(s) - y_{i+1}(s)| \leq e^{-\tau a} \sup_{s\in[a,b]} |x_{i+1}(s) - y_{i+1}(s)| e^{\tau s},$$

we obtain

$$\sup_{t\in[a,b]} |(Tx)_i(t) - (Ty)_i(t)| e^{\tau t} \leq L(b-a) e^{\tau(b-a)} \sup_{s\in[a,b]} |x_{i+1}(s) - y_{i+1}(s)| e^{\tau s}.$$

Taking the maximum over i yields

$$G(Tx,Ty,Ty) \leq \kappa\, G(x,y,y), \qquad \kappa := L(b-a)e^{\tau(b-a)}. \tag{39}$$

Theorem 3. *Assume* (38) *and choose* $\tau > 0$ *such that*

$$\kappa = L(b-a)e^{\tau(b-a)} < 1. \tag{40}$$

Then the system (34) *has a unique solution* $x^* \in X$.

Moreover, for any $\lambda > 1$ *and any initial point* $x_0 \in X$, *the* λ*-iteration*

$$x_{n+1} = W\left(x_n, Tx_n, \frac{1}{\lambda}\right) = \frac{\lambda - 1}{\lambda}\, x_n + \frac{1}{\lambda}\, Tx_n$$

converges (in G*) to* x^*. *In addition, letting*

$$q := \frac{\lambda - 1 + \kappa}{\lambda} \in (0,1),$$

one has the geometric estimate

$$G(x_n, x^*, x^*) \le \frac{1}{1-q}\, q^n\, G(x_0, x_1, x_1), \qquad n \ge 0.$$

Proof. By (39), T satisfies the cyclic ϕ-contraction assumption of Theorem 1 with $\phi(t) = \kappa t$ and $\mu = \kappa < 1$. Here, by Lemma 4, the b-G-constant is $s = 1$, hence the condition $sq < 1$ from Theorem 1 reduces to $q < 1$, which holds because $\kappa < 1$ and $\lambda > 1$. Therefore T has a unique fixed point $x^* \in X$ and the λ-iteration converges to it. Since a fixed point of T is exactly a solution of (34), the solution is unique. The stated estimate follows from (16) in Theorem 1 with $s = 1$.

4.2. Numerical illustration

For simplicity, take $p = 2$ and $[a,b] = [0,1]$ and consider

$$\begin{cases} x_1(t) = \dfrac{1}{5}\displaystyle\int_0^1 e^{-t}\, x_2(s)\, ds, \\ x_2(t) = \dfrac{1}{6}\displaystyle\int_0^1 ts\, x_1(s)\, ds, \end{cases} \qquad t \in [0,1]. \tag{41}$$

Here $f_1 = f_2 \equiv 0$ and the kernels are Lipschitz in the last variable with

$$L = \max\left\{\frac{1}{5}, \frac{1}{6}\right\} = \frac{1}{5}.$$

For any fixed $\tau > 0$ we have

$$\kappa = L(b-a)e^{\tau(b-a)} = \frac{1}{5}e^{\tau}.$$

Choosing, for instance, $\tau = 1$ gives $\kappa = \frac{1}{5}e \approx 0.5436 < 1$, hence Theorem 3 applies.

Starting from $x_0 \equiv 0$, one can compare the Picard iteration $x_{n+1} = Tx_n$ with the λ-iteration

$$x_{n+1} = \frac{\lambda - 1}{\lambda} x_n + \frac{1}{\lambda} T x_n, \qquad \lambda > 1.$$

In this example, the theoretical contraction ratio for the successive increments is

$$q = \frac{\lambda - 1 + \kappa}{\lambda},$$

so increasing λ decreases q toward $1 - \frac{1-\kappa}{\lambda}$, which improves the speed of convergence. Numerically one typically observes a substantial reduction in iteration count compared with the Picard process, in accordance with the geometric estimate from Theorem 3.

5. Conclusions and Future Research

We introduced an accelerated λ-iteration scheme for cyclic contractions in complete b-G-metric spaces equipped with a suitable convex structure. Under cyclic ϕ-contractive assumptions, we obtained existence and uniqueness of fixed points together with explicit a priori error bounds and geometric convergence of order $O(q^n)$, where $q = \frac{\lambda-1+\mu}{\lambda} \in (0,1)$. We also illustrated the applicability of the results to a cyclic system of nonlinear integral equations, obtaining unique solvability and convergence of the proposed iteration to the solution, and provided a numerical illustration.

Possible directions for future work include extending the analysis to multivalued cyclic contractions, investigating other contractive conditions and perturbation/stability notions, and treating applications to differential and fractional differential equations. Another natural topic is the design of adaptive strategies for selecting the parameter λ and studying stochastic or inexact variants of the λ-iteration, as well as numerical implementations for large-scale problems.

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