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Bipolar Fuzzy Commutative Hyper BCK-Ideals in Hyper BCK-Algebras

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Abstract. This study introduces the concept of bipolar fuzzy commutative hyper BCK-ideals (BF-CHBCKIs) within the algebraic framework of hyper BCK-algebras, offering a novel approach to modeling dual uncertainty through bipolar fuzzy sets. By defining and classifying BF-CHBCKIs across multiple types and examining their structural relationships with reflexive, strong, and weak hyper BCK-ideals, we establish a comprehensive theoretical foundation supported by formal theorems and illustrative examples. These findings extend current understandings in hyperstructure theory and fuzzy algebra, contributing to the broader landscape of abstract mathematical reasoning. Importantly, this research aligns with Sustainable Development Goal 4 (SDG-4) by promoting inclusive and equitable quality education. The formalization of BF-CHBCKIs fosters advanced mathematical thinking and provides meaningful tools for enhancing learning environments, particularly in schools and institutions that emphasize research-oriented instruction. By integrating abstract algebraic structures with uncertainty modeling, this work supports the cultivation of analytical skills, mathematical creativity, and deeper engagement with formal logic among students and emerging researchers.

2020 Mathematics Subject Classifications: 03E72, 06F35, 03G25

Key Words and Phrases: Hyper BCK-algebra (HBCKA), commutative hyper BCK-ideal (CHBCKI), fuzzy commutative hyper BCK-ideal (FCHBCKI), bipolar fuzzy commutative hyper BCK-ideal (BF-CHBCKI)

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1. Introduction

The study of algebraic structures provides a common foundation for understanding a wide range of mathematical concepts. These structures form the mathematical basis for many of the algorithms and protocols that support our digital world. These provide the way to study mathematical operations in their most general form. By concentrating on the essential properties of algebraic structures, we can gain a deeper understanding of the underlying ideas of mathematical systems.

In 1966, Imai et al. (see [1–3]) introduced the algebraic structure called a BCK-algebra as an extension of the concepts of propositional calculus and set-theoretic difference. Following their introduction, many researchers have extensively studied BCK-algebras, especially on ideals. In 1934, Marty [4] presented the idea of hyperstructure theory, or multi-algebras, during the 8th Congress of Scandinavian Mathematicians. There are numerous areas in both applied and pure research where the idea of hyperstructures can be useful. By applying hyperstructure theory to BCK-algebras, Jun et al. [5] developed a hyper BCK-algebra, expanded the BCK-algebra, and studied its related properties. Building on the foundational structure of hyper BCK-algebras, Borzooei and Bakhshi [6] systematically introduced four distinct types of commutative hyper BCK-ideals (CHBCKIs), laying the groundwork for a series of significant algebraic results. Expanding this framework into the domain of uncertainty modeling, Durga Prasad et al. [7] proposed the concepts of intuitionistic fuzzy positive implicative hyper BCK-ideals, also categorized into types 1, 2, 3, and 4, thereby bridging intuitionistic fuzzy logic with hyperstructure theory. Further advancing this trajectory, Satyanarayana et al. [8] developed the notions of intuitionistic fuzzy commutative hyper BCK-ideals across the same typological classifications, reinforcing the depth and versatility of fuzzy extensions within commutative hyper BCK-algebraic systems.

Numerous techniques extend the concept of a set, with Zadeh's fuzzy sets [9] being the most prominent example. Fuzzy sets allow elements to belong to a set with a degree of membership, known as the membership grade (between 0 and 1). They provide a framework for handling uncertainty. Lee [10] initially proposed the notion of bipolar-valued fuzzy sets as an extension of fuzzy sets. The membership function that characterizes these sets assigns each element a pair of values, one from $[0, 1]$ (indicating positive membership) and one from $[-1, 0]$ (indicating negative membership). This representation is effective when analyzing topics that require examining both positive and negative components.

The bipolar fuzzy set (BFS) framework has emerged as a powerful extension of classical fuzzy set theory, offering a dual-valued approach to uncertainty by modeling both degrees of satisfaction and dissatisfaction. This paradigm has been fruitfully applied to a wide range of algebraic structures, demonstrating its versatility and depth. In particular, BFS logic has been explored in the context of near-rings [11], where novel classes of bipolar fuzzy ideals have been introduced. In Γ-semirings, BFSs provide a flexible tool to define bipolar fuzzy ideals that generalize standard multiplicative behaviors under uncertainty [12]. The theory has also been extended to semigroups, where rough bipolar fuzzy ideals offer refined ways to handle incomplete or vague operations [13]. More abstract algebraic systems have likewise benefited from BFS modeling. For example, TM-algebras [14], prime ideals in lattices [15], and subalgebras and ideals in BCK/BCI-algebras [16] have all been studied using bipolar fuzzy logic. Within the framework of BCK-algebras, BFSs have been used to define commutative ideals [17], as well as bipolar intuitionistic fuzzy implicative [18] and positive implicative ideals [19], incorporating a richer uncertainty semantics that captures both belief and disbelief. BFSs have also played a significant role in hyperstructure theory, particularly in the study of hyper BCK-ideals [20] and implicative hyper BCK-ideals [21], where operations yield sets rather than single outcomes. Foundational work by Jun and colleagues [22–24] introduced and classified several types of bipolar fuzzy hyper BCK-ideals, developing rigorous definitions and algebraic properties based on cut-level approaches and structural inclusion. These studies laid the theoretical groundwork for further generalizations, including the integration of BFS logic with soft set theory [25], where Muhiuddin et al. introduced the notion of bipolar-valued fuzzy soft hyper BCK-ideals—a hybrid model that enables multi-criteria uncertainty reasoning in hyperalgebraic contexts. Recently, this approach has even been applied to fantastic ideals in BCK/BCI-algebras [26], further emphasizing the capacity of BFSs to capture nuanced algebraic behavior under dual uncertainty. These developments collectively highlight the growing influence of bipolar fuzzy logic in algebraic reasoning, especially in systems characterized by non-determinism, duality, and graded membership.

In this paper, we apply the concept of BFSs to CHBCKIs in HBCKAs and introduce the new notion of BF-CHBCKIs. We then present several theorems characterizing these notions in terms of level subsets. Furthermore, we establish the relationship among these notions, specifically BF-(strong, weak, reflexive)-HBCKIs and BF-CHBCKIs, and investigate some interesting properties.

Let $\mathcal{H}$ be a non-empty set endowed with a hyperoperation, that is, $\circ$ is a function from $\mathcal{H} \times \mathcal{H}$ to $\mathcal{P}^*(\mathcal{H}) = \mathcal{P}(\mathcal{H})\backslash\{\emptyset\}$. For any two subsets T and J of $\mathcal{H}$, denoted by $T \circ J$, the set $\bigcup\limits_{a\in T, b\in J} a \circ b$. We will utilize $\hbar_1 \circ \hbar_2$ instead of $\hbar_1 \circ \{\hbar_2\}$, $\{\hbar_1\} \circ \hbar_2$, or $\{\hbar_1\} \circ \{\hbar_2\}$.

2. Preliminaries

In this section, we recall fundamental concepts and notation essential to the development of the main results in this paper. These include basic definitions related to BCK-algebras, hyper BCK-algebras, fuzzy sets, and bipolar fuzzy sets, as well as relevant classes of ideals. Unless stated otherwise, all algebraic structures considered here are assumed to be non-trivial.

The notion of BCK-algebras was first introduced by Iséki and Tanaka [1] as an algebraic counterpart to certain propositional calculi. Subsequently, hyper BCK-algebras were developed to generalize BCK-algebras by allowing hyperoperations, as introduced in [5]. Meanwhile, the concept of fuzzy sets, introduced by Zadeh [9], was extended to bipolar fuzzy sets (BFSs) by Zhang [27] and Lee [10] to model duality in membership functions.

For the reader's convenience, we summarize here the essential definitions and properties that will be used throughout the rest of the paper.

Definition 1. *[5] By a hyper BCK-algebra (HBCKA), we mean a non-empty collection* $\mathcal{H}$ *possessed of a hyperoperation* $\circ$ *and a constant* 0 *fulfilling the principles listed below:*

(HBCKA-1) $(\hbar_1 \circ \hbar_3) \circ (\hbar_2 \circ \hbar_3) \ll \hbar_1 \circ \hbar_2$,

(HBCKA-2) $(\hbar_1 \circ \hbar_2) \circ \hbar_3 = (\hbar_1 \circ \hbar_3) \circ \hbar_2$,

(HBCKA-3) $\hbar_1 \circ \mathcal{H} \ll \{\hbar_1\}$,

(HBCKA-4) $\hbar_1 \ll \hbar_2$ *and* $\hbar_2 \ll \hbar_1 \Rightarrow \hbar_1 = \hbar_2$, *for all* $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$.

We denote a relationship $\ll$ on $\mathcal{H}$ by letting $\hbar_1 \ll \hbar_2 \Leftrightarrow 0 \in \hbar_1 \circ \hbar_2$ and every $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}, \mathcal{H}_1 \ll \mathcal{H}_2$ is described by $\forall r \in \mathcal{H}_1, \exists j \in \mathcal{H}_2$ such that $r \ll j$. In such a case, we call $\ll$ the hyper order in $\mathcal{H}$. Note that the scenario (HBCKA-3) is equal to the condition (P1). In any HBCKA $\mathcal{H}$, the following is true.

(P1) $\hbar_1 \circ \hbar_2 \ll \{\hbar_1\}$,

(P2) $\hbar_1 \circ 0 \ll \{\hbar_1\}$, $0 \circ \hbar_1 \ll \{\hbar_1\}$, and $0 \circ 0 \ll \{0\}$,

(P3) $(\mathcal{H}_1 \circ \mathcal{H}_2) \circ \mathcal{H}_3 = (\mathcal{H}_1 \circ \mathcal{H}_3) \circ \mathcal{H}_2$, $\mathcal{H}_1 \circ \mathcal{H}_2 \ll \mathcal{H}_1$, and $0 \circ \mathcal{H}_1 \ll \{0\}$,

(P4) $0 \circ 0 = \{0\}$,

(P5) $0 \ll \hbar_1$,

(P6) $\hbar_1 \ll \hbar_1$,

(P7) $\mathcal{H}_1 \ll \mathcal{H}_1$,

(P8) $\mathcal{H}_1 \subseteq \mathcal{H}_2 \Rightarrow \mathcal{H}_1 \ll \mathcal{H}_2$,

(P9) $\{0\} = 0 \circ \hbar_1$,

(P10) $\hbar_1 \circ 0 = \{\hbar_1\}$,

(P11) $0 \circ \mathcal{H}_1 = \{0\}$,

(P12) $\hbar_1 \ll \{0\} \Rightarrow \hbar_1 = \{0\}$,

(P13) $\mathcal{H}_1 \circ \mathcal{H}_2 \ll \mathcal{H}_1$,

(P14) $\hbar_1 \in \hbar_1 \circ 0$,

(P15) $\hbar_1 \circ 0 \ll \{\hbar_2\} \Rightarrow \hbar_1 \ll \hbar_2$,

(P16) $\hbar_2 \ll \hbar_3 \Rightarrow \hbar_1 \circ \hbar_3 \ll \hbar_1 \circ \hbar_2$,

(P17) $\hbar_1 \circ \hbar_2 = \{0\} \Rightarrow (\hbar_1 \circ \hbar_3) \circ (\hbar_2 \circ \hbar_3) = \{0\}$ and $\hbar_1 \circ \hbar_3 \ll \hbar_2 \circ \hbar_3$,

(P18) $\mathcal{H}_1 \circ 0 = \{0\} \Rightarrow \mathcal{H}_1 = \{0\}$, for everyone $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$ in addition to every non-empty subsets $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$ of $\mathcal{H}$.

Definition 2. *[5] Let $\mathcal{I}$ be a non-empty subset of an HBCKA $\mathcal{H}$ and $0 \in \mathcal{I}$. Then $\mathcal{I}$ is known as*

- *an HBCKSA of $\mathcal{H}$ if $\hbar_1 \circ \hbar_2 \subseteq \mathcal{I}$, for all $\hbar_1, \hbar_2 \in \mathcal{I}$,*
- *a hyper BCK-ideal of $\mathcal{H}$ if for all $\hbar_1, \hbar_2 \in \mathcal{H}, \hbar_1 \circ \hbar_2 \ll \mathcal{I}$ and $\hbar_2 \in \mathcal{I} \Rightarrow \hbar_1 \in \mathcal{I}$,*
- *a weak hyper BCK-ideal of $\mathcal{H}$ if for all $\hbar_1, \hbar_2 \in \mathcal{H}, \hbar_1 \circ \hbar_2 \subseteq \mathcal{I}$ and $\hbar_2 \in \mathcal{I} \Rightarrow \hbar_1 \in \mathcal{I}$,*
- *a strong hyper BCK-ideal of $\mathcal{H}$ if for all $\hbar_1, \hbar_2 \in \mathcal{H}, (\hbar_1 \circ \hbar_2) \cap \mathcal{I} \neq \emptyset$ and $\hbar_2 \in \mathcal{I} \Rightarrow \hbar_1 \in \mathcal{I}$,*
- *reflexive if $\hbar_1 \circ \hbar_1 \subseteq \mathcal{I}$, for all $\hbar_1 \in \mathcal{H}$,*
- *S-reflexive if for all $\hbar_1, \hbar_2 \in \mathcal{H}$, $(\hbar_1 \circ \hbar_2) \cap \mathcal{I} \neq \emptyset \Rightarrow \hbar_1 \circ \hbar_2 \ll \mathcal{I}$,*
- *closed if for all $\hbar_1, \hbar_2 \in \mathcal{H}$, $\hbar_1 \ll \hbar_2$ and $\hbar_2 \in \mathcal{I} \Rightarrow \hbar_1 \in \mathcal{I}$.*

It is calm to see that each S-reflexive subset of $\mathcal{H}$ is reflexive.

Definition 3. *[6] Let $\mathcal{I}$ be a non-empty subset of an HBCKA $\mathcal{H}$ and $0 \in \mathcal{I}$. Then $\mathcal{I}$ is called a CHBCKI of*

(i) Type-1 if for all $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$, $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \subseteq \mathcal{I}$ and $\hbar_3 \in \mathcal{I} \Rightarrow \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) \subseteq \mathcal{I}$,

(ii) Type-2 if for all $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$, $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \subseteq \mathcal{I}$ and $\hbar_3 \in \mathcal{I} \Rightarrow \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) \ll \mathcal{I}$,

(iii) Type-3 if for all $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$, $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \ll \mathcal{I}$ and $\hbar_3 \in \mathcal{I} \Rightarrow \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) \subseteq \mathcal{I}$,

(iv) Type-4 if for all $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$, $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \ll \mathcal{I}$ and $\hbar_3 \in \mathcal{I} \Rightarrow \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) \ll \mathcal{I}$.

Theorem 1. *[28] Let $\mathcal{H}_1$ and $\mathcal{I}$ be non-empty subsets of an HBCKA $\mathcal{H}$. Then*

(i) if $\mathcal{I}$ is an HBCKI of $\mathcal{H}$ and $\mathcal{H}_1 \ll \mathcal{I}$, then $\mathcal{H}_1 \subseteq \mathcal{I}$.

(ii) if $\mathcal{I}$ is a reflexive HBCKI of $\mathcal{H}$, then $(\hbar_1 \circ \hbar_2) \cap \mathcal{I} \neq \emptyset \Rightarrow \hbar_1 \circ \hbar_2 \ll \mathcal{I}$, for all $\hbar_1, \hbar_2 \in \mathcal{H}$.

Definition 4. *[10] A bipolar fuzzy set (BFS) A of a set I is defined as*

$$A = \{(\hbar_1, \alpha_A^+(\hbar_1), \beta_A^-(\hbar_1)) \mid \hbar_1 \in \mathrm{I}\}, \tag{1}$$

where $\alpha_A^+ : \mathrm{I} \to [0, 1]$ and $\beta_A^- : \mathrm{I} \to [-1, 0]$ are mappings. The positive membership degree α_A^+ denotes the level of satisfaction that the element of I to the property associated with the bipolar fuzzy set A, while the negative membership degree β_A^- denotes the level of satisfaction that the element of I to some implicit counter property of A.

We shall use the symbol (α_A^+, β_A^-) to denote a bipolar fuzzy set A (see (1)).

Definition 5. *[20] A BFS (α_A^+, β_A^-) in $\mathcal{H}$ is called a bipolar fuzzy hyper BCK-ideal (BFHBCKI) of $\mathcal{H}$ if it satisfies:*

(i) $\hbar_1 \ll \hbar_2 \Rightarrow \alpha_A^+(\hbar_1) \geq \alpha_A^+(\hbar_2)$ *and* $\beta_A^-(\hbar_1) \leq \beta_A^-(\hbar_2)$,

(ii) $\alpha_A^+(\hbar_1) \geq \min\left\{\inf_{a\in\hbar_1\circ\hbar_2} \alpha_A^+(a), \alpha_A^+(\hbar_2)\right\}$,

(iii) $\beta_A^-(\hbar_1) \leq \max\left\{\sup_{a\in\hbar_1\circ\hbar_2} \beta_A^-(a), \beta_A^-(\hbar_2)\right\}$, *for all* $\hbar_1, \hbar_2 \in \mathcal{H}$.

Definition 6. *[20] A BFS* (α_A^+, β_A^-) *in* $\mathcal{H}$ *is known as a bipolar fuzzy strong hyper BCK-ideal (BFSHBCKI) of* $\mathcal{H}$ *if it satisfies:*

(i) $\inf_{a\in\hbar_1\circ\hbar_1} \alpha_A^+(a) \geq \alpha_A^+(\hbar_1) \geq \min\left\{\inf_{b\in\hbar_1\circ\hbar_2} \alpha_A^+(b), \alpha_A^+(\hbar_2)\right\}$,

(ii) $\sup_{c\in\hbar_1\circ\hbar_1} \beta_A^-(c) \leq \beta_A^-(\hbar_1) \leq \max\left\{\sup_{d\in\hbar_1\circ\hbar_2} \beta_A^-(d), \beta_A^-(\hbar_2)\right\}$, *for all* $\hbar_1, \hbar_2 \in \mathcal{H}$.

Definition 7. *[20] A BFS* (α_A^+, β_A^-) *in* $\mathcal{H}$ *is known as a bipolar fuzzy s-weak hyper BCK-ideal (BFsWHBCKI) of* $\mathcal{H}$ *if it satisfies:*

(i) $\alpha_A^+(0) \geq \alpha_A^+(\hbar_1)$ *and* $\beta_A^-(0) \leq \beta_A^-(\hbar_1)$, *for all* $\hbar_1 \in \mathcal{H}$,

(ii) for every $\hbar_1, \hbar_2 \in \mathcal{H}$, *there exist* $a, b \in \hbar_1 \circ \hbar_2$ *such that* $\alpha_A^+(\hbar_1) \geq \min\{\alpha_A^+(a), \alpha_A^+(\hbar_2)\}$ *and* $\beta_A^-(\hbar_1) \leq \max\{\beta_A^-(b), \beta_A^-(\hbar_2)\}$.

Definition 8. *[20] A BFS* (α_A^+, β_A^-) *in* $\mathcal{H}$ *is known as a BF-weak HBCKI of* $\mathcal{H}$ *if it satisfies:*

(i) $\alpha_A^+(0) \geq \alpha_A^+(\hbar_1) \geq \min\left\{\inf_{a\in\hbar_1\circ\hbar_2} \alpha_A^+(a), \alpha_A^+(\hbar_2)\right\}$,

(ii) $\beta_A^-(0) \leq \beta_A^-(\hbar_1) \leq \max\left\{\sup_{a\in\hbar_1\circ\hbar_2} \beta_A^-(a), \beta_A^-(\hbar_2)\right\}$, *for all* $\hbar_1, \hbar_2 \in \mathcal{H}$.

Definition 9. *[20] A BFS* (α_A^+, β_A^-) *in* $\mathcal{H}$ *is known as a BFHBCK-subalgebra of* $\mathcal{H}$ *if it satisfies:*

(i) $\inf_{a\in\hbar_1\circ\hbar_2} \alpha_A^+(a) \geq \min\{\alpha_A^+(\hbar_1), \alpha_A^+(\hbar_2)\}$

(ii) $\sup_{a\in\hbar_1\circ\hbar_2} \beta_A^-(a) \leq \max\{\beta_A^-(\hbar_1), \beta_A^-(\hbar_2)\}$, *for all* $\hbar_1, \hbar_2 \in \mathcal{H}$.

Definition 10. *A BFS* (α_A^+, β_A^-) *in* $\mathcal{H}$ *is said to satisfy sup-inf property if for any subset* $\mathcal{H}_1$ *of* $\mathcal{H}$, *there exist* $\hbar_{10}, \hbar_{20} \in \mathcal{H}_1$ *such that* $\alpha_A^+(\hbar_{10}) = \sup_{\hbar_1\in\mathcal{H}_1} \alpha_A^+(\hbar_1)$ *and* $\beta_A^-(\hbar_{20}) = \inf_{\hbar_2\in\mathcal{H}_1} \beta_A^-(\hbar_2)$.

3. Bipolar fuzzy commutative hyper BCK-ideals

In this section, we introduce and investigate a new class of ideals in the framework of hyper BCK-algebras, namely bipolar fuzzy commutative hyper BCK-ideals. This concept is formulated by combining the structural flexibility of hyperoperations with the dual-valued semantics of bipolar fuzzy sets. Our motivation stems from the growing demand to model algebraic uncertainty in a way that simultaneously captures both supportive and opposing evidence — a task well-suited to the bipolar fuzzy paradigm.

Building upon the foundational definitions outlined in Section 2, we first define bipolar fuzzy commutative hyper BCK-ideals and examine their basic properties. Special attention is given to the interplay between commutativity and bipolarity, which provides a nuanced perspective on membership under hyperoperations. Several illustrative examples are provided to clarify the behavior of such ideals in various algebraic settings.

We also explore necessary and sufficient conditions under which these ideals exhibit structural regularity, as well as their relationships to existing classes of bipolar fuzzy and hyper BCK-ideals. These results not only generalize previous work on commutative ideals and bipolar fuzzy structures, but also pave the way for further developments in multi-valued algebraic logic.

Definition 11. *Let (α_A^+, β_A^-) be a BFS in $\mathcal{H}$ with $\alpha_A^+(0) \geq \alpha_A^+(\hbar_1)$ and $\beta_A^-(0) \leq \beta_A^-(\hbar_1)$, for all $\hbar_1 \in \mathcal{H}$. Then (α_A^+, β_A^-) is said to be a bipolar fuzzy commutative hyper BCK-ideal (BF-CHBCKI) of*

(i) Type-1 if for all $\mathfrak{t} \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$,

$$\alpha_A^+(\mathfrak{t}) \geq \min \left\{ \inf_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a) \ , \alpha_A^+(\hbar_3) \right\},$$

$$\beta_A^-(\mathfrak{t}) \leq \max \left\{ \sup_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b) \ , \beta_A^-(\hbar_3) \right\}$$

(ii) Type-2 if for all $\mathfrak{t} \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$,

$$\alpha_A^+(\mathfrak{t}) \geq \min \left\{ \inf_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a) \ , \alpha_A^+(\hbar_3) \right\},$$

$$\beta_A^-(\mathfrak{t}) \leq \max \left\{ \sup_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b) \ , \beta_A^-(\hbar_3) \right\}$$

(iii) Type-3 if for all $\mathfrak{t} \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$,

$$\alpha_A^+(\mathfrak{t}) \geq \min \left\{ \sup_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a) \ , \alpha_A^+(\hbar_3) \right\},$$

$$\beta_A^-(\mathfrak{t}) \leq \max \left\{ \inf_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b) \ , \beta_A^-(\hbar_3) \right\}$$

(iv) Type-4 if for all $\mathfrak{t} \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$,

$$\alpha_A^+(\mathfrak{t}) \geq \min \left\{ \sup_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a) \, , \alpha_A^+(\hbar_3) \right\},$$

$$\beta_A^-(\mathfrak{t}) \leq \max \left\{ \inf_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b) \, , \beta_A^-(\hbar_3) \right\},$$

for all $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$.

Example 1. *Let* $\mathcal{H} = \{0, \hbar_1, \hbar_2\}$. *Consider the following Cayley table:*

$\circ$	0	$\hbar_1$	$\hbar_2$
0	$\{0\}$	$\{0\}$	$\{0\}$
$\hbar_1$	$\{\hbar_1\}$	$\{0, \hbar_1\}$	$\{0, \hbar_1\}$
$\hbar_2$	$\{\hbar_2\}$	$\{\hbar_1, \hbar_2\}$	$\{0, \hbar_1, \hbar_2\}$

Then $(\mathcal{H}, \circ)$ *is an HBCKA. We define a BFS* (α_A^+, β_A^-) *in* $\mathcal{H}$ *as follows:*

$$\alpha_A^+(0) = \alpha_A^+(\hbar_2) = 1, \ \alpha_A^+(\hbar_1) = 0.5,$$

$$\beta_A^-(0) = \beta_A^-(\hbar_2) = -0.6, \ \beta_A^-(\hbar_1) = -0.2.$$

Then (α_A^+, β_A^-) *is a BF-CHBCKI of Type-1 and, consequently, of Type-2.*

Example 2. *Let* $\mathcal{H} = \{0, \hbar_1, \hbar_2\}$. *Consider the following Cayley table:*

$\circ$	0	$\hbar_1$	$\hbar_2$
0	$\{0\}$	$\{0\}$	$\{0\}$
$\hbar_1$	$\{\hbar_1\}$	$\{0\}$	$\{\hbar_1\}$
$\hbar_2$	$\{\hbar_2\}$	$\{\hbar_2\}$	$\{0, \hbar_2\}$

Then $(\mathcal{H}, \circ)$ *is an HBCKA. We define a BFS* (α_A^+, β_A^-) *in* $\mathcal{H}$ *as follows:*

$$\alpha_A^+(0) = 0.9, \ \alpha_A^+(\hbar_1) = 0.6, \ \alpha_A^+(\hbar_2) = 0.3,$$

$$\beta_A^-(0) = -0.29, \ \beta_A^-(\hbar_1) = -0.23, \ \beta_A^-(\hbar_2) = -0.13.$$

Then (α_A^+, β_A^-) *is a BF-CHBCKI of Type-3 and, consequently, of Type-4.*

Theorem 2. *Let* (α_A^+, β_A^-) *be a BFS in* $\mathcal{H}$. *Then the following statements are valid.*

(i) If (α_A^+, β_A^-) *is a BF-CHBCKI of Type-3, then it is a BF-CHBCKI of Type-1 and Type-4.*

(ii) If (α_A^+, β_A^-) *is a BF-CHBCKI of Type-4 (or) 1, then it is a BF-CHBCKI of Type-2.*

Proof. The proof is straightforward.

The converse of Theorem 2 is not necessarily true in general. This fact is illustrated in the Example 3.

Example 3. *Let $\mathcal{H} = \{0, \hbar_1, \hbar_2, \hbar_3\}$. Consider the following Cayley table:*

$\circ$	0	$\hbar_1$	$\hbar_2$	$\hbar_3$
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\hbar_1$	$\{\hbar_1\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\hbar_2$	$\{\hbar_2\}$	$\{\hbar_2\}$	$\{0\}$	$\{0\}$
$\hbar_3$	$\{\hbar_3\}$	$\{\hbar_3\}$	$\{\hbar_2, \hbar_3\}$	$\{0, \hbar_2, \hbar_3\}$

Then $(\mathcal{H}, \circ)$ is an HBCKA. We define a BFS (α_A^+, β_A^-) in $\mathcal{H}$ as follows:

$$\alpha_A^+(0) = \alpha_A^+(\hbar_1) = 1,\ \alpha_A^+(\hbar_2) = 0.5,\ \alpha_A^+(\hbar_3) = 0.45,$$

$$\beta_A^-(0) = \beta_A^-(\hbar_1) = -0.7,\ \beta_A^-(\hbar_2) = -0.3,\ \beta_A^-(\hbar_3) = -0.1.$$

Then (α_A^+, β_A^-) is a BF-CHBCKI of Type-1. However, it does not follow the conditions for Type-3, as $\hbar_3 \in \hbar_3 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_3)) = \{\hbar_2, \hbar_3\}$,

$$\alpha_A^+(\hbar_3) = 0.45 < 0.6 = \alpha_A^+(\hbar_2) = \min \left\{ \sup_{a \in (\hbar_3 \circ \hbar_2) \circ 0} \alpha_A^+(a), \alpha_A^+(0) \right\},$$

$$\beta_A^-(\hbar_3) = -0.1 > -0.3 = \beta_A^-(\hbar_2) = \max \left\{ \inf_{b \in (\hbar_3 \circ \hbar_2) \circ 0} \beta_A^-(b), \beta_A^-(0) \right\}.$$

Theorem 3. *If (α_A^+, β_A^-) is a BF-CHBCKI of Type-1 of $\mathcal{H}$, then it is a BF-weak HBCKI.*

Proof. The proof is straightforward.

The converse of Theorem 3 is not necessarily true in general. This fact is illustrated in Example 4.

Example 4. *Consider the Cayley table given in Example 3.2. Then $(\mathcal{H}, \circ)$ is an HBCKA. We define a BFS (α_A^+, β_A^-) in $\mathcal{H}$ as follows:*

$$\alpha_A^+(0) = 0.8,\ \alpha_A^+(\hbar_1) = 0.4,\ \alpha_A^+(\hbar_2) = 0.25,$$

$$\beta_A^-(0) = -0.7,\ \beta_A^-(\hbar_1) = -0.5,\ \beta_A^-(\hbar_2) = 0.$$

Then (α_A^+, β_A^-) is a BF-weak HBCKI of $\mathcal{H}$. However, it does not follow the conditions for BF-CHBCKI of Type-1, because $\hbar_2 \in \hbar_2 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) = \{0, \hbar_1\}$,

$$\alpha_A^+(\hbar_1) = 0.35 < 0.8 = \alpha_A^+(0) = \min \left\{ \inf_{a \in (\hbar_1 \circ \hbar_2) \circ 0} \alpha_A^+(a), \alpha_A^+(0) \right\},$$

$$\beta_A^-(\hbar_1) = -0.5 > -0.7 = \beta_A^-(0) = \max \left\{ \sup_{b \in (\hbar_1 \circ \hbar_2) \circ 0} \beta_A^-(b), \beta_A^-(0) \right\}.$$

Theorem 4. *If (α_A^+, β_A^-) is a BF-CHBCKI of Type-3 of $\mathcal{H}$, then it is a BFSHBCKI.*

Proof. Suppose (α_A^+, β_A^-) is a BF-CHBCKI of Type-3 of $\mathcal{H}$. Setting $\hbar_2 = 0$ in Definition 11 (iii), we obtain $\mathfrak{t} \in \hbar_1 \circ (0 \circ (0 \circ \hbar_1)) = \hbar_1$,

$$\begin{pmatrix} \alpha_A^+(\hbar_1) \geq \min\left\{\sup\limits_{a\in(\hbar_1\circ 0)\circ\hbar_3} \alpha_A^+(a), \alpha_A^+(\hbar_3)\right\} = \min\left\{\sup\limits_{a\in\hbar_1\circ\hbar_3} \alpha_A^+(a), \alpha_A^+(\hbar_3)\right\} \\ \beta_A^-(\hbar_1) \leq \max\left\{\inf\limits_{b\in(\hbar_1\circ 0)\circ\hbar_3} \beta_A^-(b), \beta_A^-(\hbar_3)\right\} = \max\left\{\inf\limits_{b\in\hbar_1\circ\hbar_3} \beta_A^-(b), \beta_A^-(\hbar_3)\right\} \end{pmatrix} \tag{2}$$

for all $\hbar_1, \hbar_3 \in \mathcal{H}$.

First, we show that for $\hbar_1, \hbar_2 \in \mathcal{H}$, if $\hbar_1 \ll \hbar_2$, then $\alpha_A^+(\hbar_1) \geq \alpha_A^+(\hbar_2)$ and $\beta_A^-(\hbar_1) \leq \beta_A^-(\hbar_2)$. To show this, let $\hbar_1, \hbar_2 \in \mathcal{H}$ be such that $\hbar_1 \ll \hbar_2$. Then $0 \in \hbar_1 \circ \hbar_2$, and by (2), we have

$$\begin{pmatrix} \alpha_A^+(\hbar_1) \geq \min\left\{\sup\limits_{a\in\hbar_1\circ\hbar_2} \alpha_A^+(a), \alpha_A^+(\hbar_2)\right\} = \min\{\alpha_A^+(0), \alpha_A^+(\hbar_2)\} = \alpha_A^+(\hbar_2) \\ \beta_A^-(\hbar_1) \leq \max\left\{\inf\limits_{b\in\hbar_1\circ\hbar_2} \beta_A^-(b), \beta_A^-(\hbar_2)\right\} = \max\{\beta_A^-(0), \beta_A^-(\hbar_2)\} = \beta_A^-(\hbar_2) \end{pmatrix} \tag{3}$$

Let $\hbar_1 \in \mathcal{H}$ and $a \in \hbar_1 \circ \hbar_1$. Since $\hbar_1 \circ \hbar_1 \ll \hbar_1$, we have $a \ll \hbar_1$, for all $a \in \hbar_1 \circ \hbar_1$. Hence, by (3), we deduce

$$\alpha_A^+(a) \geq \alpha_A^+(\hbar_1) \text{ and } \beta_A^-(a) \leq \beta_A^-(\hbar_1), \text{ for all } a \in \hbar_1 \circ \hbar_1.$$

Hence,

$$\inf_{a\in\hbar_1\circ\hbar_1} \alpha_A^+(a) \geq \alpha_A^+(\hbar_1) \text{ and } \sup_{c\in\hbar_1\circ\hbar_1} \beta_A^-(c) \leq \beta_A^-(\hbar_1). \tag{4}$$

From the combination of (2) and (4), we obtain

$$\inf_{a\in\hbar_1\circ\hbar_1} \alpha_A^+(a) \geq \alpha_A^+(\hbar_1) \geq \min\left\{\sup_{b\in\hbar_1\circ\hbar_2} \alpha_A^+(b), \alpha_A^+(\hbar_2)\right\},$$

$$\sup_{c\in\hbar_1\circ\hbar_1} \beta_A^-(c) \leq \beta_A^-(\hbar_1) \leq \max\left\{\inf_{d\in\hbar_1\circ\hbar_2} \beta_A^-(d), \beta_A^-(\hbar_2)\right\}.$$

Thus, (α_A^+, β_A^-) is a BFSHBCKI of $\mathcal{H}$.

The converse of Theorem 4 is not necessarily true in general. This fact is illustrated in Example 5.

Example 5. *Consider the Cayley table given in Example 2. Then $(\mathcal{H}, \circ)$ is an HBCKA. We define a BFS (α_A^+, β_A^-) in $\mathcal{H}$ as follows:*

$$\alpha_A^+(0) = 0.8, \ \alpha_A^+(\hbar_1) = 0.6, \ \alpha_A^+(\hbar_2) = 0.3,$$

$$\beta_A^-(0) = -0.23, \ \beta_A^-(\hbar_1) = -0.19, \ \beta_A^-(\hbar_2) = -0.13.$$

Then (α_A^+, β_A^-) is a BFSHBCKI of $\mathcal{H}$. However, it does not follow the conditions for BF-CHBCKI of Type-3, because $\hbar_2 \in \hbar_2 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_2)) = \{0, \hbar_2\}$,

$$\alpha_A^+(\hbar_2) = 0.3 < 0.8 = \alpha_A^+(0) = \min\left\{\inf_{a \in (\hbar_2 \circ \hbar_2) \circ 0} \alpha_A^+(a)\,, \alpha_A^+(0)\right\},$$

$$\beta_A^-(\hbar_2) = -0.13 > -0.23 = \beta_A^-(0) = \max\left\{\sup_{b \in (\hbar_2 \circ \hbar_2) \circ 0} \beta_A^-(b), \beta_A^-(0)\right\}.$$

Corollary 1. *Let (α_A^+, β_A^-) be a BF-CHBCKI of Type-3, and let $\hbar_1, \hbar_2 \in \mathcal{H}$. Then*

(i) $\hbar_1 \ll \hbar_2 \Rightarrow \alpha_A^+(\hbar_1) \geq \alpha_A^+(\hbar_2)$ *and* $\beta_A^-(\hbar_1) \leq \beta_A^-(\hbar_2)$*,*

(ii) $\alpha_A^+(\hbar_1) \geq \min\{\alpha_A^+(a), \alpha_A^+(\hbar_2)\}$*,*

(iii) $\beta_A^-(\hbar_1) \leq \max\{\beta_A^-(b), \beta_A^-(\hbar_2)\}$*, for all* $a, b \in \hbar_1 \circ \hbar_2$*.*

Corollary 2. *If (α_A^+, β_A^-) is a BF-CHBCKI of Type-3, then*

(i) $\alpha_A^+(\hbar_1) \geq \min\left\{\inf_{a \in \hbar_1 \circ \hbar_2} \alpha_A^+(a), \alpha_A^+(\hbar_2)\right\}$*,*

(ii) $\beta_A^-(\hbar_1) \leq \max\left\{\sup_{b \in \hbar_1 \circ \hbar_2} \beta_A^-(b), \beta_A^-(\hbar_2)\right\}$*, for all* $\hbar_1, \hbar_2 \in \mathcal{H}$*.*

Proposition 1. *Let (α_A^+, β_A^-) be a BF-CHBCKI of Type-3. If (α_A^+, β_A^-) satisfies the inf-sup property, then it is a BFsWHBCKI of $\mathcal{H}$.*

Proof. The proof is straightforward.

Definition 12. *Let (α_A^+, β_A^-) be a BFS in $\mathcal{H}$. Then (α_A^+, β_A^-) is closed if for all $\hbar_1, \hbar_2 \in \mathcal{H}$ such that $\hbar_1 \ll \hbar_2$, we have $\alpha_A^+(\hbar_1) \geq \alpha_A^+(\hbar_2)$ and $\beta_A^-(\hbar_1) \leq \beta_A^-(\hbar_2)$.*

Definition 13. *[16] For a BFS (α_A^+, β_A^-), the positive $\mathfrak{p}$-cut (where $\mathfrak{p} \in [0,1]$) and the negative $\mathfrak{n}$-cut (where $\mathfrak{n} \in [-1,0]$) are defined as follows:*

$$\mathcal{U}(\alpha_A^+; \mathfrak{p}) = \{\hbar_1 \in \mathcal{H} \mid \alpha_A^+(\hbar_1) \geq \mathfrak{p}\},$$

$$\mathcal{L}(\beta_A^-; \mathfrak{n}) = \{\hbar_1 \in \mathcal{H} \mid \beta_A^-(\hbar_1) \leq \mathfrak{n}\}.$$

Theorem 5. *Let (α_A^+, β_A^-) be a BFS in $\mathcal{H}$. Then the following statements are hold:*

(i) A BFS (α_A^+, β_A^-) is a BF-CHBCKI of Type-1 of $\mathcal{H}$ if and only if for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, the non-empty cut sets $\mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $\mathcal{L}(\beta_A^-; \mathfrak{n})$ are CHBCKIs Type-1 of $\mathcal{H}$.

(ii) If (α_A^+, β_A^-) is a BF-CHBCKI of Type-3 of $\mathcal{H}$, then for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, the non-empty cut sets $\mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $\mathcal{L}(\beta_A^-; \mathfrak{n})$ are CHBCKIs Type-3 of $\mathcal{H}$.

(iii) If (α_A^+, β_A^-) satisfies the sup-inf property and for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, the non-empty cut sets $\mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $\mathcal{L}(\beta_A^-; \mathfrak{n})$ are reflexive-CHBCKIs of Type-3 of $\mathcal{H}$, then (α_A^+, β_A^-) is a BF-CHBCKI of Type-3 of $\mathcal{H}$.

Proof. (i) Suppose (α_A^+, β_A^-) is a BF-CHBCKI of Type-1 of $\mathcal{H}$ and for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, $\mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $\mathcal{L}(\beta_A^-; \mathfrak{n})$ are non-empty. By definition, $0 \in \mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $0 \in \mathcal{L}(\beta_A^-; \mathfrak{n})$. Therefore, $0 \in \mathcal{U}(\alpha_A^+; \mathfrak{p}) \cap \mathcal{L}(\beta_A^-; \mathfrak{n})$. Let $\hbar_1, \hbar_2, \hbar_3$ be elements of $\mathcal{H}$ such that $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \subseteq \mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $\hbar_3 \in \mathcal{U}(\alpha_A^+; \mathfrak{p})$. Then, for all $a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3$, we have $a \in \mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $\hbar_3 \in \mathcal{U}(\alpha_A^+; \mathfrak{p})$. This implies $\alpha_A^+(a) \geq \mathfrak{p}$, for all $a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3$ and $\alpha_A^+(\hbar_3) \geq \mathfrak{p}$. Consequently, $\inf\limits_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a) \geq \mathfrak{p}$ and $\alpha_A^+(\hbar_3) \geq \mathfrak{p}$. Thus, for all $k \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$, we have

$$\alpha_A^+(k) \geq \min\left\{ \inf_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a), \alpha_A^+(\hbar_3) \right\} \geq \min\{\mathfrak{p}, \mathfrak{p}\} = \mathfrak{p}.$$

This implies $k \in \mathcal{U}(\alpha_A^+; \mathfrak{p})$, for all $k \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$. Therefore, $\hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) \subseteq \mathcal{U}(\alpha_A^+; \mathfrak{p})$.

Let $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \subseteq \mathcal{L}(\beta_A^-; \mathfrak{n})$ and $\hbar_3 \in \mathcal{L}(\beta_A^-; \mathfrak{n})$. Then, for all $b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3$, we have $b \in \mathcal{L}(\beta_A^-; \mathfrak{n})$ and $\hbar_3 \in \mathcal{L}(\beta_A^-; \mathfrak{n})$. This implies $\beta_A^-(b) \leq \mathfrak{n}$, for all $b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3$ and $\beta_A^-(\hbar_3) \leq \mathfrak{n}$. Consequently, $\sup\limits_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b) \leq \mathfrak{n}$ and $\beta_A^-(\hbar_3) \leq \mathfrak{n}$. Thus, for all $k \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$, we have

$$\beta_A^-(k) \leq \max\left\{ \sup_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b), \beta_A^-(\hbar_3) \right\} = \max\{\mathfrak{n}, \mathfrak{n}\} = \mathfrak{n}.$$

This implies $k \in \mathcal{L}(\beta_A^-; \mathfrak{n})$, for all $t \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$. Therefore, $\hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) \subseteq \mathcal{L}(\beta_A^N; \mathfrak{n})$. Thus, for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, the cut sets $\mathcal{U}(\alpha_A^-; \mathfrak{p})$ and $(\beta_A^-; \mathfrak{n})$ are CHBCKIs of Type-1 of $\mathcal{H}$.

Conversely, let us assume that for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, the cut sets $\mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $\mathcal{L}(\beta_A^-; \mathfrak{n})$ are CHBCKIs of Type-1 of $\mathcal{H}$. Let $\hbar_1, \hbar_2, \hbar_3$ be elements of $\mathcal{H}$. We define $\mathfrak{p} = \min\left\{ \inf\limits_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a), \alpha_A^+(\hbar_3) \right\}$. Then

$$\begin{aligned}
&\inf_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a) \geq \mathfrak{p} \text{ and } \alpha_A^+(\hbar_3) \geq \mathfrak{p}. \\
&\Rightarrow \alpha_A^+(a) \geq \mathfrak{p} \text{ for all } a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3 \text{ and } \alpha_A^+(\hbar_3) \geq \mathfrak{p} \\
&\Rightarrow a \in \mathcal{U}(\alpha_A^+; \mathfrak{p}) \text{ for all } a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3 \text{ and } \hbar_3 \in \mathcal{U}(\alpha_A^+; \mathfrak{p}) \\
&\Rightarrow (\hbar_1 \circ \hbar_2) \circ \hbar_3 \subseteq \mathcal{U}(\alpha_A^+; \mathfrak{p}) \text{ and } \hbar_3 \in \mathcal{U}(\alpha_A^+; \mathfrak{p}).
\end{aligned}$$

By hypothesis, we have $\hbar_1\circ(\hbar_2\circ(\hbar_2\circ\hbar_1)) \subseteq \mathcal{U}(\alpha_A^+;\mathfrak{p})$. Thus, for all $k_1 \in (\hbar_1\circ(\hbar_2\circ(\hbar_2\circ\hbar_1)),$ $\alpha_A^+(k_1) \geq \mathfrak{p} = \min\left\{\inf_{a\in(\hbar_1\circ\hbar_2)\circ\hbar_3}\alpha_A^+(a)\ ,\alpha_A^+(\hbar_3)\right\}$.

Define $\mathfrak{n} = \max\left\{\sup_{b\in(\hbar_1\circ\hbar_2)\circ\hbar_3}\beta_A^-(b)\ ,\beta_A^-(\hbar_3)\right\}$. Then

$$\sup_{b\in(\hbar_1\circ\hbar_2)\circ\hbar_3}\beta_A^-(b)\ \leq \mathfrak{n} \text{ and } \beta_A^-(\hbar_3) \leq \mathfrak{n}.$$
$$\Rightarrow \beta_A^-(b) \leq \mathfrak{n} \text{ for all } b \in (\hbar_1\circ\hbar_2)\circ\hbar_3 \text{ and } \beta_A^-(\hbar_3) \leq \mathfrak{n}$$
$$\Rightarrow b \in \mathcal{L}(\beta_A^-;\mathfrak{n}) \text{ for all } b \in (\hbar_1\circ\hbar_2)\circ\hbar_3 \text{ and } \hbar_3 \in \mathcal{L}(\beta_A^-;\mathfrak{n})$$
$$\Rightarrow (\hbar_1\circ\hbar_2)\circ\hbar_3 \subseteq \mathcal{L}(\beta_A^-;\mathfrak{n}) \text{ and } \hbar_3 \in \mathcal{L}(\beta_A^-;\mathfrak{n}).$$

By hypothesis, we have $\hbar_1\circ(\hbar_2\circ(\hbar_2\circ\hbar_1)) \subseteq \mathcal{L}(\beta_A^-;\mathfrak{n})$. Thus, for all $k_2 \in (\hbar_1\circ(\hbar_2\circ(\hbar_2\circ\hbar_1)),$ $\beta_A^-(k_2) \leq \mathfrak{n} = \max\left\{\sup_{b\in(\hbar_1\circ\hbar_2)\circ\hbar_3}\beta_A^-(b)\ ,\beta_A^-(\hbar_3)\right\}$. Consider the case where $\alpha_A^+(\hbar_1) = \mathfrak{p}$ and $\beta_A^-(\hbar_2) = \mathfrak{n}$ for some $\hbar_1, \hbar_2 \in \mathcal{H}$. Since $0 \in \mathcal{U}(\alpha_A^+;\mathfrak{p}) \cap \mathcal{L}(\beta_A^-;\mathfrak{n})$, we obtain $\alpha_A^+(0) \geq \mathfrak{p} = \alpha_A^+(\hbar_1)$ and $\beta_A^-(0) \leq \mathfrak{n} = \beta_A^-(\hbar_2)$, for all $\hbar_1, \hbar_2 \in \mathcal{H}$. Therefore, (α_A^+, β_A^-) is a BF-CHBCKI of Type-1 of $\mathcal{H}$.

(ii) Assume (α_A^+, β_A^-) is a BF-CHBCKI of Type-3 of $\mathcal{H}$, and for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, the cut sets $\mathcal{U}(\alpha_A^+;\mathfrak{p})$ and $\mathcal{L}(\beta_A^-;\mathfrak{n})$ are non-empty. By definition, we have $0 \in \mathcal{U}(\alpha_A^+;\mathfrak{p}) \cap \mathcal{L}(\beta_A^-;\mathfrak{n})$. Let $\hbar_1, \hbar_2, \hbar_3$ be elements of $\mathcal{H}$ such that $(\hbar_1\circ\hbar_2)\circ\hbar_3 \ll \mathcal{U}(\alpha_A^+;\mathfrak{p})$ and $\hbar_3 \in \mathcal{U}(\alpha_A^+;\mathfrak{p})$. Then, for every $a \in (\hbar_1\circ\hbar_2)\circ\hbar_3$, we can find $k \in \mathcal{U}(\alpha_A^+;\mathfrak{p})$ such that $a \ll k$. By Corollary 1, we obtain $\alpha_A^+(a) \geq \alpha_A^+(k) \geq \mathfrak{p}$. This implies $\alpha_A^+(a) \geq \mathfrak{p}$, for all $a \in (\hbar_1\circ\hbar_2)\circ\hbar_3$ and $\hbar_3 \in \mathcal{U}(\alpha_A^+;\mathfrak{p})$, Consequently, $\sup_{a\in(\hbar_1\circ\hbar_2)\circ\hbar_3}\alpha_A^+(a)\ \geq \mathfrak{p}$ and $\alpha_A^+(\hbar_3) \geq \mathfrak{p}$. Thus, by hypothesis, for all $k \in \hbar_1\circ(\hbar_2\circ(\hbar_2\circ\hbar_1)),$

$$\alpha_A^+(k) \geq \min\left\{\sup_{a\in(\hbar_1\circ\hbar_2)\circ\hbar_3}\alpha_A^+(a)\ ,\alpha_A^+(\hbar_3)\right\} \geq \min\{\mathfrak{p},\mathfrak{p}\} = \mathfrak{p}.$$

Therefore, it follows that $\hbar_1\circ(\hbar_2\circ(\hbar_2\circ\hbar_1)) \subseteq \mathcal{U}(\alpha_A^+;\mathfrak{p})$.

Let $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$ be such that $(\hbar_1\circ\hbar_2)\circ\hbar_3 \ll \mathcal{L}(\beta_A^-;\mathfrak{n})$ and $\hbar_3 \in \mathcal{L}(\beta_A^-;\mathfrak{n})$. Then, for every $b \in (\hbar_1\circ\hbar_2)\circ\hbar_3$, we can find $l \in \mathcal{L}(\beta_A^-;\mathfrak{n})$ such that $b \ll l$. By Corollary 1, we obtain $\beta_A^-(b) \leq \beta_A^-(l) \leq \mathfrak{n}$. This implies $\beta_A^-(b) \leq \mathfrak{n}$, for all $b \in (\hbar_1\circ\hbar_2)\circ\hbar_3$ and $\hbar_3 \in \mathcal{L}(\beta_A^-;\mathfrak{n})$. Consequently, $\inf_{b\in(\hbar_1\circ\hbar_2)\circ\hbar_3}\beta_A^-(b)\ \leq \mathfrak{n}$ and $\beta_A^-(\hbar_3) \leq \mathfrak{n}$. Thus, by hypothesis, for all $k \in \hbar_1\circ(\hbar_2\circ(\hbar_2\circ\hbar_1)),$

$$\beta_A^-(k) \leq \max\left\{\inf_{b\in(\hbar_1\circ\hbar_2)\circ\hbar_3}\beta_A^-(b)\ ,\beta_A^-(\hbar_3)\right\} \leq \max\{\mathfrak{n},\mathfrak{n}\} = \mathfrak{n}.$$

Therefore, it follows that $\hbar_1\circ(\hbar_2\circ(\hbar_2\circ\hbar_1)) \subseteq \mathcal{L}(\beta_A^-;\mathfrak{n})$. Therefore, we conclude that for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, the cut sets $\mathcal{U}(\alpha_A^+;\mathfrak{p})$ and $\mathcal{L}(\beta_A^-;\mathfrak{n})$ are CHBCKIs of Type-3 of $\mathcal{H}$.

(iii) Assume that $\mathcal{H}$ satisfies the sup-inf property, and for all $(\mathfrak{p}, \mathfrak{n}) \in [0,1] \times [-1,0]$, the cut sets $\mathcal{U}(\alpha_A^+; \mathfrak{p})$ and $\mathcal{L}(\beta_A^-; \mathfrak{n})$ are reflexive CHBCKIs of Type-3 of $\mathcal{H}$. Let $\hbar_1, \hbar_2, \hbar_3 \in \mathcal{H}$. Define $\mathfrak{p} = \min\left\{\sup\limits_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a), \alpha_A^+(\hbar_3)\right\}$. Then $\sup\limits_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a) \geq \mathfrak{p}$ and $\alpha_A^+(\hbar_3) \geq \mathfrak{p}$. Since α_A^+ satisfies the sup property, there exists $a_0 \in (\hbar_1 \circ \hbar_2) \circ \hbar_3$ such that $\alpha_A^+(a_0) = \sup\limits_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a) \geq \mathfrak{p}$. This implies $\alpha_A^+(a_0) \geq \mathfrak{p}$. That is, the set $((\hbar_1 \circ \hbar_2) \circ \hbar_3) \cap \mathcal{U}(\alpha_A^+; \mathfrak{p})$ is non-empty. We know that every CHBCKI of Type-3 is an HBCKI of $\mathcal{H}$ (see Theorem 4.4 in [6]). Therefore, $\mathcal{U}(\alpha_A^+; \mathfrak{p})$ is a reflexive-HBCKI of $\mathcal{H}$ and $((\hbar_1 \circ \hbar_2) \circ \hbar_3) \cap \mathcal{U}(\alpha_A^+; \mathfrak{p}) \neq \emptyset$. By Theorem 1 (ii), it follows that $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \ll \mathcal{U}(\alpha_A^+; \mathfrak{p})$. Since $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \ll \mathcal{U}(\alpha_A^+; \mathfrak{p})$, $\hbar_3 \in \mathcal{U}(\alpha_A^+; \mathfrak{p})$, and $\mathcal{U}(\alpha_A^+; \mathfrak{p})$ is a CHBCKI of Type-3 of $\mathcal{H}$, it follows that $\hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) \subseteq \mathcal{U}(\alpha_A^+; \mathfrak{p})$. This implies that for all $k \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$, we have $k \in \mathcal{U}(\alpha_A^+; \mathfrak{p})$. Thus, $\alpha_A^+(a) \geq \mathfrak{p}$, where $\mathfrak{p} = \min\left\{\sup\limits_{a \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \alpha_A^+(a), \alpha_A^+(\hbar_3)\right\}$.

Define $\mathfrak{n} = \max\left\{\inf\limits_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b), \beta_A^-(\hbar_3)\right\}$. Then, $\inf\limits_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b) \leq \mathfrak{n}$ and $\beta_A^-(\hbar_3) \leq \mathfrak{n}$. Since β_A^- satisfies the inf property, there exists $b_0 \in (\hbar_1 \circ \hbar_2) \circ \hbar_3$ such that $\beta_A^-(b_0) = \inf\limits_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b) \leq \mathfrak{n}$. This implies $\beta_A^-(b_0) \leq \mathfrak{n}$. That is, the set $((\hbar_1 \circ \hbar_2) \circ \hbar_3) \cap \mathcal{L}(\beta_A^-; \mathfrak{n})$ is non-empty. We know that every CHBCKI of Type-3 is an HBCKI of $\mathcal{H}$ (see Theorem 4.4 in [6]). Therefore, $\mathcal{L}(\beta_A^-; \mathfrak{n})$ is a reflexive-HBCKI of $\mathcal{H}$ and $((\hbar_1 \circ \hbar_2) \circ \hbar_3) \cap \mathcal{L}(\beta_A^-; \mathfrak{n}) \neq \emptyset$. By Theorem 1 (ii), it follows that $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \ll \mathcal{L}(\beta_A^-; \mathfrak{n})$. Since $(\hbar_1 \circ \hbar_2) \circ \hbar_3 \ll \mathcal{L}(\beta_A^-; \mathfrak{n})$, $\hbar_3 \in \mathcal{L}(\beta_A^-; \mathfrak{n})$, and $\mathcal{L}(\beta_A^-; \mathfrak{n})$ is a CHBCKI of Type-3 of $\mathcal{H}$, it follows that $\hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1)) \subseteq \mathcal{L}(\beta_A^-; \mathfrak{n})$. This implies that for all $k \in \hbar_1 \circ (\hbar_2 \circ (\hbar_2 \circ \hbar_1))$, we have $k \in \mathcal{L}(\beta_A^-; \mathfrak{n})$. Thus, $\beta_A^-(t_2) \leq \mathfrak{n}$, where $\mathfrak{n} = \max\left\{\inf\limits_{b \in (\hbar_1 \circ \hbar_2) \circ \hbar_3} \beta_A^-(b), \beta_A^-(\hbar_3)\right\}$. Therefore, we conclude that (α_A^+, β_A^-) is a BF-CHBCKI of Type-3 of $\mathcal{H}$.

4. Conclusion

In this paper, we propose a comprehensive framework for the bipolar fuzzification of commutative hyper BCK-ideals (CHBCKIs) within the context of hyper BCK-algebras (HBCKAs). By formalizing bipolar fuzzy commutative hyper BCK-ideals (BF-CHBCKIs), we introduced new definitions, classifications, and theorems that elucidate their structural properties and their interrelations with various forms of hyper BCK-ideals, including reflexive, strong, and weak types. The use of bipolar fuzzy subsets as analytical tools not only enhances the flexibility of algebraic reasoning under uncertainty but also strengthens the foundation for advanced research in fuzzy hyperstructure theory. Moreover, the findings of this study contribute to the development of research-oriented mathematical pedagogy, offering students in inquiry-driven learning environments access to complex yet structured models of logic and abstraction. In alignment with Sustainable Development

Goal 4 (SDG-4), this work supports the creation of equitable and inclusive educational pathways by enabling learners to engage with high-level mathematical constructs early in their academic journey. Future research may expand on these ideas by exploring analogous ideals in other algebraic systems, further enriching both theoretical knowledge and educational practice.

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