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A Linear Subdivision Scheme with Sixth-Order Precision and C^5 Smoothness

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Abstract. Subdivision schemes are a crucial component of geometric modeling, widely applied in curve and surface design. Classical schemes such as the six-point interpolatory scheme and the quintic B-spline are well known for their efficiency and smoothness. Yet, both have inherent drawbacks, particularly with respect to approximation order and smoothness. This study introduces a novel subdivision scheme that blends the strengths of the six-point interpolatory and the quintic B-spline schemes. The proposed scheme achieves sixth-order approximation and C^5 smoothness while preserving the support size of the six-point scheme. A key feature of the scheme is the introduction of a tension parameter, which provides flexibility to control the trade-off between smoothness and approximation order. Moreover, the scheme preserves essential properties such as monotonicity and convexity under mild conditions. Despite the other higher-order shape-preserving schemes, which are non-linear and computationally complex, this scheme is linear and stationary. Experimental results confirm that the proposed scheme consistently produces accurate and visually elegant curves, outperforming existing schemes in both approximation order and smoothness.

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1. Introduction

Subdivision schemes (SSs) are iterative processes that refine an initial control polygon or mesh to generate smooth curves or surfaces, using predefined refinement rules. These schemes have become fundamental tools in computer graphics due to their extensive applications in areas such as medical imaging, engineering, visual computing, and image analysis. The two major types of SSs are approximating and interpolating schemes. Interpolatory schemes ensure that the resulting limit curve passes through the initial control points, while approximating schemes produce curves that may not necessarily interpolate the original data points.

Mathematically, a stationary univariate subdivision scheme can be described using a refinement rule that updates the sequence of points at each level. Let $g^n = \{g_m^n : m \in \mathbb{Z}\}$ represent the set of points at level n. The new refined value at position i is given by:

$$(S_p g^n)_i = \sum_{m\in\mathbb{Z}} c_{i-2m} g_m^n,$$

where $g^n = \{g_m^n : m \in \mathbb{Z}\}$ represents the sequence of data at refinement level n, S_p denotes the subdivision operator, which can be represented as a bi-infinite matric
$M_c = (c_{i-2m} : i, m \in \mathbb{Z})$ defined by the mask $c = \{c_m : m \in \mathbb{Z}\}$. The mask $'c'$ is usually compactly supported, which guarantees that the refinement process remains localized. A SS is deemed uniformly convergent when the sequence of refined data g^n converges uniformly to a continuous limiting function [1]. The convergence of the limit function is influenced by the mask $'c'$ and is typically evaluated using the Laurent polynomial [2], denoted as $C(v) = \sum_{i\in\mathbb{Z}} c_i v^i,\ i \in \mathbb{C}$ and joint spectral radius [3] which together help determine the smoothness and stability of the scheme.

The design of novel SSs remains an important area of research in curve and surface modeling. De Rham [4] and Chaikin [5] were renowned as the founding fathers of SSs. They pioneered corner-cutting schemes, which yielded C^1-limit curves. A significant advancement came with the four-point binary interpolatory scheme developed by Dyn et al. [6], which also achieved C^1 continuity. Later, Deslauriers and Dubuc introduced a widely used interpolatory scheme known as the DD scheme [7]. More recently, researchers explored the idea of shape-controlling parameters. For instance, Mustafa et al. [8] introduced a framework for constructing binary approximating schemes, allowing better control over curve behavior. Zheng et al. [9] extended this idea by designing multi-parameter schemes with high levels of continuity. Similarly, Tan et al. [10] proposed a four-point scheme that incorporates two shape parameters, achieving C^3 smoothness while preserving the shape of the original data.

SSs are valued for key features such as smoothness and shape preservation. For deeper discussion of these properties, readers are referred to works such as [11–19].

The quintic B-spline (Q_B-scheme) [20] and DD six-point interpolatory schemes (S_D-scheme) [21] are stationary SSs. Based on quintic polynomials, these schemes have distinct advantages and limitations. The quintic B-spline maintains high-order continuity while preserving the monotonicity and convexity of the initial sequence, having minimal support of the fundamental limit curve [3]. The scheme achieves an order of approximation limited to ’two’ because it can only reproduce polynomials up to linear accuracy. However, the six-point interpolatory scheme only provides C^2-smoothness and, in general, lacks the shape-preserving property, even though it provides sixth-order accuracy. Moreover, a significant drawback of the S_D-scheme is its interpolatory nature. Artifacts like self-intersections and inflection points may arise if the edges of the control polygon differ substantially in length. Furthermore, the refinement mask length, smoothness, and accuracy are essential criteria for selecting a subdivision algorithm in practical applications. Multiple attempts are made to increase the smoothness and accuracy by using non-uniform, non-linear, and non-stationary SSs [22–24]. However, it increases the complexity and demands for a preliminary smoothing process. To improve the smoothness of the interpolatory scheme, Choi et. al.[25] introduced a family of quasi-interpolatory schemes by enhancing the support length of the mask. Recently, Yang et al. [26] have presented a family of linear, stationary SS with C^2-smoothness and fourth-order accuracy. However, this scheme in [26] achieves C^2-smoothness; it still relies on a similar construction to the cubic B-spline, which may limit its potential for further improving approximation accuracy.

Addressing the limitations mentioned above, this work introduces a novel class of stationary SSs, combining the benefits of Q_B and S_D-schemes. The presented scheme features a parameter μ_0, bridging the gap between these two schemes and offering a versatile range of subdivision rules to balance smoothness, precision, and shape-preservation. The main advancements of our work are: First, by compromising on the interpolation property, our scheme achieves enhanced smoothness of C^5, having the equal support width $[-5, 5]$ as the S_D-scheme. Second, despite only reproducing linear polynomials, our scheme can attain sixth-order precision using the function that belongs to the Sobolev Space, such as $e \in W^6(\mathbb{R})$. Next, we prove that our proposed scheme maintains monotonicity and convexity under certain mild conditions. A key advantage of our scheme is its linearity and stationarity, which distinguishes it from other shape-preserving schemes that are often non-linear, non-uniform, and computationally complex. To demonstrate the effectiveness of the proposed SS, several numerical examples are presented.

The design of this paper is as follows. Section 2 discusses fundamental concepts and definitions. A novel class of binary 6-point SS (S_P-scheme) is constructed in Section 3. The convergence and accuracy of the S_P-scheme are analyzed in Section 4. The monotonicity and convexity preservation of the S_P-scheme is discussed in Section 5. Lastly, in Section 6, we present various experimental results.

2. Fundamental concepts and definitions

This section provides fundamental concepts and essential background on stationary SSs.

Definition 1. *A stationary binary SS is a process that iteratively generates a sequence of control point grids. The control points at level* $n + 1$ *are computed from an initial set as* *follows:*

$$g_j^{n+1} = \sum_{i\in\mathbb{Z}} c_{j-2i} g_i^n, j \in \mathbb{Z},$$

where different masks are used for even and odd-indexed points. In contrast, non-stationary SS apply recursive refinements using masks that may vary at different levels. The subdivision operator is assumed to be bounded and is characterized by the sequence $c = \{c_i : i \in \mathbb{Z}\}$, *referred to as the subdivision mask, which is generally assumed to have finite support.*

We define $\mathbb{N}_0$ as the set of non-negative integers. The notation $\prod_i$ represents the space of polynomials with a degree up to i. For a sequence $g \in L^\infty(\mathbb{Z})$, the following difference operators are defined:

$$(D_n g^n)_i = 2^n (g_i^n - g_{i-1}^n), \qquad (D_n^2 g^n)_i = 2^{2n}(g_{i-1}^n - 2g_i^n + g_{i+1}^n),\ i \in \mathbb{Z}.$$

In this study, we consider a sequence $g^n = \{g_i^n : i \in \mathbb{Z}\}$ at level $n \in \mathbb{N}_0$ associated with grid points $2^{-n}\mathbb{Z}$.

Definition 2. *A binary SS* S_P *applied to a given initial sequence* $g_0 \in \mathbb{C}^\infty(\mathbb{Z})$ *is called uniformly convergent when it generates a continuous limit curve* $S_P^\infty g_0 \in \mathbb{C}(\mathbb{R})$ *such that for at least one nonzero initial data, the sequence of linear interpolants* G^n, *at grid points* $2^{-n}\mathbb{Z}$, *satisfies:*

$$lim_{n\to\infty} ||G^n - S_P^\infty g_0||_{L^\infty(N)} = 0,$$

where $N \subset \mathbb{R}$. *If* $S_P^\infty g_0$ *has continuous derivatives up to order e, the* $\{S_P\}$ *scheme is termed* C^e*-convergent.*
Specifically, as the provided data set $\delta = \{\delta_{0,i} : i \in \mathbb{Z}\}$, *at zeroth level with kronecker delta the fundamental limit function of* S_P*-scheme is given as,*

$$\Omega = S_P^\infty \delta.$$

Since the subdivision mask c has finite support, the basic limit function Ω *also has finite support. By the linearity of the refinement rule, it follows that*

$$S_P^\infty g_0 = \sum_{i\in\mathbb{Z}} g_0^n \Omega(\cdot - i).$$

A well-established condition for the convergence of an SS S_P *with mask* $c = \{c_i : i \in \mathbb{Z}\}$ *is that its even and odd values satisfy the partition of unity property:*

$$\sum_{i\in\mathbb{Z}} c_{j-2i} = 1, \qquad j = 0, 1.$$

Indeed, for an SS defined by mask c, its essential properties, including convergence, regularity, and polynomial reproduction (or generation), are encapsulated in its symbol, defined by the v-transform:

$$C(v) = \sum_{i\in\mathbb{Z}} c_i v^i, \qquad v \in \mathbb{C}\backslash\{0\}.$$

Since c is finitely supported, the symbol $C(v)$ forms a Laurent polynomial. The norm of the associated SS S_P is given by:

$$||S_P||_\infty = max\{\sum_{i\in\mathbb{Z}} |c_{2i}|, \sum_{i\in\mathbb{Z}} |c_{1+2i}|\}.$$

Additionally, the Laurent polynomial corresponding to the m-iterated rule S_P^m is:

$$C^{[m]}(v) = \prod_{l=0}^{m-1} C(v^{2^l}) = \sum_{i\in\mathbb{Z}} c_i^{[m]} v^i,$$

Where $S_P^0 = S_P$ and the norm of S_P^m is given by:

$$||S_P^m||_\infty = max\{\sum_{i\in\mathbb{Z}} |c_{j+2^m i}^{[m]}| : \ j = 0, 1, ..., 2^m - 1\}.$$

A crucial property of an SS is its order of approximation, which is related to its ability to reproduce polynomials.

Definition 3. *Let $Q^0 = \{q(m) : m \in \mathbb{Z}\}$ where $q \in \prod_e$ with $e \in \mathbb{N}_0$. We say that a Stationary SS S_P reproduces polynomials in $\prod_e$ if S_P is convergent and $p = S_P^\infty Q^0$. Also, the SS S_P is said to be $\prod_e$-generating $S_P^\infty Q^0 \in \prod_e$ [26].*

Polynomial reproduction is a valuable property, as it guarantees that a convergent SS S_P with a compactly supported mask reproduces polynomials of degree e, ensuring order of approximation $e+1$ [27]. However, generating polynomials of degree e doesn't necessarily imply the same level of approximation accuracy $e+1$. For example, the Q_B-scheme of degree e generates polynomials in $\prod_e$, but only reproduces linear polynomials ($\prod_1$), yielding an order of approximation 2. According to Levin's work [28], an SS that generates polynomials of degree e can be modified to reproduce polynomials in $\prod_e$ by applying a specific operator to the initial sequence. Nevertheless, it is noted that the resulting order of approximation is often not optimal. However, we introduce a new SS using a tension parameter, offering improved smoothness C^5 and sixth-order of approximation, while reproducing linear polynomials and generating polynomials in $\prod_5$.

3. The proposed subdivision scheme

The following section introduces a new six-point binary subdivision schemes. We also discuss the properties of polynomial generation.

3.1. Construction

We construct a novel class of stationary SSs that combining the advantages of the Q_B-scheme and the S_D-scheme via a tension parameter. Sequence of data is given by

$g^k = \{g_i^n : i \in \mathbb{Z}\}$ at level n, sequence of refined data $g^{n+1} = \{g_i^{n+1} : i \in \mathbb{Z}\}$ is computed as a linear combination of previous value of g^n in the following two rules:

$$\begin{cases} g_{2i}^{n+1} = \dfrac{1}{64}((\mu_0)g_{i-2}^n + (16\mu_0)g_{i-1}^n + (64-34\mu_0)g_i^n + (16\mu_0)g_{i+1}^n + (\mu_0)g_{i+2}^n), \\ g_{2i+1}^{n+1} = \dfrac{1}{256}((3-3\mu_0)g_{i-2}^n + (-25+49\mu_0)g_{i-1}^n + (150-46\mu_0)g_i^n \\ \qquad + (150-46\mu_0)g_{i+1}^n + (-25+49\mu_0)g_{i+2}^n + (3-3\mu_0)g_{i+3}^n). \end{cases} \tag{1}$$

The construction of our subdivision mask relies crucially on the parameter μ_0. The choice of μ_0 significantly affects the regularity, order of approximation, and shape-preservation of the resulting scheme. Suppose that the sequence of initial data is sampled from a function on the grid $2^{-n_0}\mathbb{Z}$, we investigate the optimal selection of μ_0 as

$$\mu_0 = \tau 2^{-2n_0}, \qquad \tau \geq 0. \tag{2}$$

The key characteristics of our construction in (1) are as follows. First of all, the mask of S_D-scheme is $\{\frac{3}{256}, 0, \frac{-25}{256}, 0, \frac{75}{128}, 1, \frac{75}{128}, 0, \frac{-25}{256}, 0, \frac{3}{256}\}$, our S_P-scheme refinement rule can be interpreted as a linear modification of the S_D-scheme presented below:

$$S_P g^n = S_D g^n + 2^{-2n}\frac{\mu_0}{64}\mathbb{D}_2(g^n). \tag{3}$$

where linear operator is defined as $\mathbb{D}_2$

$$\begin{cases} \mathbb{D}_2(g^n)_{2i} = (D^2g^n)_{i-1} + 18(D^2g^n)_i + (D^2g^n)_{i+1}, \\ \mathbb{D}_2(g^n)_{2i+1} = \dfrac{-3(D^2g^n)_{i-1} + 43(D^2g^n)_i + 43(D^2g^n)_{i+1} - 3(D^2g^n)_{i+2}}{4}. \end{cases} \tag{4}$$

The new S_P-scheme enhances the S_D-scheme by integrating the discrete 2nd-derivative information of g^n. Additionally, the parameter μ_0 is chosen based on the initial data density, as specified in (2). This enables the new S_P-scheme to attain sixth-order of approximation, enhancing smoothness without expanding support length. Additionally, because it exactly reproduces all polynomials up to degree five, i.e., $\prod_5$, the scheme ensures the presence of the smoothing factor $(1+v)^6$ in its symbol. For further use, the explicit form of the scheme's symbol is provided in equation (5):

$$\begin{aligned} C(v) \;=\; & \frac{(1+v)^6}{2^7}\left\{\frac{1}{2v}(3-3\mu_0) + \frac{1}{v^2}(-9+11\mu_0) + \frac{1}{2v^3}(38-38\mu_0)\right. \\ & \left. + \frac{1}{v^4}(-9+11\mu_0) + \frac{1}{2v^5}(3-3\mu_0)\right\}. \end{aligned} \tag{5}$$

Remark 1. *The suggested scheme is a generalization of the S_D-scheme and the Q_B-scheme. If $\mu_0 = 0$, the scheme is the S_D-scheme. Furthermore, when $\mu_0 = 1$ it changes to the quintic B-spline (represented by Q_B-scheme) with the mask $\{\frac{1}{64}, \frac{3}{32}, \frac{1}{4}, \frac{13}{32}, \frac{15}{32}, \frac{13}{32}, \frac{1}{4}, \frac{3}{32}, \frac{1}{64}\}$. Then, the quintic B-spline Q_B may also be used to illustrate the refinement rule in (1) as follows:*

$$S_P g^n = Q_B g^n - 2^{-2n}\frac{(1-\mu_0)}{64}\mathbb{D}_2(g^n). \tag{6}$$

with the operator $\mathbb{D}_2$ in (4). This representation will help analyze our SS shape-preserving characteristics, as we shall see later.

3.2. Property of polynomial generation

When the corresponding fundamental limiting function Ω of S_P meets the Strang-Fix condition of order of approximation e, it is equivalent to the SS symbol having the smoothing parameter $(1+v)^e, e \in \mathbb{N}$. Up to degree e, it ensures that the suggested scheme produces polynomials. At a later stage, we shall discover that the S_P-scheme in (3) can achieve sixth-order approximation even though it reproduces polynomials in $\prod_1$.
For convenience, the notation used is as follows: for $q \in \prod_5$ and $n \in \mathbb{N}_0$,

$$q_i^n = q(i2^{-n}) \text{ and } q^{(4),n+1} = \mathbb{D}_2(q^n).$$

where given operator $\mathbb{D}_2$ is defined in (4).

Theorem 1. *Assume that S_P is the subdivision scheme in (1) with parameter μ_0. Each of the polynomials q represents a part of $\prod_5$. Polynomial generation satisfies the following property.*

$$S_P^\infty q^0 = q + \frac{\mu_0}{62} q^{(4)}$$

Proof. By mathematical induction, to begin, let for any $n \in \mathbb{N}_0$,

$$S_P^{i+1} q^0 = q^{n+1} + \frac{\mu_0}{64} q^{(4),\ n+1} \sum_{i=0}^{n} 2^{-2i} \tag{7}$$

Through (3), this identity is true for $n = 0$ as S_P-scheme reproducing polynomials in $\prod_5$. Next, let that Eq.(7) is valid for $n \in \mathbb{N}_0$. Since $q \in \prod_5$, clearly, $(D^2 q_i^{(4),\ n}) = 0$. Consequently, based on the expression in (3), by induction hypothesis, we derive the equation

$$\begin{aligned} S_P^{n+1} q^0 &= S_P(q^n + \frac{\mu_0}{64} q^{(4),n} \sum_{i=0}^{i-1} 2^{-2i}) \\ &= (S_D + 2^{-2n} \frac{\mu_0}{64} \mathbb{D}_2)(q^n + \frac{\mu_0}{64} q^{(4),n} \sum_{i=0}^{i-1} 2^{-2i}) \\ &= q^{n+1} + \frac{\mu_0}{64} q^{(4),\ n+1} \sum_{i=0}^{n} 2^{-2i} \end{aligned}$$

So, mathematical induction has been proved. Then, we can easily find that,

$$q_{2i+m}^{(4),\ n+1} = q^{(4)}(i2^{-n}) + O(2^{-n}) \text{ for m=0,1.}$$

It precisely ends the proof.

4. Evaluation of the proposed scheme

In this section, we discuss the convergence rate, smoothness, and order of approximation of the proposed scheme.

4.1. Evaluation of convergence and smoothness

The convergence and smoothness of a subdivision scheme, along with the length of its refinement mask, are fundamental criteria for choosing a suitable subdivision scheme. To examine such properties for the proposed scheme defined in equation (1), we begin by introducing some essential concepts. Given an S_P-scheme, let $S_{P,1}$-scheme denotes the scheme for the divided differences of the initial control points, which possesses the following property.

$$Dg^{n+1} = S_{P,1}Dg^n,$$

Where $g^n = S_P^n g^0$. The characteristic Laurent polynomial $C_1(v)$ for $S_{P,1}$-scheme is given as:

$$C_1(v) = \frac{2}{1+v}C(v),$$

Here, $C(v)$ represents the symbol of the scheme S_P. More, for the γ-th order difference scheme $S_{P,\gamma}$ derived from S_P, the associated symbol is given as

$$C_\gamma(v) = \sum_{i\in\mathbb{Z}} c_i^{[\gamma]} v^i = \left(\frac{2}{1+v}\right)^\gamma C(v). \tag{8}$$

Here, $C_0(v) = C(v)$ and $c_i^{[0]} = c_i$. This result offers a standard approach for determining whether a stationary SS converges.

Theorem 2. *[26] Consider two stationary subdivision schemes, S_P and $S_{P,1}$, with respective symbols $C(v)$ and $C_1(v)$. Then, the S_P-scheme is uniformly convergence (i.e., C^0) of S_P-scheme is the existence of a positive integer r satisfying $\left\|\left(\frac{1}{2}S_{P,1}\right)^r\right\|_\infty < 1$, where $\frac{1}{2}S_{P,1}$ is the scheme denoted by the symbol $\frac{1}{2}C_1(v)$.*

Moreover, a sufficient condition for a S_P-scheme to be C^m, $m \geq 1$, is given below.

Theorem 3. *[26] Suppose that S_P be a stationary SS with the symbol*

$$C(v) = \frac{1}{2}(1+v)c(v)$$

with a Laurent polynomial $c(v)$. If the S_c-scheme corresponding with $c(v)$ is C^m, then the S_P-scheme is $C^{m+1}(\mathbb{R})$.

We will now demonstrate that the scheme in (1) achieves C^5 smoothness for a specific range of μ_0 values. Note that when $\mu_0 = 0$, the mask in (1) reduces to the S_D-scheme, which is known to be C^2. Therefore, we focus on the case where $\mu_0 > 0$.

Theorem 4. *Consider the SS S_P using the refinement rule is given as (1), having the tension parameter μ_0 specified in (2). Then, S_P-scheme generates limit functions of class C^5 for $\frac{9}{11} < \mu_0 < 1$.*

Proof. We begin by investigating the uniform convergence of the S_P-scheme. To continue, we note that the Laurent polynomial $C(v)$ from equation (5) corresponds to the refinement mask c introduced in equation (1). Using the relation in equation (8), the Laurent polynomial for the 1st-order difference $S_{P,1}$-scheme is expressed as follows:

$$\begin{aligned} C_1(v) &= \frac{(1+v)^5}{64}\{\frac{1}{2}(3-3\mu_0)+\frac{1}{v}(-9+11\mu_0)+\frac{1}{2v^2}(38-38\mu_0) \\ &\quad +\frac{1}{v^3}(-9+11\mu_0)+\frac{1}{2v^4}(3-3\mu_0)\} \end{aligned}$$

The mask $c^{[1]} = (c_i^{[1]} : i \in \mathbb{Z})$ which is used in the $S_{P,1}$-scheme is defined as follows:

$$\begin{aligned} c^{[1]} = \frac{1}{128}\{&3-3\mu_0, -3+7\mu_0, -22+42\mu_0, 22+22\mu_0, 128-68\mu_0, 128-68\mu_0, \\ &22+22\mu_0, -22+42\mu_0, -3+7\mu_0, 3-3\mu_0\}. \end{aligned}$$

Now, $\|\frac{1}{2}S_{P,1}\| < 1$ for $\frac{-39}{49} < \mu_0 < \frac{195}{71}$. This implies that, it has C^0-continuity.
Next, we will check for C^1 smoothness for S_P, consider the 2nd-order difference $S_{P,2}$-scheme is,

$$\begin{aligned} C_2(v) &= \frac{(1+v)^4}{32}\{\frac{v}{2}(3-3\mu_0)+(-9+11\mu_0)+\frac{1}{2v}(38-38\mu_0) \\ &\quad +\frac{1}{v^2}(-9+11\mu_0)+\frac{1}{2v^3}(3-3\mu_0)\} \end{aligned}$$

The mask $c^{[2]} = (c_i^{[2]} : i \in \mathbb{Z})$ which is used in the $S_{P,2}$-scheme is defined as follows:

$$\begin{aligned} c^{[2]} = \frac{1}{64}\{&3-3\mu_0, -6+10\mu_0, -16+32\mu_0, 38-10\mu_0, 90-58\mu_0, 38-10\mu_0, \\ &16+32\mu_0, -6+10\mu_0, 3-3\mu_0\}. \end{aligned}$$

Now, $\|\frac{1}{2}S_{P,2}\| < 1$ for $0 < \mu_0 < 2$. This implies that, it has C^1-continuity.
Next, we will check for C^2 smoothness for S_P, consider the 3rd-order difference $S_{P,3}$-scheme is,

$$\begin{aligned} C_3(v) &= \frac{(1+z)^3}{16}\{\frac{v^2}{2}(3-3\mu_0)+(v)(-9+11\mu_0)+\frac{1}{2}(38-38\mu_0) \\ &\quad +\frac{1}{v}(-9+11\mu_0)+\frac{1}{2v^2}(3-3\mu_0)\} \end{aligned}$$

The mask $c^{[3]} = (c_i^{[3]} : i \in \mathbb{Z})$ which is used in the $S_{P,3}$-scheme is defined as follows:

$$\begin{aligned} c^{[3]} = \frac{1}{32}\{&3-3\mu_0, -9+13\mu_0, -7+19\mu_0, 45-29\mu_0, 45-29\mu_0, -7+19\mu_0, \\ &-9+13\mu_0, 3-3\mu_0\}. \end{aligned}$$

Now, $\|\frac{1}{2}S_{P,3}\| < 1$ for $0 < \mu_0 < 2$. This implies that, it has C^2-continuity.
Next, we will check for C^3 smoothness for S_P, consider the 4th-order difference $S_{P,4}$-scheme is,

$$\begin{aligned} C_4(v) &= \frac{(1+z)^2}{8}\{\frac{v^3}{2}(3-3\mu_0) + (v^2)(-9+11\mu_0) + \frac{v}{2}(38-38\mu_0) \\ &\quad +(-9+11\mu_0) + \frac{1}{2v}(3-3\mu_0)\} \end{aligned}$$

The mask $c^{[4]} = (c_i^{[4]} : i \in \mathbb{Z})$ which is used in the $S_{P,4}$-scheme is defined as follows:

$$c^{[4]} = \frac{1}{16}\{3-3\mu_0, 5+3\mu_0, -12+16\mu_0, 40-32\mu_0, -12+16\mu_0, 5+3\mu_0, \\ 3-3\mu_0, \}.$$

Now, $\|\frac{1}{2}S_{P,4}\| < 1$ for $\frac{1}{2} < \mu_0 < \frac{3}{2}$. This implies that, it has C^3-continuity.
Next, we will check for C^4 smoothness for S_P, consider the 5th-order difference $S_{P,5}$-scheme is,

$$\begin{aligned} C_5(v) &= \frac{(1+z)}{4}\{\frac{v^4}{2}(3-3\mu_0) + (v^3)(-9+11\mu_0) + \frac{v^2}{2}(38-38\mu_0) \\ &\quad +(v)(-9+11\mu_0) + \frac{1}{2}(3-3\mu_0)\} \end{aligned}$$

The mask $c^{[5]} = (c_i^{[5]} : i \in \mathbb{Z})$ which is used in the $S_{P,5}$-scheme is defined as follows:

$$c^{[5]} = \frac{1}{8}\{3-3\mu_0, -15+19\mu_0, 20-16\mu_0, 20-16\mu_0, -15+19\mu_0, 3-3\mu_0\}.$$

Now, $\|\frac{1}{2}S_{P,5}\| < 1$ for $\frac{11}{19} < \mu_0 < \frac{27}{19}$. This implies that, it has C^4-continuity.

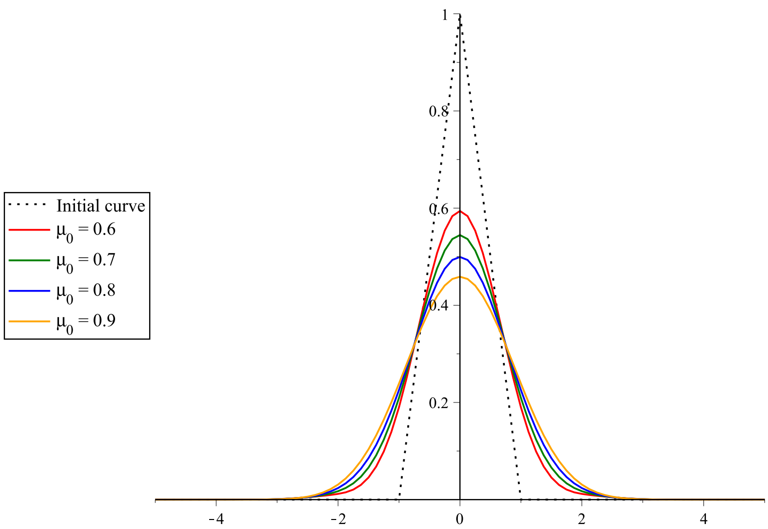


Figure 1: Basic limit function for $\mu_0 = 0.6, 0.7, 0.8, 0.9$.

Next, we will check for C^5 smoothness for S_P, consider the 6th-order difference $S_{P,6}$-scheme is,

$$C_6(v) = \frac{1}{2}\{\frac{v^5}{2}(3-3\mu_0)+(v^4)(-9+11\mu_0)+\frac{v^3}{2}(38-38\mu_0) \\ +(v^2)(-9+11\mu_0)+\frac{v}{2}(3-3\mu_0)\}$$

The mask $c^{[6]} = (c_i^{[6]} : i \in \mathbb{Z})$ which is used in the $S_{P,6}$-scheme is defined as follows:

$$c^{[6]} = \frac{1}{4}\{3-3\mu_0, -18+22\mu_0, 38-38\mu_0, -18+22\mu_0, 3-3\mu_0\}.$$

Now, $\|\frac{1}{2}S_{P,6}\| < 1$ for $\frac{9}{11} < \mu_0 < 1$. This implies that, it has C^5-continuity.
Next, for C^6 smoothness, the divided difference of 7th-order does not exist making S_P-scheme of C^5-continuity.

Remark 2. *The fundamental limit function of the S_P-scheme is expressed as $\phi = S_P^\infty\delta$, where $\delta = \{\delta_{0,i} : i \in \mathbb{Z}\}$ is a sequence at level 0, and $\delta_{i,0}$ denotes the Kronecker delta. It is important to note that the support of this function is [-5,5], which is equivalent to the S_D-scheme. A visualization of the basic limit function of the S_P-scheme for varying values of the tension parameter μ_0 is provided in Figure 1.*

4.2. Order of approximation

We investigate the order of approximation of the S_P-scheme with the mask in (1), focusing on its connection to the parameter μ_0. Specifically, our objective is to demonstrate that the S_P-scheme achieves sixth-order of approximation with an appropriate choice of μ_0. Subsequently, we will prove that if the initial data sequence is given by $g^0 = \{g(i2^{-n_0}) : i \in \mathbb{Z}\}$, where g is a smooth function in $\mathbb{W}^6_\infty(\mathbb{R})$, then the scheme S_P generates a limit function $S_P^\infty g^{n_0}$ such that

$$\|g - S_P^\infty g^0\| \le k_g 2^{-6n_0} \tag{9}$$

To simplify our presentation, we use the notation $\hat{g}^n$, for $n \in \mathbb{Z}_+$, to represent the sequence obtained by sampling a smooth function g at the grid points $2^{-n}\mathbb{Z}$. That is,

$$\hat{g}^n = \{g(i2^{-n} : i \in \mathbb{Z}\}, n \in \mathbb{Z}_+. \tag{10}$$

Lemma 1. *Let S_P be the SPSS with the parameter $\mu_0 = \tau 2^{-2n_0}$. Let $\hat{g}^n, n \ge n_0$ be a sequence of data points obtained by sampling a function $g \in \mathbb{W}^6_\infty(\mathbb{R})$ with density 2^{-n}. Then*

$$\|S_p\hat{g}^n - \hat{g}^{n+1}\|_\infty \le k_g 2^{-2(2n+n_0)}$$

for some constant $k_g > 0$ depending on g .

Proof. Let us first estimate the case of odd rule, i.e.,$|S_P\hat{g}^n_{2i+1} - \hat{g}^{n+1}_{2i+1}|$. To do this, the proposed scheme is represented using S_D-scheme:

$$(S_P\hat{g}^n)_{2i+1} = S_D\hat{g}^n_{2i+1} - \frac{\tau}{64}2^{-2n_0}.2^{-2n}(\mathbb{D}_2 g^n_{2i+1}) \tag{11}$$

Clearly $D_2g^n_{2i+1} = O(2^{-4n})$ for $i \in \mathbb{Z}$ because g is C^5 and g^{iv} is bounded. It follows that

$$|(S_P\hat{g}^n)_{2i+1} - \hat{g}^{n+1}_{2i+1}| \leq |(S_D\hat{g}^n)_{2i+1} - \hat{g}^{n+1}_{2i+1}| + O(2^{-2(2n+n_0)}) = O(2^{-2(2n+n_0)}) \tag{12}$$

Since, the S_D-scheme provides the sixth-order approximation and $g \in \mathbb{W}^6_\infty(\mathbb{R})$, it readily holds that

$$|(S_D\hat{g}^n)_{2i+1} - \hat{g}^{n+1}_{2i+1}| = O(2^{-6n}).$$

Combining this with (4) verifies the lemmas claim,

$$\|S_p\hat{g}^n - \hat{g}^{n+1}\|_\infty \leq k_g 2^{-2(2n+n_0)}$$

In a similar way, we can verify the even rule.

Theorem 5. *Assume that the initial data sequence $g^0 := \{g(2^{-n_0}i) : i \in \mathbb{Z}\}$ is obtained by sampling a function $g \in W^6_\infty(\mathbb{R})$, i.e., $g^0 := \hat{g}^{n_0}$. Let $\{S_P\}$ denote the SPSS defined by the refinement rule given in equation (2.1), and suppose the shape parameter μ_0 satisfies $0 \leq \mu_0 \leq \tau 2^{-n_0}$, for some constant $\tau > 0$. Under these conditions, the following error estimate holds:*

$$\|S_P^\infty g_0 - g\|_\infty \leq k_{g,\tau}2^{-6n_0},$$

where $k_{g,\tau} > 0$ is a constant that depends only on the function g and the parameter τ, but is independent of the refinement level n_0.

Proof. Let $D = \{i2^{-n} : i \in \mathbb{Z},\, n \in \mathbb{Z}_+\}$ be the set of dyadic points. Suppose the initial data g^0 is defined on the grid $2^{-n_0}\mathbb{Z}$, and let the iterated subdivision sequence be given by $g^n = S_P^{n-n_0}\hat{g}^{n_0}$. Then the sequence g^n satisfies the telescoping identity

$$g^n = S_P\hat{g}^{n-1} + \sum_{l=n_0}^{n-2} S^{n-l-1}(S_P\hat{g}^l - \hat{g}^{l+1}).$$

Assume the proposed S_P-scheme is stable i.e., there exists a constant $k > 0$ such that $||S_P^m||_\infty \leq k$ for any $m \in \mathbb{N}$. Then, for any $n \geq n_0$ and $i \in \mathbb{Z}$, the following estimate holds:

$$\begin{aligned} |g^n_i - g(i2^{-n})| &\leq ||S_P\hat{g}^{n-1} - \hat{g}^n||_\infty + \sum_{l=n_0}^{n-2} ||S_P^{n-l-1}||_\infty ||S_P\hat{g}^l - \hat{g}^{l-1}||_\infty \\ &\leq k\sum_{l=n_0}^{n-1} ||S_P\hat{g}^n - \hat{g}^{l+1}||_\infty. \end{aligned}$$

Our next step is to approximate the term $||S\hat{g}^l - \hat{g}^{l+1}||_\infty$. By assumption, $0 \le \mu_0 \le \tau 2^{-2n_0}$ for some $\tau > 0$. Thus

$$||S\hat{g}^l - \hat{g}^{l+1}||_\infty \le k_g, \tau 2^{-2(n_0+l)}$$

for some $l \ge g_0$ with a fixed value $k_g, \tau > 0$ depending on g and τ. It leads to the bound

$$\sum_{l=n_0}^{n-1} ||S_P\hat{g}^n - \hat{g}^{l+1}||_\infty \le k_g \sum_{l=n_0}^{n-1} 2^{-2(n_0+l)} \le k_g 2^{-6n_0}.$$

5. Shape-preserving properties

Smoothness, precision, and shape preservation, such as monotonicity and convexity, are all important aspects of an SS in practical and commercial contexts. Thus, this section's goal is to examine our SS shape-preserving characteristics. To proceed, we express the refinement rules given in equation (1) in the following rewritten form:

$$S_P g^n = Q_B g^n - 2^{-2n-6}(1-\mu_0)\mathbb{D}_2(g^n). \tag{13}$$

Where $\mathbb{D}_2$ is given in (4) and Q_B is the quintic B-spline with the mask $\{\frac{1}{64}, \frac{3}{32}, \frac{1}{4}, \frac{13}{32}, \frac{15}{32}, \frac{13}{32}, \frac{1}{4}, \frac{3}{32}, \frac{1}{64}\}$. If $\mu_0 = 0$, our scheme reduces to the S_D-scheme. It is a known fact that the S_D-scheme has the inherent drawback of being interpolatory. If the initial control points are irregularly distributed, this property can lead to undesirable artifacts in the resulting curve, such as self-intersections. Thus, it is appropriate to examine the parameter μ_0 away from zero. However, if $\mu_0 = 1$, the scheme transforms into the Q_B-scheme. In essence, the parameter μ_0 acts as a balancing factor between the S_D-scheme and Q_B-scheme. By conducting numerical experiments on various μ_0, we note that a suitable choice of μ_0 is in the range

$$\frac{9}{11} < \mu_0 < 1. \tag{14}$$

5.1. Preservation of monotonicity

We first demonstrate that S_P-scheme in (13) preserves monotonicity under a mild condition. Specifically, we show that $(Dg^{n+1})_i \ge 0$ whenever $(Dg^n)_i \ge 0$ for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. It is important to note that the Q_B-scheme fulfills the following equations before proceeding:

$$\begin{cases} (DQ_Bg^n)_{2i} = \frac{5}{32}(Dg^n)_{i-1} + \frac{15}{32}(Dg^n)_i + \frac{11}{32}(Dg^n)_{i+1}\frac{1}{32}(Dg^n)_{i+2}, \\ (DQ_Bg^n)_{2i+1} = \frac{1}{32}(Dg^n)_{i-1} + \frac{11}{32}(Dg^n)_i + \frac{15}{32}(Dg^n)_{i+1}\frac{5}{32}(Dg^n)_{i+2}. \end{cases} \tag{15}$$

which means that Q_B-scheme is preserves the monotonicity . Then, given (13) and (15), we can derive the following identities.

$$\begin{cases} (Dg^n)_{2i} = (\dfrac{5(Dg^n)_{i-1} + 15(Dg^n)_i + 11(Dg^n)_{i+1} + (Dg^n)_{i+2}}{32}) \\ \qquad - 2^{-2n}\dfrac{(1-\mu_0)}{128}(-3(D^3g^n)_{i-1} + 36(D^3g^n)_i + 7(D^3g^n)_{i+1}), \\ (Dg^n)_{2i+1} = (\dfrac{(Dg^n)_{i-1} + 11(Dg^n)_i + 15(Dg^n)_{i+1} + 5(Dg^n)_{i+2}}{32}) \\ \qquad - 2^{-2n}\dfrac{(1-\mu_0)}{128}(7(D^3g^n)_i + 36(D^3g^n)_{i+1} - 3(D^3g^n)_{i+2}). \end{cases} \tag{16}$$

Definition 4. *(Condition-M). Suppose given a number μ_0 and g^{n_0} be an initial sequence defined on the grid $2^{-n_0}\mathbb{Z}$. Let the sequence g^{n_0} satisfies 'condition-M' when*

$$(Dg^{n_0})_i \geq 2^{-2n_0}\frac{(1-\mu_0)}{k}|(D^3g^{n_0})_i|, \tag{17}$$

Where $k > 0$.

Lemma 2. *Let g^{n_0} denote the initial data associated with the grid points $2^{-n_0}Z$, and suppose that $g^{n+1} = S_P g^n, n \geq n_0$, with S_P in (1). If g^{n_0} fulfills the condition-M. Thus for all $n \geq n_0$,*

$$(Dg^n)_i \geq 2^{-2n}\frac{(1-\mu_0)}{k}|(D^3g^n)_i|, \tag{18}$$

where $k > 0$.

Proof. Using mathematical induction, we establish this lemma. Given that the initial sequence fulfills 'condition-M', inequality (18) holds. We now assume that (18) is true for $n > n_0$ and strive to prove that

$$(Dg^{n+1})_i \geq 2^{-2n+1}\frac{(1-\mu_0)}{k}|(D^3g^{n+1})_i|.$$

To prove the result, we consider the following two cases: (I) $i = 2j$ and (II) $i = 2j+1$. We begin with Case (I), $i = 2j$, and we can readily obtain the expression using equation (16)

$$\begin{aligned} (D^3g^{n+1})_{2j} &= 2^{2(n+1)}\frac{1}{2}(-3(D^3g^{n+1})_{2j-1} + 36(D^3g^{n+1})_{2j} + 7(D^3g^{n+1})_{2j+1}) \\ &= (\frac{27(D^3g^n)_{j-1} + 153(D^3g^n)_j + 133(D^3g^n)_{j+1} + 7(D^3g^n)_{j+2}}{32}) - \frac{(1-\mu_0)}{128} \\ &\quad (417(D^3g^n)_{j-1} - 851(D^3g^n)_j + 451(D^3g^n)_{j+1} - 17(D^3g^n)_{j+2}) \end{aligned}$$

Thus, it follows that

$$|(D^3g^{n+1})_{2j}| \leq \frac{618}{128}|(D^3g^n)_{j-1}| + \frac{2926}{128}|(D^3g^n)_j| + \frac{162}{128}|(D^3g^n)_{j+1}| + \frac{90}{128}|(D^3g^n)_{j+2}|.$$

This, together with (9), induces the relation

$$\begin{aligned}(Dg^{n+1})_{2j}-2^{-2(n+1)}\tfrac{(1-\mu_0)}{k}|(D^3g^{n+1})_{2j}| \geq\ & \tfrac{5}{32}((Dg^n)_{j-1}-2^{-2(n)}\tfrac{(1-\mu_0)}{k}|(D^3g^n)_{j-1}|)\\ &+\tfrac{15}{32}((Dg^n)_{j}-2^{-2(n)}\tfrac{(1-\mu_0)}{k}|(D^3g^n)_{j}|)\\ &+\tfrac{11}{32}((Dg^n)_{j+1}-2^{-2(n)}\tfrac{(1-\mu_0)}{k}|(D^3g^n)_{j+1}|)\\ &+\tfrac{1}{32}((Dg^n)_{j+2}-2^{-2(n)}\tfrac{(1-\mu_0)}{k}|(D^3g^n)_{j+2}|).\end{aligned}$$

Based on the inductive assumption for n, the nonnegativity of the final term follows. We now turn our attention to the case $i = 2j + 1$, where the identities in (16) enable us to

$$\begin{aligned}|(D^3g^{n+1})_{2j+1}| &= |(\frac{7}{16}(D^3g^n)_{j-1}+\frac{133}{16}(D^3g^n)_{j}+\frac{153}{16}(D^3g^n)_{j+1}+\frac{27}{16}(D^3g^n)_{j+2})\\ &\quad+\frac{(1-\mu_0)}{128}(-17(D^3g^n)_{j-1}+451(D^3g^n)_{j}-851(D^3g^n)_{j+1}\\ &\quad+417(D^3g^n)_{j+2})|\\ &\leq \frac{90}{128}|(D^3g^n)_{j-1}|+\frac{162}{128}|(D^3g^n)_{j}|+\frac{2926}{128}|(D^3g^n)_{j+1}|+\frac{618}{128}|(D^3g^n)_{j+2}|.\end{aligned}$$

By the same argument as given above, a direct calculation using (16) shows that

$$(Dg^{n+1})_{2j+1}-2^{-2(n+1)}\frac{(1-\mu_0)}{k}|(D^3g^{n+1})_{2j+1}| \geq 0.$$

Where $k > 0$.

Theorem 6. *Let the initial sequence g^{n_0} be defined on the grid $2^{-n_0}\mathbb{Z}$, and consider the subdivision process given by*

$$g^{n+1} = \mathcal{S}_P g^n, \quad \textit{for all } n \geq n_0,$$

where $\mathcal{S}_P$ is the subdivision scheme defined in equation (1). If the initial data g^{n_0} satisfies Condition-M*, then the discrete first differences satisfy*

$$(Dg^n)_i \geq 0, \quad \textit{for all } n \geq n_0 \textit{ and } i \in \mathbb{Z}.$$

Therefore, the scheme $\mathcal{S}_P$ preserves monotonicity.

Proof. By lemma 2, we have already shown that $(Dg^n)_i \geq 2^{-2n}\frac{(1-\mu_0)}{k}|(D^3g^n)_i| \geq 0$ for any $n \geq n_0$. Hence, the result follows directly.

5.2. Preservation of convexity

Next, we show that S_P-scheme maintains convexity under a mild condition. For simplicity, we introduce the following notation

$$\nabla g_i^n = g_i^n - g_{i-1}^n \qquad \textit{and} \qquad \Delta g_i^n = g_{i-1}^n - 2g_i^n + g_{i+1}^n.$$

Lemma 3. *Let Q_B denote the quintic B-spline. Thus, for any $i \in \mathbb{Z}$,*

$$\begin{cases} (D^2Q_Bg^n)_{2i} = \frac{1}{16}(D^2g^n)_{i-1} + \frac{7}{16}(D^2g^n)_i + \frac{7}{16}(D^2g^n)_{i+1} + \frac{1}{16}(D^2g^n)_{i+2}, \\ (D^2Q_Bg^n)_{2i+1} = \frac{4}{16}(D^2g^n)_i + \frac{8}{16}(D^2g^n)_{i+1}\frac{4}{16}(D^2g^n)_{i+2}. \end{cases} \tag{19}$$

This shows that the Q_B-scheme preserves convexity.

Lemma 4. *Assume S_P is the SS using the refinement rule given in (1), suppose that μ_0 is given in (2). When $g^{n+1} = S_Pg^n$ for $n \geq n_0$. Next, we obtain*

$$\begin{aligned} (D^2g^{n+1})_{2i} &= (\frac{1}{16}(D^2g^n)_{i-1} + \frac{7}{16}(D^2g^n)_i + \frac{7}{16}(D^2g^n)_{i+1} + \frac{1}{16}(D^2g^n)_{i+2}) \\ &\quad -\frac{(1-\mu_0)}{32}2^{-n}(10\nabla(D^3g^n)_{i+1} + 10\nabla(D^3g^n)_{i+2}), \\ (D^2g^{n+1})_{2i+1} &= (\frac{4}{16}(D^2g^n)_i + \frac{8}{16}(D^2g^n)_{i+1}\frac{4}{16}(D^2g^n)_{n+2}) - \frac{(1-\mu_0)}{32}2^{-n} \\ &\quad (-3\nabla(D^3g^n)_{i+1} + 26\nabla(D^3g^n)_{i+2} - 3\nabla(D^3g^n)_{i+3}). \end{aligned} \tag{20}$$

Proof. We now concentrate on proving the first equation in (20). We note that Q_B implies that the quintic B-spline, using (13) and Lemma 3 to obtain

$$\begin{aligned} (D^2g^{n+1})_{2n} &= (\frac{1}{16}(D^2g^n)_{i-1} + \frac{7}{16}(D^2g^n)_i + \frac{7}{16}(D^2g^n)_{i+1} + \frac{1}{16}(D^2g^n)_{i+2}) \\ &\quad -\frac{(1-\mu_0)}{32}(10(D^2g^n)_{i-1} - 10(D^2g^n)_i - 10(D^2g^n)_{i+1} + 10(D^2g^n)_{i+2}) \\ &= (\frac{1}{16}(D^2g^n)_{i-1} + \frac{7}{16}(D^2g^n)_i + \frac{7}{16}(D^2g^n)_{i+1} + \frac{1}{16}(D^2g^n)_{i+2}) \\ &\quad -\frac{(1-\mu_0)}{32}2^{-n}(10\nabla(D^3g^n)_{i+1} + 10\nabla(D^3g^n)_{i+2}). \end{aligned}$$

By applying equation (13) and Lemma 3, we obtain the following expression through basic calculation

$$\begin{aligned} (D^2g^{n+1})_{2n+1} &= (\frac{4}{16}(D^2g^n)_i + \frac{8}{16}(D^2g^n)_{i+1} + \frac{4}{16}(D^2g^n)_{i+2}) - \frac{(1-\mu_0)}{32} \\ &\quad (-3D^2g^n_{i-1} + 32D^2g^n_i - 58D^2g^n_{i+1} + 32D^2g^n_{i+2} - 3D^2g^n_{i+3}) \\ &= (\frac{4}{16}(D^2g^n)_i + \frac{8}{16}(D^2g^n)_{i+1}\frac{4}{16}(D^2g^n)_{n+2}) - \frac{(1-\mu_0)}{32}2^{-n} \\ &\quad (-3\nabla(D^3g^n)_{i+1} + 26\nabla(D^3g^n)_{i+2} - 3\nabla(D^3g^n)_{i+3}). \end{aligned}$$

Lemma 5. *Suppose S_p be the SS given by (1), acting on the initial data g^{n_0} at grid points $2^{-n_0}\mathbb{Z}$. We then define the sequence $g^{n+1} = S_pg^n$ for $n \geq n_0$, we obtain*

$$\nabla(D^3g^{n+1})_{2i} = (\frac{-5}{32}\nabla(D^3g^n)_{i-1} + \frac{11}{32}\nabla(D^3g^n)_i + \frac{5}{16}\nabla(D^3g^n)_{i+1}),$$

$$\begin{aligned}\nabla(D^3g^{n+1})_{2i+1} &= (\frac{3}{64}\nabla(D^3g^n)_{i-1} - \frac{24}{64}\nabla(D^3g^n)_i + \frac{85}{64}\nabla(D^3g^n)_{i+1}\\ &\quad -\frac{32}{64}\nabla(D^3g^n_{i+2})).\end{aligned} \tag{21}$$

Proof. From Lemma 4, it follows by direct calculation that

$$\begin{aligned}2^{-(n+1)}\nabla(D^3g^{n+1})_{2i+1} &= (-3(D^2g^{n+1})_{2i-1} + 32(D^2g^{n+1})_{2i} - 58(D^2g^{n+1})_{2i+1}\\ &\quad +32(D^2g^{n+1})_{2i+2} - 3(D^2g^{n+1})_{2i+3})\\ &= (\frac{10}{16}(D^2g^n)_{i-1} + \frac{20}{16}(D^2g^n)_i - \frac{40}{16}(D^2g^n)_{i+1} - \frac{20}{16}(D^2g^n)_{i+2}\\ &\quad +\frac{30}{16}(D^2g^n)_{i+3}) - \frac{(1-\mu_0)}{32}2^{-n}(130\nabla D^3f^n_i\\ &\quad -290\nabla D^3f^n_{i+1} + 190\nabla D^3g^n_{i+2} - 30\nabla D^3g^n_{i+3})\\ \nabla(D^3g^{n+1})_{2i+1} &= (-\frac{30}{32}\nabla(D^3g^n)_{i-1} - \frac{40}{32}\nabla(D^3g^n)_i + \frac{62}{32}\nabla(D^3g^n)_{i+1}\\ &\quad -\frac{2}{32}\nabla(D^3g^n_{i+2})).\end{aligned}$$

$$\begin{aligned}2^{-(n+1)}\nabla(D^3g^{n+1})_{2i} &= (10(D^2g^{n+1})_{2i-1} - 10(D^2g^{n+1})_{2i} - 10(D^2g^{n+1})_{2i+1}\\ &\quad +10(D^2g^{n+1})_{2i+2})\\ &= (\frac{30}{16}(D^2g^n)_{i-1} - \frac{20}{16}(D^2g^n)_i - \frac{40}{16}(D^2g^n)_{i+1} + \frac{20}{16}(D^2g^n)_{i+2}\\ &\quad +\frac{10}{16}(D^2g^n)_{i+3}) - \frac{(1-\mu_0)}{32}2^{-n}(-30\nabla D^3f^n_i\\ &\quad +190\nabla D^3f^n_{i+1} - 290\nabla D^3g^n_{i+2} + 130\nabla D^3g^n_{i+3})\\ \nabla(D^3g^{n+1})_{2i} &= (-\frac{2}{32}\nabla(D^3g^n)_{i-1} + \frac{62}{32}\nabla(D^3g^n)_i - \frac{40}{32}\nabla(D^3g^n)_{i+1}\\ &\quad -\frac{30}{32}\nabla(D^3g^n)_{i+1}).\end{aligned}$$

Definition 5. *(Condition-C). Given a number μ_0 as in (2), and an initial sequence g^{n_0} defined on $2^{-n_0}\mathbb{Z}$, we say g^{n_0} fulfills 'condition-C' if*

$$(D^2g^{n_0})_j \geq \frac{(1-\mu_0)}{k}2^{-n_0}|\nabla(D^3g^n)_{j+1}|, \qquad \forall j \in \mathbb{Z}. \tag{22}$$

Where, $k > 0$.

Theorem 7. *Suppose S_P be the SS in (1) with μ_0 in (14). Assume that the initial sequence g^{n_0} be connected to $2^{-n_0}\mathbb{Z}$, and suppose that it fulfills the condition-C. Then, for any $n > n_0$, $(\Delta g^n)_i \geq 0$ for $i \in \mathbb{Z}$, which implies that the S_P-scheme preserves the convexity.*

Table 1: Maximum errors in L^∞-norm and corresponding order of approximations for g_1 and g_2 given in Example 1.

Initial sampling interval	$g_1 \in W^6_\infty(\mathbb{R})$		$g_2 \in W^6_\infty(\mathbb{R})$	
	$\|\|.\|\|_\infty$ error	Order	$\|\|.\|\|_\infty$ error	Order
2^{-1}	5.9981E−05	–	5.8756E−05	–
2^{-2}	9.37E−07	6.0	9.32E−07	5.9
2^{-3}	1.5E−08	6.0	1.5E−08	6.0
2^{-4}	2.288E−10	6.0	2.3E−10	6.0
2^{-5}	3.6E−12	6.0	3.7E−12	6.0
2^{-6}	5.59E−14	6.0	5.6E−14	6.0

Note: The results confirm that both test functions g_1 and g_2 achieve sixth-order convergence in the L^∞-norm as the sampling interval decreases.

Proof. Applying the principle of mathematical induction, prove that for any $n \geq n_0$,

$$(D^2 g^n)_j \geq \frac{(1-\mu_0)}{k} 2^{-n} |\nabla (D^3 g^n)_{j+1}|, \qquad \forall\, j \in \mathbb{Z}. \tag{23}$$

The initial sequence g^{n_0} satisfies by hypothesis. Assuming the inequality is true for some i. Now, we check if it is true for $i+1$, starting with the case where $j = 2i$. By applying Lemma 4 and 5, we deduce

$$\begin{aligned}(D^2 g^{n+1})_{2i} - 2^{-n-1}\tfrac{(1-\mu_0)}{k} \mid \nabla (D^3 g^{n+1})_{2i+1} \mid \geq\ & \tfrac{4}{16}((D^2 g^n)_{i-1} - 2^{-n}\tfrac{(1-\mu_0)}{k}|\nabla (D^3 g^n)_{i-1}|) \\ & + \tfrac{8}{16}((D^2 g^n)_{i} - 2^{-n}\tfrac{(1-\mu_0)}{k}|\nabla (D^3 g^n)_{i}|) \\ & + \tfrac{4}{16}((D^2 g^n)_{i+1} - 2^{-n}\tfrac{(1-\mu_0)}{k}|\nabla (D^3 g^n)_{i+1}|).\end{aligned}$$

According to our inductive hypothesis, the final result is positive. We now turn $j = 2i-1$, and use Lemma 5 to drive

$$\begin{aligned}(D^2 g^{n+1})_{j} - 2^{-n-1}\tfrac{(1-\mu_0)}{k} \mid \nabla (D^3 g^{n+1})_{j} \mid \geq\ & \tfrac{1}{16}((D^2 g^n)_{i-1} - 2^{-n}\tfrac{(1-\mu_0)}{k}|\nabla (D^3 g^n)_{i-1}|) \\ & + \tfrac{7}{16}((D^2 g^n)_{i} - 2^{-n}\tfrac{(1-\mu_0)}{k}|\nabla (D^3 g^n)_{i}|) \\ & + \tfrac{7}{16}((D^2 g^n)_{i+1} - 2^{-n}\tfrac{(1-\mu_0)}{k}|\nabla (D^3 g^n)_{i+1}|) \\ & + \tfrac{1}{16}((D^2 g^n)_{i+2} - 2^{-n}\tfrac{(1-\mu_0)}{k}|\nabla (D^3 g^n)_{i+2}|).\end{aligned}$$

Based on the assumption for the case n, we can conclude that the final term is positive, thereby completing the proof.

6. Numerical Applications

This section presents numerical experiments to demonstrate the effectiveness of the proposed subdivision schemes.

Example 1. ***(Order of approximation)***
To verify the order of approximation of S_P-scheme given in equation (1), we consider the following two smooth functions:

$$g_1(x) = x^6, \quad g_2(x) = x^6 \cos(x).$$

The initial data sequences are sampled from each function over the interval $[0, 0.2]$ at sampling densities 2^{-n_0}, where $n_0 = 0, \ldots, 6$. For every refinement level, the parameter μ_0 is selected as $\mu_0 = 2^{-n_0}$, corresponding to the choice $\tau = 1$ in equation (2).

Table 1 presents the computed maximum errors, based on the L^∞-norms, along with the estimated rate of convergence. The functions g_1 and g_2 belong to the Sobolev space and achieve the optimal order of approximation six. These numerical results validate the theoretical order of approximations described in Theorem 5.

Example 2. ***(Tension parameter)***
To show the effect of the parameter μ_0 on the limit curves, we provide a numerical example. The dotted lines represent the control polygon, and the control points are marked with circles. Figure 2 presents the curves produced by the proposed scheme for $\mu_0 = 3/10, 5/10, 8/11, 10/11$. As μ_0 increases, the curves become smoother and deviate more from the original control polygon. It shows how the tension parameter affects the limit curve, resulting in different rates of convergence.

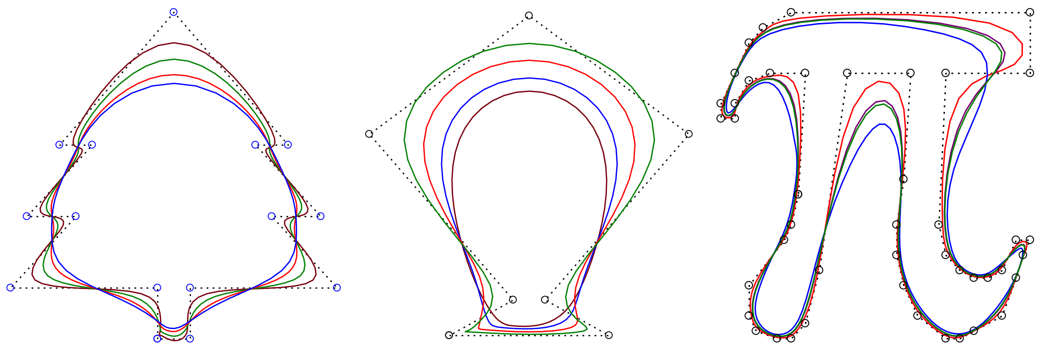
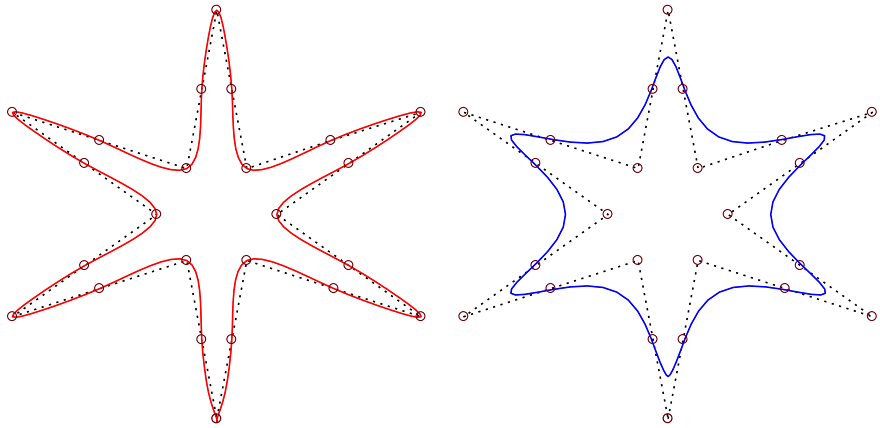
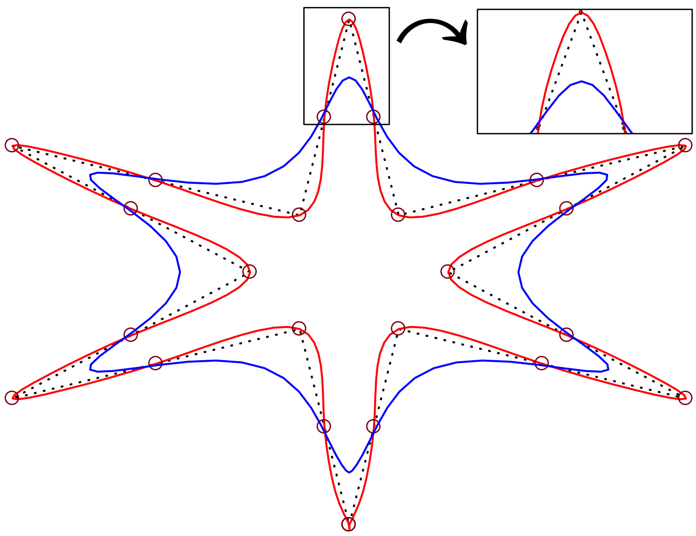


Figure 2: Subdivision limit curves produced with the parameter $\mu_0 = 3/10, 5/10, 8/11$ and $10/11$, respectively.

Example 3. ***(Curve design)***
This example illustrates the numerical results of our subdivision S_P-scheme with its parent schemes. Figure 3 and 4 show the limit curve when μ_0 is set to zero, the curve corresponds to the limit curve of the S_D-scheme algorithm through this results in an undesirable artifact and when μ_0 increases toward 1 it becomes Q_B-scheme the curves deviate more significantly from the control polygon.

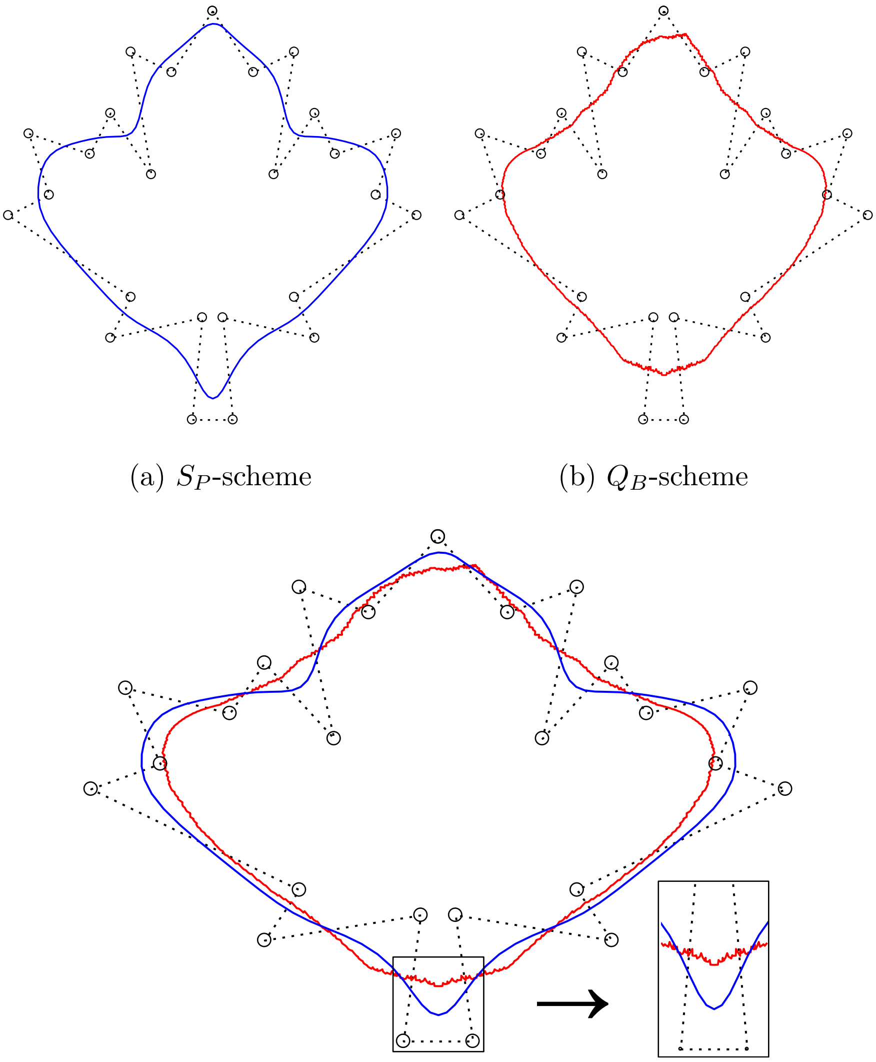


(a) S_P-scheme (b) S_D-scheme



(c) Zoomed area

Figure 3: 2D-model generated by S_D-scheme and S_P-scheme at $\mu_0 = 10/11$.

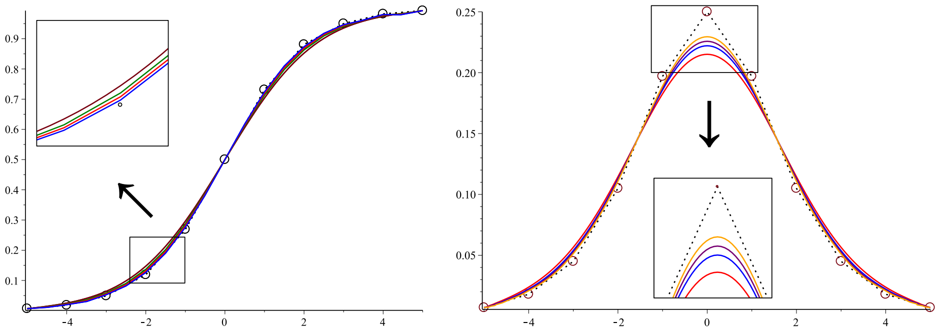


(a) S_P-scheme (b) Q_B-scheme

(c) Zoomed area

Figure 4: 2D-model generated by S_P-scheme and Q_B-scheme at $\mu_0 = 10/11$.

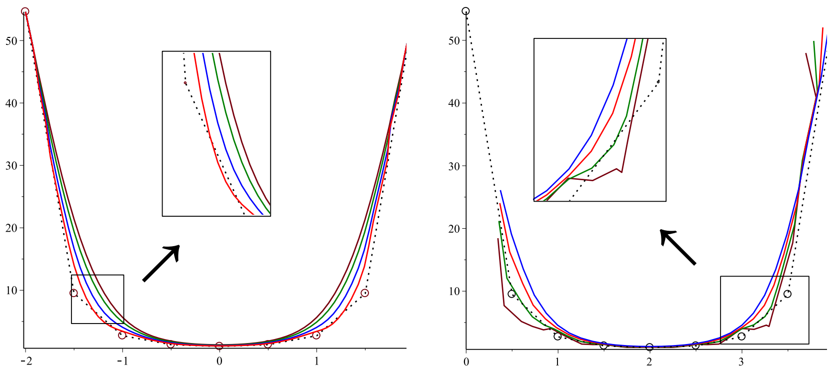
Example 4. ***(Monotonicity preservation)***
This example illustrates the effectiveness of the SS S_P in Monotonicity preservation. Suppose that initial data set g^{n_0} be given on [-5,5] as follows:
$g^{n_0} = \{0.007, 0.018, 0.047, 0.119, 0.269, 0.500, 0.731, 0.881, 0.952, 0.982, 0.993\}$.
A strictly monotone initial dataset g^{n_0} is provided, and the scheme is applied with varying tension parameter values $\mu_0 = 0.5, 0.6, 0.7, 0.8, 0.9$. The results demonstrate that the limit curves generated by the scheme preserve the monotonicity of the initial data in Figure 5(a), and their first derivatives in Figure 5(b) are also examined. The scheme's performance is showcased within the applicable domain, confirming its ability to maintain monotonicity.



(a) (b)

Figure 5: Monotonicity preservation. Limit function for (a) $\mu_0 = 0.5, 0.6, 0.7, 0.8, 0.9$ and (b) their first derivatives.

Example 5. ***(Convexity preservation)***
This example demonstrates the effectiveness of the SS S_P in Convexity preservation. Suppose that initial data set g^{n_0} be given on [-5,5] as follows:
$g^{n_0} = \{0.00, 0.0816, 0.3265, 0.7347, 1.3061, 2.0408, 2.9388, 4.00\}$.
A strictly convex initial dataset g^{n_0} is provided, and the scheme is applied with varying tension parameter values $mu_0 = (0.5, 0.6, 0.7, 0.8, 0.9)$. The results illustrate that the limit curves generated by the scheme preserve the Convexity of the initial data in Figure 6 (a), and their second derivatives in Figure 6 (b) are also examined. The scheme's performance is showcased within the applicable domain, confirming its ability to maintain Convexity.



(a) (b)

Figure 6: Convexity preservation. Limit functions for (a) $\mu_0 = 0.5, 0.6, 0.7, 0.8, 0.9$ and (b) their second derivatives.

Example 6. ***(Discontinuous function)***
In this example, we analyze the performance of the proposed scheme on a discontinuous

Table 2: Comparison with existing schemes.

Scheme	Type	Support	Order	C^n
Binary 3-point [29]	Approximating	5	3	C^3
Binary 4-point [30]	Interpolating	6	4	C^1
Binary 4-point [31]	Approximating	7	4	C^2
Binary 5-point [32]	Approximating	9	2	C^4
Binary 4-point [26]	Approximating	6	4	C^2
Binary 6-point [21]	Interpolating	10	6	C^2
S_P-scheme	Approximating	10	6	C^5

function defined as:

$$f(x) = \begin{cases} 1, & \textit{if } \; 0 \leq x < 0.4 \\ 0, & \textit{if } \; 0.4 \leq x \leq 0.8 \end{cases}$$

This step function is sampled uniformly at nine points from x=0 to x=0.8. The scheme is applied using a fixed tension parameter $\mu_0 = 9/11$. In Figure 7, The results show that the scheme handles the jump without introducing oscillations and preserves locality near the discontinuity, demonstrating its robustness on non-smooth data.

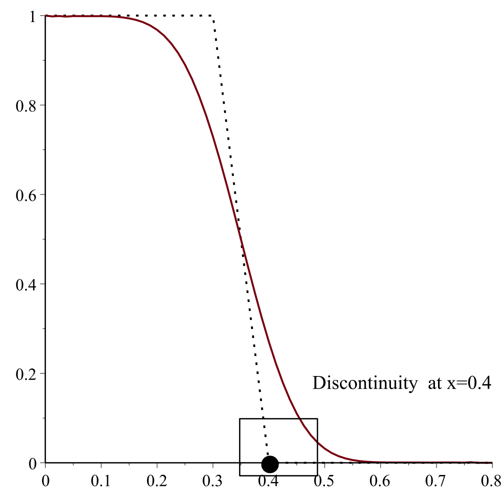


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Figure 7: Graph of the discontinuous function $g(x)$.

7. Comparison and conclusion

In conclusion, we have proposed a new linear, stationary subdivision scheme that unifies the strengths of the quintic B-spline and six-point interpolatory scheme. The proposed scheme achieves C^5-smoothness and sixth-order approximation with same support as of interpolatory scheme. The inclusion of a tension parameter allows for shape control, making the scheme flexible for preserving essential properties such as convexity and monotonicity. A comparative analysis in Table 2 with existing schemes highlights that the proposed scheme offers superior smoothness (C^5 continuity) and higher approximation order (sixth order), while maintaining same support of interpolatory scheme. Numerical experiments further confirm its effectiveness in producing smooth, accurate, and visually elegant curves, demonstrating its potential for practical applications in geometric modeling and computer-aided design. Despite these strengths, the scheme has some limitations. At present, it is restricted to univariate curve subdivision, and the choice of the tension parameter may vary depending on the data set or application. As future work, the scheme can be extended to bivariate surfaces using the tensor product approach, which is expected to further enhance smoothness and approximation order. Another promising direction is the development of adaptive strategies for selecting the tension parameter automatically, along with exploring applications in CAD, image processing, and scientific visualization.

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