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Fixed Point Theorems in Neutrosophic $\mathcal{F}$-Metric Spaces and Their Application

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Abstract. This study examines the framework of neutrosophic $\mathcal{F}$-metric spaces and highlights their role in nonlinear analysis. Within this extended setting, we establish a fixed point theorem that broadens traditional results to the neutrosophic domain. The applicability of the proposed theorem is demonstrated through its use in modeling a satellite web coupling problem. To substantiate the theoretical contributions, we also present concrete examples along with graphical illustrations capturing the nature of the contraction condition.

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Key Words and Phrases: Neutrosophic $\mathcal{F}$-metric space, neutrosophic φ-contraction, fixed point, boundary value problem

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1. Introduction

Fuzzy set theory, introduced by Zadeh [1] in 1965, revolutionized the mathematical treatment of uncertainty by allowing elements to possess degrees of membership between 0 and 1. This flexible framework laid the groundwork for significant advancements in various disciplines, including decision-making, control theory, and pattern recognition. The mathematical foundations of fuzzy intersection and union were further enriched by Schweizer and Sklar [2] through the introduction of continuous t-norms and t-conorms in probabilistic metric spaces.

Building upon this foundation, Kramosil and Michalek [3] introduced the notion of fuzzy metric spaces, which was later refined by George and Veeramani [4] using continuous t-norms to ensure topological consistency. Subsequent contributions by Grabiec [5] and

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Gregori and Sapena [6] established fundamental fixed point results within these spaces. Later, Turkoglu and Sangurlu [7], as well as Sedghi and Shobe [8], extended the concept to fuzzy- ψ and b-fuzzy contractive mappings. The seminal work of Wardowski [9] on F-contractions inspired further generalizations such as (F, φ)- and φ-contractive mappings explored by Sezen and Türkoglu [10] and Nadaban et al. [11], respectively.

Das et al. [12] advanced the theory by introducing fuzzy $\mathcal{F}$-metric spaces through a special class of functions $f : [0,1] \to [0,1]$, thereby relaxing the traditional axioms of fuzzy metric spaces. Their framework offered a broader analytical structure for capturing uncertainty and distance.

Parallel developments emerged from Atanassov's [13] introduction of intuitionistic fuzzy sets, which assign each element both membership and non-membership degrees. Park [14] extended this idea to intuitionistic fuzzy metric spaces, later studied further by Xia et al. [15]. These structures enabled a more refined representation of vagueness compared to classical fuzzy models.

The notion of neutrosophy, proposed by Smarandache [16], incorporated an additional degree of indeterminacy, paving the way for neutrosophic sets and logic. Kirisci and Simsek [17] unified the membership, non-membership, and indeterminacy components to formulate neutrosophic metric spaces, while Ahmad et al. [18] extended the idea to neutrosophic b-metric spaces and derived associated fixed point theorems. Further advancements include the works of Jeyaraman et al. [19–21], who developed fixed point results under generalized contraction conditions. More recently, Akram et al. [22] presented new classes of generalized neutrosophic metric spaces, highlighting their structural richness and applications.

Motivated by these foundational and modern developments, as well as the extensive literature on fixed point theory across generalized fuzzy settings ([23–26]), we introduce a new class of neutrosophic $\mathcal{F}$-metric spaces and establish fixed point theorems under generalized contraction conditions. These results extend and unify the existing work in fuzzy, intuitionistic fuzzy, and neutrosophic frameworks. To illustrate the practical significance, we apply our theoretical findings to a nonlinear satellite web coupling problem, supported by numerical examples and graphical analysis. The outcomes demonstrate both theoretical advancement and potential applications in complex systems modeling.

2. Preliminaries

This section presents the basic definitions and notions of neutrosophic metric spaces and neutrosophic $\mathcal{F}$-metric spaces that form the foundation for our main results.

Definition 1. *[17]A binary operation* $\star : [0,1] \times [0,1] \to [0,1]$ *is referred to as a continous t-norm [$\mathcal{CTN}$] if it meets the following conditions:*
(i) $\star$ *is associative and commutative;*
(ii) $\star$ *is continuous;*
(iii) $\mathfrak{p} \star 1 = \mathfrak{p}$, $\forall\, \mathfrak{p} \in [0,1]$;
(iv) $\mathfrak{p} \star \mathfrak{r} \leq \mathfrak{q} \star \mathfrak{s}$ *whenever* $\mathfrak{p} \leq \mathfrak{q}$ *and* $\mathfrak{r} \leq \mathfrak{s}$, $\forall\, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s} \in [0,1]$.

Example 1. *[15] A few illustrative instances of $\mathcal{CTN}$ are given below:*
(i) $\mathfrak{p} \star \mathfrak{q} = \min\{\mathfrak{p}, \mathfrak{q}\}$.
(ii) $\mathfrak{p} \star \mathfrak{q} = \mathfrak{p}\mathfrak{q}$.
(iii) $\mathfrak{p} \star \mathfrak{q} = \max\{0, \mathfrak{p} + \mathfrak{q} - 1\}$, *where* $\mathfrak{p}, \mathfrak{q} \in [0, 1]$.

Definition 2. *[17] A binary operation $\Diamond : [0,1]\times[0,1] \to [0,1]$ is referred to as a continuous t-conorm [$\mathcal{CTCN}$] if it meets the following conditions:*
(i) $\Diamond$ is associative and commutative;
(ii) $\Diamond$ is continuous;
(iii) $\mathfrak{p}\Diamond 0 = \mathfrak{p}$, $\forall\, \mathfrak{p} \in [0, 1]$;
(iv) $\mathfrak{p}\Diamond\mathfrak{r} \leq \mathfrak{q}\Diamond\mathfrak{s}$ *whenever* $\mathfrak{p} \leq \mathfrak{q}$ *and* $\mathfrak{r} \leq \mathfrak{s}$, $\forall\, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s} \in [0, 1]$.

Example 2. *[15]The following are examples of $\mathcal{CTCN}$*
(i) $\mathfrak{p}\Diamond\mathfrak{q} = \max\{\mathfrak{p}, \mathfrak{q}\}$.
(ii) $\mathfrak{p}\Diamond\mathfrak{q} = \min\{\mathfrak{p} + \mathfrak{q}, 1\}$, *where* $\mathfrak{p}, \mathfrak{q} \in [0, 1]$.

Definition 3. *[16] Let Ξ be a non-empty fixed set. A neutrosophic set $\aleph$ is defined as an object of the form: $\aleph = \{\varrho, \mathcal{A}_{\aleph}(\varrho), \mathcal{B}_{\aleph}(\varrho), \mathcal{C}_{\aleph}(\varrho)\}$, where*
(i) $\mathcal{A}_{\aleph}(\varrho)$: Degree of membership of ϱ to the set $\aleph$,
(ii) $\mathcal{B}_{\aleph}(\varrho)$: Degree of indeterminacy,
(iii) $\mathcal{C}_{\aleph}(\varrho)$: Degree of non-membership.

Definition 4. *[20] Let $\Xi \neq \emptyset$. For a six tuple $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \star, \Diamond)$, where $\star$ is a $\mathcal{CTN}$, $\Diamond$ is a $\mathcal{CTCN}$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are neutrosophic sets on $\Xi \times \Xi \times (0, \infty)$, if $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \star, \Diamond)$ enjoys the conditions listed below, for every $\varrho, \delta, \mathfrak{z} \in \Xi$ and $\tau, \iota > 0$,*
1. $0 \leq \mathcal{A}(\varrho, \delta, \tau) \leq 1; 0 \leq \mathcal{B}(\varrho, \delta, \tau) \leq 1; 0 \leq \mathcal{C}(\varrho, \delta, \tau) \leq 1$;
2. $\mathcal{A}(\varrho, \delta, \tau) + \mathcal{B}(\varrho, \delta, \tau) + \mathcal{C}(\varrho, \delta, \tau) \leq 3$;
3. $\mathcal{A}(\varrho, \delta, \tau) = 1 \Leftrightarrow \varrho = \delta$;
4. $\mathcal{A}(\varrho, \delta, \tau) = \mathcal{A}(\delta, \varrho, \tau)$;
5. $\mathcal{A}(\varrho, \mathfrak{z}, \tau + \iota) \geq \mathcal{A}(\varrho, \delta, \tau) \star \mathcal{A}(\delta, \mathfrak{z}, \iota)$;
6. $\mathcal{A}(\varrho, \delta, .)$ *is neutrosophic continuous from* $[0, \infty) \to [0, 1]$;
7. $\lim_{\tau\to\infty} \mathcal{A}(\varrho, \delta, \tau) = 1$;
8. $\mathcal{B}(\varrho, \delta, \tau) = 0 \Leftrightarrow \varrho = \delta$;
9. $\mathcal{B}(\varrho, \delta, \tau) = \mathcal{B}(\delta, \varrho, \tau)$;
10. $\mathcal{B}(\varrho, \mathfrak{z}, \tau + \iota) \leq \mathcal{B}(\varrho, \delta, \tau)\Diamond\mathcal{B}(\delta, \mathfrak{z}, \iota)$;
11. $\mathcal{B}(\varrho, \delta, .)$ *is neutrosophic continuous from* $[0, \infty) \to [0, 1]$;
12. $\lim_{\tau\to\infty} \mathcal{B}(\varrho, \delta, \tau) = 0$;
13. $\mathcal{C}(\varrho, \delta, \tau) = 0 \Leftrightarrow \varrho = \delta$;
14. $\mathcal{C}(\varrho, \delta, \tau) = \mathcal{C}(\delta, \varrho, \tau)$;
15. $\mathcal{C}(\varrho, \mathfrak{z}, \tau + \iota) \leq \mathcal{C}(\varrho, \delta, \tau)\Diamond\mathcal{C}(\delta, \mathfrak{z}, \iota)$;
16. $\mathcal{C}(\varrho, \delta, .)$ *is neutrosophic continuous from* $[0, \infty) \to [0, 1]$;
17. $\lim_{\tau\to\infty} \mathcal{C}(\varrho, \delta, \tau) = 0$.
Then $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \star, \Diamond)$ is named to be a neutrosophic metric space.

Example 3. *Let Ξ be a non-void set. Define the operations $\varrho_1 * \varrho_2 = \min\{\varrho_1, \varrho_2\}$, $\varrho_1 \Diamond \varrho_2 = \max\{\varrho_1, \varrho_2\}$. Consider the neutrosophic sets $\mathcal{A}(\varrho, \delta, \tau) = \frac{\tau}{\tau + d(\varrho,\delta)}$, where d is a metric on Ξ, $\mathcal{B}(\varrho, \delta, \tau) = \frac{d(\varrho,\delta)}{\tau + d(\varrho,\delta)}$ and $\mathcal{C}(\varrho, \delta, \tau) = \frac{d(\varrho,\delta)}{\tau + 2d(\varrho,\delta)}$. Then, $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ is a neutrosophic metric space.*

Remark 1. *[12] The class $\mathcal{F}$ originates from Wardowski's notion of F-contractions [9], and was further adapted to fuzzy metric frameworks by Das et.al[12]. Here, $\mathfrak{f}$ acts as a control function, while $\lambda \in (0, 1]$ provides a scaling parameter that determines the contraction strength. Formally, we consider*

$$\mathcal{F} = \left\{ \mathfrak{f} : [0,1] \to [0,1] \;\middle|\; \begin{array}{l} \text{(i) } \mathfrak{f} \text{ is strictly increasing on } [0,1), \\ \text{(ii) } \forall \{\tau_n\} \subset [0,1],\ \tau_n \to 1 \ \Leftrightarrow\ \mathfrak{f}(\tau_n) \to 1 \end{array} \right\}.$$

Example 4. *[12] Below are examples of functions that are members of $\mathcal{F}$:*
(i) $\mathfrak{f}(\varrho) = \varrho^i$, $\forall\, \varrho \in [0, 1], i \in \mathbb{N}$.
(ii) $\mathfrak{f}(\varrho) = \sqrt{\varrho}$, $\forall\ 0 \leq \varrho \leq 1$.

Definition 5. *Let Ξ be a non empty set and $\mathcal{A}, \mathcal{B}, \mathcal{C} : \Xi \times \Xi \times (0, \infty) \to [0, 1]$ be a neutrosophic sets on Ξ and $\star$ is a $\mathcal{CTN}$, $\Diamond$ is a $\mathcal{CTCN}$.*
Suppose there exists $(\mathfrak{f}, \lambda) \in \mathcal{F} \times (0, 1]$ such that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ fulfill the following conditions:
$(\mathcal{N}1)$ $0 \leq \mathcal{A}(\varrho, \delta, \tau) \leq 1; 0 \leq \mathcal{B}(\varrho, \delta, \tau) \leq 1; 0 \leq \mathcal{C}(\varrho, \delta, \tau) \leq 1;$
$(\mathcal{N}2)$ $\mathcal{A}(\varrho, \delta, \tau) + \mathcal{B}(\varrho, \delta, \tau) + \mathcal{C}(\varrho, \delta, \tau) \leq 3$
$(\mathcal{N}3)$ $\mathcal{A}(\varrho, \delta, \tau) = 1, \forall \tau > 0$ *iff* $\varrho = \delta$;
$(\mathcal{N}4)$ $\mathcal{A}(\varrho, \delta, \tau) = \mathcal{A}(\delta, \varrho, \tau),\ \forall\, \varrho, \delta \in \Xi, \tau > 0;$
$(\mathcal{N}5)$ *For every $(\varrho, \delta) \in \Xi \times \Xi$, for all $k \in \mathbb{N}, k \geq 2$ and for all $\{\vartheta_i\}_i^k \subseteq \Xi$ with $\vartheta_1 = \varrho$ and $\vartheta_N = \delta$, we have*

$$(\mathfrak{f}(\mathcal{A}(\varrho, \delta, \tau)))^\lambda \geq \mathfrak{f}\left(\mathcal{A}(\vartheta_1, \vartheta_2, \tau_1) \star \mathcal{A}(\vartheta_2, \vartheta_3, \tau_2) \star \cdots \star \mathcal{A}(\vartheta_{N-1}, \vartheta_N, \tau_{k-1})\right);$$

$(\mathcal{N}6)$ *$\mathcal{A}(\varrho, \delta, .)$ is neutrosophic continuous from $[0, \infty) \to [0, 1]$;*
$(\mathcal{N}7)$ $\lim_{\tau \to \infty} \mathcal{A}(\varrho, \delta, \tau) = 1;$
$(\mathcal{N}8)$ $\mathcal{B}(\varrho, \delta, \tau) = 0, \forall \tau > 0$ *iff* $\varrho = \delta$;
$(\mathcal{N}9)$ $\mathcal{B}(\varrho, \delta, \tau) = \mathcal{B}(\delta, \varrho, \tau),\ \forall\, \delta, \varrho \in \Xi, \tau > 0;$
$(\mathcal{N}10)$ *For every $(\varrho, \delta) \in \Xi \times \Xi$, for all $k \in \mathbb{N}, k \geq 2$ and for every $\{\vartheta_i\}_i^k \subseteq \Xi$ with $\vartheta_1 = \varrho$ and $\vartheta_N = \delta$, we have*

$$(\mathfrak{f}(1 - \mathcal{B}(\varrho, \delta, \tau)))^\lambda \geq \mathfrak{f}\left(1 - \{\mathcal{B}(\vartheta_1, \vartheta_2, \tau_1) \Diamond \mathcal{B}(\vartheta_2, \vartheta_3, \tau_2) \Diamond \cdots \Diamond \mathcal{B}(\vartheta_{N-1}, \vartheta_N, \tau_{k-1})\}\right);$$

$(\mathcal{N}11)$ *$\mathcal{B}(\varrho, \delta, .)$ is neutrosophic continuous from $[0, \infty) \to [0, 1]$;*
$(\mathcal{N}12)$ $\lim_{\tau \to \infty} \mathcal{B}(\varrho, \delta, \tau) = 0;$
$(\mathcal{N}13)$ $\mathcal{C}(\varrho, \delta, \tau) = 0, \forall \tau > 0$ *iff* $\varrho = \delta$;
$(\mathcal{N}14)$ $\mathcal{C}(\varrho, \delta, \tau) = \mathcal{C}(\delta, \varrho, \tau),\ \forall\, \varrho, \delta \in \Xi, \tau > 0;$
$(\mathcal{N}15)$ *For every $(\varrho, \delta) \in \Xi \times \Xi$, for all $k \in \mathbb{N}, k \geq 2$ and for all $\{\vartheta_i\}_i^k \subseteq \Xi$ with $\vartheta_1 = \varrho$ and $\vartheta_N = \delta$, we have*

$$(\mathfrak{f}(1 - \mathcal{C}(\varrho, \delta, \tau)))^\lambda \geq \mathfrak{f}\left(1 - \{\mathcal{C}(\vartheta_1, \vartheta_2, \tau_1) \Diamond \mathcal{C}(\vartheta_2, \vartheta_3, \tau_2) \Diamond \cdots \Diamond \mathcal{C}(\vartheta_{k-1}, \vartheta_k, \tau_{k-1})\}\right)$$

where $\tau = \tau_1 + \tau_2 + \ldots + \tau_{k-1}; \tau_i > 0$ *for* $i = 1, 2, \cdots, (k-1)$*;*
$(\mathcal{N}16)$ $\mathcal{C}(\varrho, \delta, .)$ *is neutrosophic continuous from* $[0, \infty) \to [0, 1]$ *;*
$(\mathcal{N}17)$ $\lim_{\tau \to \infty} \mathcal{C}(\varrho, \delta, \tau) = 0$*;*
Then $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{F}, \lambda, \star, \Diamond)$ *is referred to as a neutrosophic* $\mathcal{F}$*-metric space* $[\mathcal{NFMS}]$.

Example 5. *Let* $\Xi = \mathbb{R}$, $\mathfrak{a} \star \mathfrak{b} = min\{\mathfrak{a}, \mathfrak{b}\}, \mathfrak{a} \Diamond \mathfrak{b} = max\{\mathfrak{a}, \mathfrak{b}\}$ *and define the functions* $\mathcal{A}, \mathcal{B}, \mathcal{C} : \Xi \times \Xi \times (0, \infty) \to [0, 1]$ *by*

$$\mathcal{A}(\varrho, \delta, \tau) = e^{\frac{-|\varrho - \delta|}{\tau}}, \mathcal{B}(\varrho, \delta, \tau) = 1 - e^{\frac{-|\varrho - \delta|}{\tau}}, \mathcal{C}(\varrho, \delta, \tau) = 1 - e^{\frac{-|\varrho - \delta|}{2\tau}},$$

$\forall\ (\varrho, \delta) \in \Xi \times \Xi$ *and* $\tau > 0$*. Then* $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathfrak{f}, \lambda, \star, \Diamond)$ *is a* $\mathcal{NFMS}$*.*

Proposition 1. *Let* $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathfrak{f}, \lambda, \star, \Diamond)$ *be a* $\mathcal{NFMS}$*,* $\{\varrho_i\} \subseteq \Xi$ *be a sequence and* $\varrho \in \Xi$*. Then*
(i) $\{\varrho_i\}$ *is convergent to* ϱ *iff*

$$\lim_{i \to \infty} \mathcal{A}(\varrho_i, \varrho, \tau) = 1, \lim_{i \to \infty} \mathcal{B}(\varrho_i, \varrho, \tau) = 0, \lim_{i \to \infty} \mathcal{C}(\varrho_i, \varrho, \tau) = 0, \forall \tau > 0.$$

(ii) $\{\varrho_i\}$ *is Cauchy iff*

$$\lim_{j,i \to \infty} \mathcal{A}(\varrho_i, \varrho_j, \tau) = 1, \lim_{j,i \to \infty} \mathcal{B}(\varrho_i, \varrho_j, \tau) = 0, \lim_{j,i \to \infty} \mathcal{C}(\varrho_i, \varrho_j, \tau) = 0, \forall \tau > 0.$$

Definition 6. *Consider the class* Υ *consists of all functions* $\varphi : [0, 1] \to [0, 1]$ *that meet these criteria:*
(i) φ *is continuous and monotonically nondecreasing;*
(ii) For all $\tau \in (0, 1)$*, the inequality* $\varphi(\tau) > \tau$ *holds.*

Example 6. *Define a function* $\varphi : [0, 1] \to [0, 1]$ *by* $\varphi(\tau) = \frac{\tau}{\tau + s(1-\tau)}, \tau \in [0, 1]$ *where* $s \in (0, 1)$*. Then* $\varphi \in \Upsilon$*.*

Lemma 1. *If* $\varphi \in \Upsilon$ *then* $\varphi(1) = 1$*.*

Lemma 2. *If* $\varphi \in \Upsilon$ *then* $\lim_{i \to \infty} \varphi^i(\tau) = 1$ *for all* $\tau \in (0, 1)$*.*

3. Main Results

This section presents the main fixed point results in neutrosophic $\mathcal{F}$-metric spaces under generalized contraction conditions, extending related results in fuzzy settings.

Definition 7. *Consider the* $\mathcal{NFMS}$ $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathfrak{f}, \lambda, \star, \Diamond)$*. A mapping* $\mathcal{L} : \Xi \to \Xi$ *is called a neutrosophic* φ*-contraction mapping with respect to the function* $\varphi \in \Upsilon$ *if*

$$\begin{aligned} \mathcal{A}(\mathcal{L}\varrho, \mathcal{L}\delta, \tau) &\geq \varphi(\mathcal{A}(\varrho, \delta, \tau)), \\ 1 - \mathcal{B}(\mathcal{L}\varrho, \mathcal{L}\delta, \tau) &\geq \varphi(1 - \mathcal{B}(\varrho, \delta, \tau)), \\ 1 - \mathcal{C}(\mathcal{L}\varrho, \mathcal{L}\delta, \tau) &\geq \varphi(1 - \mathcal{C}(\varrho, \delta, \tau)) \end{aligned} \tag{1}$$

for all $\varrho, \delta \in \Xi$ *and* $\tau > 0$*.*

Theorem 1. *Let* $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathfrak{f}, \lambda, \star, \Diamond)$ *be a complete* $\mathcal{NFMS}$ *and* $\mathcal{L}$ *satisfying equation (1). Then* $\mathcal{L}$ *admits unique fixed point in* Ξ.

Proof. Let $\varrho \in \Xi$ and define $\varrho_i = \mathcal{L}^i \varrho, i \in \mathbb{N} \cup \{0\}$. If $\mathcal{L}(\varrho_r) = \varrho_r$ for some $r \in \mathbb{N} \cup \{0\}$ then the proof is complete. Otherwise, assume that $\mathcal{L}(\varrho_r) \neq \varrho_r \ \forall \ r \in \mathbb{N} \cup \{0\}$.
Since $\mathcal{L}$ satisfying equation (1) then the sequence $\{\varrho_i\}$ satisfies:

$$\begin{aligned}
\mathcal{A}(\varrho_{i+2}, \varrho_{i+1}, \tau) &\geq \varphi(\mathcal{A}(\varrho_{i+1}, \varrho_i, \tau)), \\
1 - \mathcal{B}(\varrho_{i+2}, \varrho_{i+1}, \tau) &\geq \varphi(1 - \mathcal{B}(\varrho_{i+1}, \varrho_i, \tau)), \\
1 - \mathcal{C}(\varrho_{i+2}, \varrho_{i+1}, \tau) &\geq \varphi(1 - \mathcal{C}(\varrho_{i+1}, \varrho_i, \tau)), \forall i \in \mathbb{N} \cup \{0\}, \tau > 0.
\end{aligned}$$

By iterating the above inequality, it follows that for every $j \geq 2$, we get

$$\begin{aligned}
\mathcal{A}(\varrho_{j+1}, \varrho_j, \tau) &\geq \varphi(\mathcal{A}(\varrho_j, \varrho_{j-1}, \tau)) \\
&\geq \varphi^2(\mathcal{A}(\varrho_{j-1}, \varrho_{j-2}, \tau)) \\
&\geq \cdots \geq \varphi^j(\mathcal{A}(\varrho_1, \varrho_0, \tau)), \\
1 - \mathcal{B}(\varrho_{j+1}, \varrho_j, \tau) &\geq \varphi(1 - \mathcal{B}(\varrho_j, \varrho_{j-1}, \tau)) \\
&\geq \varphi^2(1 - \mathcal{B}(\varrho_{j-1}, \varrho_{j-2}, \tau)) \\
&\geq \cdots \geq \varphi^j(1 - \mathcal{B}(\varrho_1, \varrho_0, \tau)) \\
1 - \mathcal{C}(\varrho_{j+1}, \varrho_j, \tau) &\geq \varphi(1 - \mathcal{C}(\varrho_j, \varrho_{j-1}, \tau)) \\
&\geq \varphi^2(1 - \mathcal{C}(\varrho_{j-1}, \varrho_{j-2}, \tau)) \\
&\geq \cdots \geq \varphi^j(1 - \mathcal{C}(\varrho_1, \varrho_0, \tau)).
\end{aligned}$$

Letting $j \to \infty$ and applying Lemma (2) to the above inequalities, we obtain

$$\lim_{j\to\infty} \mathcal{A}(\varrho_{j+1}, \varrho_j, \tau) \geq \lim_{j\to\infty} \varphi^j(\mathcal{A}(\varrho_1, \varrho_0, \tau)) = 1 \Rightarrow \lim_{j\to\infty} \mathcal{A}(\varrho_{j+1}, \varrho_j, \tau) = 1 \tag{2}$$

$$\lim_{j\to\infty} 1 - \mathcal{B}(\varrho_{j+1}, \varrho_j, \tau) \geq \lim_{j\to\infty} \varphi^j(1 - \mathcal{B}(\varrho_1, \varrho_0, \tau)) = 1 \Rightarrow \lim_{j\to\infty} \mathcal{B}(\varrho_{j+1}, \varrho_j, \tau) = 0 \tag{3}$$

$$\lim_{j\to\infty} 1 - \mathcal{C}(\varrho_{j+1}, \varrho_j, \tau) \geq \lim_{j\to\infty} \varphi^j(1 - \mathcal{C}(\varrho_1, \varrho_0, \tau)) = 1 \Rightarrow \lim_{j\to\infty} \mathcal{C}(\varrho_{j+1}, \varrho_j, \tau) = 0. \tag{4}$$

$\forall \tau > 0$. Continuing this process, we arrive at

$$\begin{aligned}
&\lim_{j\to\infty} \mathcal{A}(\varrho_{j+i}, \varrho_{j+i-1}, \tau) = 1, \\
&\lim_{j\to\infty} \mathcal{B}(\varrho_{j+i}, \varrho_{j+i-1}, \tau) = 0, \\
&\lim_{j\to\infty} \mathcal{C}(\varrho_{j+i}, \varrho_{j+i-1}, \tau) = 0 \ \forall \tau > 0, i \geq 1.
\end{aligned} \tag{5}$$

Next we show that $\{\varrho_r\}$ is a Cauchy sequence in $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathfrak{f}, \lambda, \star)$.
Now using the inequality $(\mathcal{N}5), (\mathcal{N}10), (\mathcal{N}15)$, for every $j \in \mathbb{N}$ and $p = 1, 2, 3, \ldots$, we get

$$(\mathfrak{f}(\mathcal{A}(\varrho_{j+p}, \varrho_j, \tau)))^\lambda$$

$$\geq \mathfrak{f}\left(\mathcal{A}\left(\varrho_{j+p}, \varrho_{j+p-1}, \frac{\tau}{p}\right) \star \cdots \star \mathcal{A}\left(\varrho_{j+2}, \varrho_{j+1}, \frac{\tau}{p}\right) \star \mathcal{A}\left(\varrho_{j+1}, \varrho_{j}, \frac{\tau}{p}\right)\right), \tag{6}$$

$$\left(\mathfrak{f}\left(1-\mathcal{B}\left(\varrho_{j+p}, \varrho_{j}, \tau\right)\right)\right)^{\lambda}$$
$$\geq \mathfrak{f}\left(1-\left\{\mathcal{B}\left(\varrho_{j+p}, \varrho_{j+p-1}, \frac{\tau}{p}\right) \Diamond \cdots \Diamond \mathcal{B}\left(\varrho_{j+2}, \varrho_{j+1}, \frac{\tau}{p}\right) \Diamond \mathcal{B}\left(\varrho_{j+1}, \varrho_{j}, \frac{\tau}{p}\right)\right\}\right) \tag{7}$$

$$\left(\mathfrak{f}\left(1-\mathcal{C}\left(\varrho_{j+p}, \varrho_{j}, \tau\right)\right)\right)^{\lambda}$$
$$\geq \mathfrak{f}\left(1-\left\{\mathcal{C}\left(\varrho_{j+p}, \varrho_{j+p-1}, \frac{\tau}{p}\right) \Diamond \cdots \Diamond \mathcal{C}\left(\varrho_{j+2}, \varrho_{j+1}, \frac{\tau}{p}\right) \Diamond \mathcal{C}\left(\varrho_{j+1}, \varrho_{j}, \frac{\tau}{p}\right)\right\}\right) \tag{8}$$

Applying the relation (5) on (6), (7) and (8) we get

$$\lim_{j\to\infty}\left\{\mathcal{A}\left(\varrho_{j+p}, \varrho_{j+p-1}, \frac{\tau}{p}\right) \star \cdots \star \mathcal{A}\left(\varrho_{j+1}, \varrho_{j}, \frac{\tau}{p}\right)\right\} = 1,$$
$$\lim_{j\to\infty}\left\{\mathcal{B}\left(\varrho_{j+p}, \varrho_{j+p-1}, \frac{\tau}{p}\right) \Diamond \cdots \Diamond \mathcal{B}\left(\varrho_{j+1}, \varrho_{j}, \frac{\tau}{p}\right)\right\} = 0 \text{ and}$$
$$\lim_{j\to\infty}\left\{\mathcal{C}\left(\varrho_{j+p}, \varrho_{j+p-1}, \frac{\tau}{p}\right) \Diamond \cdots \Diamond \mathcal{C}\left(\varrho_{j+1}, \varrho_{j}, \frac{\tau}{p}\right)\right\} = 0,\ \forall \tau > 0, p = 1, 2, 3, \ldots$$

It follows that

$$\lim_{j\to\infty}\left(\mathfrak{f}\left(\mathcal{A}\left(\varrho_{j+p}, \varrho_{j}, \tau\right)\right)\right)^{\lambda} \geq 1,$$
$$\lim_{j\to\infty}\left(\mathfrak{f}\left(1-\mathcal{B}\left(\varrho_{j+p}, \varrho_{j}, \tau\right)\right)\right)^{\lambda} \geq 1 \text{ and}$$
$$\lim_{j\to\infty}\left(\mathfrak{f}\left(1-\mathcal{C}\left(\varrho_{j+p}, \varrho_{j}, \tau\right)\right)\right)^{\lambda} \geq 1$$
$$\lim_{j\to\infty}\left(\mathfrak{f}\left(\mathcal{A}\left(\varrho_{j+p}, \varrho_{j}, \tau\right)\right)\right) = 1,$$
$$\lim_{j\to\infty}\left(\mathfrak{f}\left(1-\mathcal{B}\left(\varrho_{j+p}, \varrho_{j}, \tau\right)\right)\right) = 1 \text{ and}$$
$$\lim_{j\to\infty}\left(\mathfrak{f}\left(1-\mathcal{C}\left(\varrho_{j+p}, \varrho_{j}, \tau\right)\right)\right) = 1.$$

Therefore
$\lim_{j\to\infty}\mathcal{A}\left(\varrho_{j+p}, \varrho_{j}, \tau\right) = 1$, $\lim_{j\to\infty}\mathcal{B}\left(\varrho_{j+p}, \varrho_{j}, \tau\right) = 0$, and $\lim_{j\to\infty}\mathcal{C}\left(\varrho_{j+p}, \varrho_{j}, \tau\right) = 0\,\forall \tau > 0$.
This shows that $\{\varrho_j\}$ forms a Cauchy sequence in $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathfrak{f}, \lambda, \star, \Diamond)$.
Since Ξ is complete, so $\{\varrho_j\}$ converges to some $\vartheta \in \Xi$. Hence,

$$\lim_{j\to\infty}\mathcal{A}\left(\varrho_{j}, \vartheta, \tau\right) = 1, \lim_{j\to\infty}\mathcal{B}\left(\varrho_{j}, \vartheta, \tau\right) = 0, \text{ and } \lim_{j\to\infty}\mathcal{C}\left(\varrho_{j}, \vartheta, \tau\right) = 0\,\forall \tau > 0.$$

We now assert that ϑ is a fixed point of $\mathcal{L}$. Assume, for the sake of contradiction, that there exists $\tau_0 > 0$ such that

$$\mathcal{A}\left(\vartheta, \mathcal{L}\vartheta, \tau_0\right) < 1, \mathcal{B}\left(\vartheta, \mathcal{L}\vartheta, \tau_0\right) > 0 \,\text{and}\, \mathcal{C}\left(\vartheta, \mathcal{L}\vartheta, \tau_0\right) > 0. \tag{9}$$

Then we have,

$$(\mathfrak{f}(\mathcal{A}(\vartheta,\mathcal{L}\vartheta,\tau_0)))^{\lambda} \geq \mathfrak{f}\left(\mathcal{A}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\star\mathcal{A}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)\right), \tag{10}$$

$$(\mathfrak{f}(1-\mathcal{B}(\vartheta,\mathcal{L}\vartheta,\tau_0)))^{\lambda} \geq \mathfrak{f}\left(1-\left\{\mathcal{B}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\Diamond\mathcal{B}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)\right\}\right) \text{ and} \tag{11}$$

$$(\mathfrak{f}(1-\mathcal{C}(\vartheta,\mathcal{L}\vartheta,\tau_0)))^{\lambda} \geq \mathfrak{f}\left(1-\left\{\mathcal{C}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\Diamond\mathcal{C}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)\right\}\right) \ \forall\, j\in\mathbb{N}. \tag{12}$$

Again,

$$\begin{aligned}
\mathcal{A}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right) &\geq \varphi\left(\mathcal{A}\left(\varrho_{j-1},\vartheta,\frac{\tau_0}{2}\right)\right),\\
1-\mathcal{B}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right) &\geq \varphi\left(1-\mathcal{B}\left(\varrho_{j-1},\vartheta,\frac{\tau_0}{2}\right)\right) \text{ and}\\
1-\mathcal{C}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right) &\geq \varphi\left(1-\mathcal{C}\left(\varrho_{j-1},\vartheta,\frac{\tau_0}{2}\right)\right) \ \forall\, j\in\mathbb{N}\\
\Rightarrow \lim_{j\to\infty}\mathcal{A}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right) &\geq \lim_{j\to\infty}\varphi\left(\mathcal{A}\left(\varrho_{j-1},\vartheta,\frac{\tau_0}{2}\right)\right)=1,\\
\lim_{j\to\infty}(1-\mathcal{B}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)) &\geq \lim_{j\to\infty}\varphi\left(1-\mathcal{B}\left(\varrho_{j-1},\vartheta,\frac{\tau_0}{2}\right)\right)=1 \text{ and}\\
\lim_{j\to\infty}(1-\mathcal{C}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)) &\geq \lim_{j\to\infty}\varphi\left(1-\mathcal{C}\left(\varrho_{j-1},\vartheta,\frac{\tau_0}{2}\right)\right)=1 \text{(using Lemma (2))}\\
\Rightarrow \lim_{j\to\infty}\mathcal{A}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right) &=1, \lim_{j\to\infty}\mathcal{B}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)=0, \text{ and } \lim_{j\to\infty}\mathcal{C}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)=0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{j\to\infty}\mathcal{A}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\star\mathcal{A}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)=1, \lim_{j\to\infty}\mathcal{B}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\Diamond\mathcal{B}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)=0 \text{ and}\\
&\lim_{j\to\infty}\mathcal{C}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\Diamond\mathcal{C}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)=0\\
&\Rightarrow \lim_{j\to\infty}\mathfrak{f}\left(\mathcal{A}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\star\mathcal{A}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)\right)=1,\\
&\lim_{j\to\infty}\mathfrak{f}\left(1-\left\{\mathcal{B}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\Diamond\mathcal{B}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)\right\}\right)=1,\\
&\lim_{j\to\infty}\mathfrak{f}\left(1-\left\{\mathcal{C}\left(\vartheta,\varrho_j,\frac{\tau_0}{2}\right)\Diamond\mathcal{C}\left(\varrho_j,\mathcal{L}\vartheta,\frac{\tau_0}{2}\right)\right\}\right)=1. \text{ (Using remark (1))}\\
&\Rightarrow \lim_{j\to\infty}(\mathfrak{f}(\mathcal{A}(\vartheta,\mathcal{L}\vartheta,\tau_0)))^{\lambda}\geq 1, \lim_{j\to\infty}(\mathfrak{f}(1-\mathcal{B}(\vartheta,\mathcal{L}\vartheta,\tau_0)))^{\lambda}\geq 1 \text{ and}\\
&\lim_{j\to\infty}(\mathfrak{f}(1-\mathcal{C}(\vartheta,\mathcal{L}\vartheta,\tau_0)))^{\lambda}\geq 1\\
&\Rightarrow \mathcal{A}(\vartheta,\mathcal{L}\vartheta,\tau_0)=1, \mathcal{B}(\vartheta,\mathcal{L}\vartheta,\tau_0)=0, \mathcal{C}(\vartheta,\mathcal{L}\vartheta,\tau_0)=0.
\end{aligned}$$

Hence, we reach a contradiction with (9), implying that ϑ must be a fixed point of $\mathcal{L}$. To establish that this fixed point is unique, assume there exists $\nu\in\Xi$ such that $\nu\neq\vartheta$ and $\mathcal{L}(\nu)=\nu$. Because $\nu\neq\vartheta$, there exists $\mathfrak{s}>0$ for which

$\mathcal{A}(\vartheta,\nu,\mathfrak{s}) < 1, \mathcal{B}(\vartheta,\nu,\mathfrak{s}) > 0$ and $\mathcal{C}(\vartheta,\nu,\mathfrak{s}) > 0$.
Under this assumption, we have

$$\begin{aligned}
&\mathcal{A}(\vartheta,\nu,\mathfrak{s}) = \mathcal{A}(\mathcal{L}\vartheta,\mathcal{L}\nu,\mathfrak{s}) \geq \varphi(\mathcal{A}(\vartheta,\nu,\mathfrak{s})) > \mathcal{A}(\vartheta,\nu,\mathfrak{s}),\\
&1-\mathcal{B}(\vartheta,\nu,\mathfrak{s}) = 1-\mathcal{B}(\mathcal{L}\vartheta,\mathcal{L}\nu,\mathfrak{s}) \geq \varphi(1-\mathcal{B}(\vartheta,\nu,\mathfrak{s})) > 1-\mathcal{B}(\vartheta,\nu,\mathfrak{s}) \text{ and}\\
&1-\mathcal{C}(\vartheta,\nu,\mathfrak{s}) = 1-\mathcal{C}(\mathcal{L}\vartheta,\mathcal{L}\nu,\mathfrak{s}) \geq \varphi(1-\mathcal{C}(\vartheta,\nu,\mathfrak{s})) > 1-\mathcal{C}(\vartheta,\nu,\mathfrak{s})
\end{aligned}$$

which is impossible. Therefore, $\mathcal{L}$ admits exactly one fixed point in Ξ.

Example 7. *Let $\Xi = \mathbb{R}$. Define $\mathcal{A},\mathcal{B},\mathcal{C}$ by $\mathcal{A}(\varrho,\delta,\tau) = e^{\frac{-|\varrho-\delta|}{\tau}}, \mathcal{B}(\varrho,\delta,\tau) = 1-e^{\frac{-|\varrho-\delta|}{\tau}}$ and $\mathcal{C}(\varrho,\delta,\tau) = 1-e^{\frac{-|\varrho-\delta|}{2\tau}}$. Then $(\Xi,\mathcal{A},\mathcal{B},\mathcal{C},\mathfrak{f},\lambda,\star,\Diamond)$ is a $\mathcal{NFMS}$. Consider the mappings by $\mathcal{L}(\varrho) = \frac{\varrho}{4}\ \forall\ \varrho \in \Xi$ and $\varphi(\tau) = \frac{\tau}{2}\ \forall\ \tau \in [0,1]$.*
Then we have
$\mathcal{A}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau) = \mathcal{A}\left(\frac{\varrho}{4},\frac{\delta}{4},\tau\right) = e^{\frac{-|\varrho-\delta|}{4t}}$ *and* $\varphi(\mathcal{A}(\varrho,\delta,\tau)) = \frac{\mathcal{A}(\varrho,\delta,\tau)}{2} = \frac{e^{\frac{-|\varrho-\delta|}{\tau}}}{2}$
$\Rightarrow \mathcal{A}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau) \geq \varphi(\mathcal{A}(\varrho,\delta,\tau))$.
Similarly,
$1-\mathcal{B}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau) = 1-\mathcal{B}\left(\frac{\varrho}{4},\frac{\delta}{4},\tau\right) = e^{\frac{-|\varrho-\delta|}{4t}}$ *and* $\varphi(1-\mathcal{B}(\varrho,\delta,\tau)) = \frac{1-\mathcal{B}(\varrho,\delta,\tau)}{2} = \frac{e^{\frac{-|\varrho-\delta|}{\tau}}}{2}$
$\Rightarrow 1-\mathcal{B}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau) \geq \varphi(1-\mathcal{B}(\varrho,\delta,\tau))$.
On the other hand,
$1-\mathcal{C}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau) = 1-\mathcal{C}\left(\frac{\varrho}{4},\frac{\delta}{4},\tau\right) = e^{\frac{-|\varrho-\delta|}{8t}}$ *and* $\varphi(1-\mathcal{C}(\varrho,\delta,\tau)) = \frac{1-\mathcal{C}(\varrho,\delta,\tau)}{2} = \frac{e^{\frac{-|\varrho-\delta|}{2\tau}}}{2}$
$\Longrightarrow 1-\mathcal{C}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau) \geq \varphi(1-\mathcal{C}(\varrho,\delta,\tau))$.
Hence, $\mathcal{L}$ satisfies equation (1). So by Theorem (1), $\mathcal{L}$ has a unique fixed point in Ξ which is $\varrho = 0$. The following figures which depict the behavior of contraction mapping.

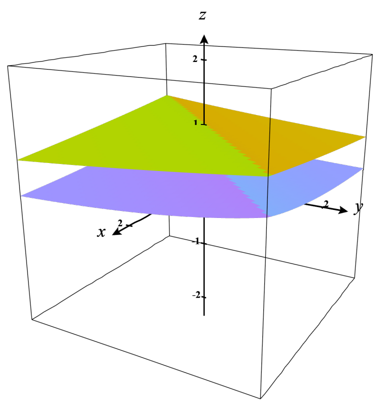


Figure 1: Figure presents the graphical depiction of the inequality $\mathcal{A}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau) \geq \varphi(\mathcal{A}(\varrho,\delta,\tau))$, wherein the left-hand side is illustrated in blue, while the right-hand side is illustrated in yellow.

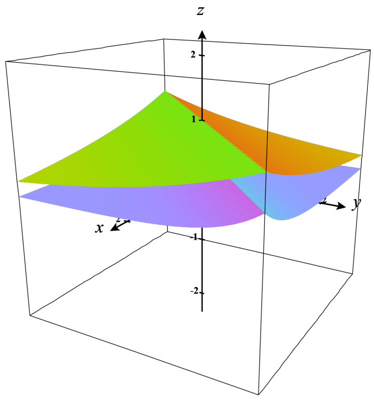


Figure 2: Figure presents the graphical depiction of the inequality $1-\mathcal{B}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau)\geq\varphi(1-\mathcal{B}(\varrho,\delta,\tau))$, wherein the left-hand side is illustrated in blue, while the right-hand side is illustrated in yellow.

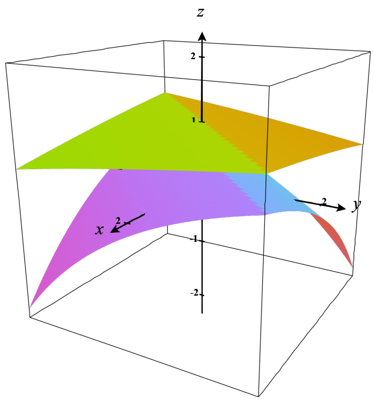


Figure 3: Figure presents the graphical depiction of the inequality $1-\mathcal{C}(\mathcal{L}\varrho,\mathcal{L}\delta,\tau)\geq\varphi(1-\mathcal{C}(\varrho,\delta,\tau))$,,wherein the left-hand side is illustrated in blue, while the right-hand side is illustrated in yellow.

4. Application

Fixed-point theory is fundamental in many disciplines, such as optimization, control systems, and nonlinear analysis. A remarkable application can be found in satellite web coupling, which aims to enhance the stability and performance of satellite networks utilized for communication, navigation, and remote sensing. By leveraging fixed-point techniques, researchers have developed algorithms that refine satellite trajectory optimization and improve the stability of inter-satellite connections.

Drawing on the demonstrated effectiveness of fixed-point techniques in tackling practical problems, this work applies Theorem (1) to solve a satellite web coupling problem. This problem considers a thin sheet that connects two cylindrical satellites, forming a web-like structure. The thermal radiation transfer between the satellites via this web leads to a

nonlinear boundary value problem governing the temperature distribution.
The governing differential equation for the radiation temperature is given by

$$-\frac{d^2\varpi}{d\tau^2} = \mu\varpi^4, 0 < \tau < a, a < 1, \varpi(0) = \varpi(a) = 0 \tag{13}$$

In this setting, $\varpi(\tau)$ denotes the radiation temperature at the position $\tau \in [0, a]$. The parameter

$$\mu = \frac{2al^2K^3}{\zeta h} > 0$$

is a positive, dimensionless constant, where K is the absolute temperature of both satellites and the web surface is assumed to radiate at absolute zero. Here, l represents the distance between the satellites, a is a positive constant describing the radiative characteristics of the web surface, the factor 2 arises from radiation emitted from both its upper and lower faces, ζ is the thermal conductivity, and h denotes the web thickness.
We now recall the equivalent integral equation:

$$\varpi(\tau) = 1 - \mu \int_0^a \mathfrak{Q}(\tau, \zeta)\varpi^4(\zeta)d\zeta,$$

where a<1,$\mathfrak{Q}(\tau, \zeta)$ denotes the associated Green's function, given by

$$\mathfrak{Q}(\lambda, \zeta) = \begin{cases} \lambda(1-\zeta), & 0 < \lambda < \zeta \\ \zeta(1-\lambda), & \zeta < \lambda < a \end{cases}$$

Let $\Xi = C[0, a]$, $a < 1$ be the space of all real valued continuous functions on $[0, a]$. We efine the mappings $\mathcal{A}, \mathcal{B}, \mathcal{C} : \Xi \times \Xi \times (0, \infty) \to [0, a]$ as follows:

$$\mathcal{A}(\zeta, \eta, \tau) = e^{-\frac{\sup_{s\in[0,a]} |\zeta(s) - \eta(s)|^2}{\tau}}, \mathcal{B}(\zeta, \eta, \tau) = 1 - e^{-\frac{\sup_{s\in[0,a]} |\zeta(s) - \eta(s)|^2}{\tau}} \text{ and}$$
$$\mathcal{C}(\zeta, \eta, \tau) = 1 - e^{-\frac{\sup_{s\in[0,a]} |\zeta(s) - \eta(s)|^2}{2\tau}}$$

for all $\zeta, \eta \in X$ and $\tau > 0$. With these, the structure $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathfrak{f}, \lambda, \star, \diamondsuit)$ forms a complete $\mathcal{NFMS}$, where $\mathfrak{a} \star \mathfrak{b} = \min\{\mathfrak{a}, \mathfrak{b}\}, \mathfrak{a}\diamondsuit\mathfrak{b} = \max\{\mathfrak{a}, \mathfrak{b}\}$.

Theorem 2. *Consider the complete $\mathcal{NFMS}$ ($\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathfrak{f}, \lambda, \star, \diamondsuit$) described earlier. Assume that the boundary value problem satisfies the inequality*

$$\sup_{\zeta\in[0,a]} \left|\left(\varpi^2(\zeta) + v^2(\zeta)\right)\left(\varpi(\zeta) + v(\zeta)\right)\right| \le \frac{k}{\mu} \text{ where } k \in (0, 4)$$

Under these conditions the problem (I) admits a unique solution.
Proof. Define mapping $\mathfrak{H} : \Xi \to \Xi$ by

$$\mathfrak{H}(\varpi(\tau)) = 1 - \mu \int_0^a \mathfrak{Q}(\tau, \zeta)\varpi^4(\zeta)d\zeta, \zeta \in [0, a], a < 1.$$

It is evident that solving problem (I) is equivalent to finding a fixed point of $\mathfrak{H}$. *Now for any* $\varpi, v \in \Xi$ *and* $\tau \in [0, a]$,

$$\begin{aligned}
|\mathfrak{H}\varpi(\tau) - \mathfrak{H}v(\tau)|^2 &= \mu^2 \left| \int_0^a \left(\varpi^4(\zeta) - v^4(\zeta)\right) \mathfrak{Q}(\tau, \zeta) d\zeta \right|^2 \\
&= \mu^2 \left| \int_0^a \left\{ \left(\varpi^2(\zeta) + v^2(\zeta)\right) (\varpi(\zeta) + v(\zeta))(\varpi(\zeta) - v(\zeta)) \right\} \mathfrak{Q}(\tau, \zeta) d\zeta \right|^2 \\
&\leq \mu^2 \sup_{\zeta \in [0,a]} \left| \left(\varpi^2(\zeta) + v^2(\zeta)\right) (\varpi(\zeta) + v(\zeta)) \right|^2 \left| \int_0^a (\varpi(\zeta) - v(\zeta)) \mathfrak{Q}(\tau, \zeta) d\zeta \right|^2 \\
&\leq \mu^2 \frac{k^2}{\mu^2} \sup_{\zeta \in [0,a]} |\varpi(\zeta) - v(\zeta)|^2 \left| \int_0^a \mathfrak{Q}(\tau, \zeta) d\zeta \right|^2 \\
&\leq k^2 \sup_{\zeta \in [0,a]} |\varpi(\zeta) - v(\zeta)|^2 \sup_{\tau \in [0,a]} \left| \int_0^a \mathfrak{Q}(\tau, \zeta) d\zeta \right|^2 \\
&= \frac{k^2}{16} \cdot \sup_{\zeta \in [0,a]} |\varpi(\zeta) - v(\zeta)|^2 \\
&= \frac{1}{\beta} \cdot \sup_{\zeta \in [0,a]} |\varpi(\zeta) - v(\zeta)|^2 \text{ where } \frac{1}{\beta} = \frac{k^2}{16} \in (0, 1), \beta > 0
\end{aligned}$$

It follows that, $\sup_{\tau \in [0,a]} |\mathfrak{H}\varpi(\tau) - \mathfrak{H}v(\tau)|^2 \leq \frac{1}{\beta} \cdot \sup_{\zeta \in [0,a]} |\varpi(\zeta) - v(\zeta)|^2$
Therefore $e^{-\{\sup_{\tau \in [0,a]} |\mathfrak{H}\varpi(\tau) - \mathfrak{H}v(\tau)|^2\}} \geq e^{-\frac{1}{\beta} \cdot \sup_{\zeta \in [0,a]} |\varpi(\zeta) - v(\zeta)|^2}$.
Let $\varphi(\tau) = \tau^{\frac{1}{\beta}} \, \forall \, \tau \in [0, a]$ *where* $\frac{1}{\beta} = \frac{k^2}{16} \in (0, 1)$.
Under this definition, the preceding inequality becomes

$$\begin{aligned}
\mathcal{A}(\mathfrak{H}\varpi, \mathfrak{H}v, \tau) &\geq (\mathcal{A}(\varpi, v, \tau))^{\frac{1}{\beta}}, \\
1 - \mathcal{B}(\mathfrak{H}\varpi, \mathfrak{H}v, \tau) &\geq (1 - \mathcal{B}(\varpi, v, \tau))^{\frac{1}{\beta}} \\
1 - \mathcal{C}(\mathfrak{H}\varpi, \mathfrak{H}v, \tau) &\geq (1 - \mathcal{C}(\varpi, v, \tau))^{\frac{1}{\beta}} \, \forall \, \tau > 0 \\
\Rightarrow \mathcal{A}(\mathfrak{H}\varpi, \mathfrak{H}v, \tau) &\geq \varphi(\mathcal{A}(\varpi, v, \tau)), \\
1 - \mathcal{B}(\mathfrak{H}\varpi, \mathfrak{H}v, \tau) &\geq \varphi(1 - \mathcal{B}(\varpi, v, \tau)), \\
1 - \mathcal{C}(\mathfrak{H}\varpi, \mathfrak{H}v, \tau) &\geq \varphi(1 - \mathcal{C}(\varpi, v, \tau)) \, \forall \, \tau > 0.
\end{aligned}$$

Hence, the mapping $\mathfrak{H}$ *satisfies the hypotheses of Theorem (1), implying that* $\mathfrak{H}$ *possesses a unique invariant point in* Ξ. *Therefore, the boundary value problem (I) admits a solution in* Ξ.

Example 8. *Consider the nonlinear integral equation :*

$$\varrho(\xi) = \xi^2 + \int_0^a \frac{\varrho(\Upsilon)}{1 + \xi\Upsilon} d\Upsilon, \xi \in [0, a], a < 1 \tag{14}$$

where the unknown function $\varrho : [0, a] \to \mathbb{R}$ *describes the signal coupling intensity between satellite nodes depending on the normalized distance* ξ*. Define the operator*

$$(\mathcal{L}\varrho)(\xi) = \xi^2 + \int_0^a \frac{\varrho(\Upsilon)}{1 + \xi\Upsilon}\, d\Upsilon.$$

We set up the neutrosophic $\mathcal{F}$*-metric space* $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, f, \lambda, \star, \diamond)$*, where for any* $\varrho, \nu \in \Xi$ *and* $\tau > 0$*,*

$$\mathcal{A}(\varrho, \nu, \tau) = \sup_{\xi\in[0,a]} \min\left\{1, \frac{|\varrho(\xi) - \nu(\xi)|}{\tau}\right\},$$
$$\mathcal{B}(\varrho, \nu, \tau) = \sup_{\xi\in[0,a]} \max\left\{0, 1 - \frac{|\varrho(\xi) - \nu(\xi)|}{\tau}\right\},$$
$$\mathcal{C}(\varrho, \nu, \tau) = \sup_{\xi\in[0,a]} \max\left\{0, 1 - \frac{|\varrho(\xi) - \nu(\xi)|}{2\tau}\right\},$$

and define the contraction function $\varphi : [0, 1) \to [0, 1)$ *by* $\varphi(\tau) = a\tau$*. Contraction Verification: For any* $\varrho, \nu \in \Xi$ *and* $\tau > 0$*, we compute*

$$\mathcal{A}(\mathcal{L}\varrho, \mathcal{L}\nu, \tau) = \sup_{\xi\in[0,1]} \min\left\{1, \frac{|(\mathcal{L}\varrho)(\xi) - (\mathcal{L}\nu)(\xi)|}{\tau}\right\}.$$

Since

$$|(\mathcal{L}\varrho)(\xi) - (\mathcal{L}\nu)(\xi)| = \left|\int_0^a \frac{\varrho(\Upsilon) - \nu(\Upsilon)}{1 + \xi\Upsilon}\, d\lambda\right| \leq \int_0^a |\varrho(\Upsilon) - \nu(\Upsilon)|\, d\lambda,$$

we have

$$\mathcal{A}(\mathcal{L}\varrho, \mathcal{L}\nu, \tau) \leq a\, \mathcal{A}(\varrho, \nu, \tau) = \varphi\big(\mathcal{A}(\varrho, \nu, \tau)\big).$$

Similarly, one can show
$1 - \mathcal{B}(\mathcal{L}\varrho, \mathcal{L}\nu, \tau) \geq \varphi\big(1 - \mathcal{B}(\varrho, \nu, \tau)\big), 1 - \mathcal{C}(\mathcal{L}\varrho, \mathcal{L}\nu, \tau) \geq \varphi\big(1 - \mathcal{C}(\varrho, \nu, \tau)\big).$
i.e., the operator $\mathcal{L}$ *satisfies equation (1). Hence, Theorem (1) guarantees the existence and uniqueness of the fixed point* $\varrho^*(\xi)$ *of* $\mathcal{L}$ *in* Ξ*, which is the unique solution of the integral equation. We approximate the solution using successive Picard iterations:*

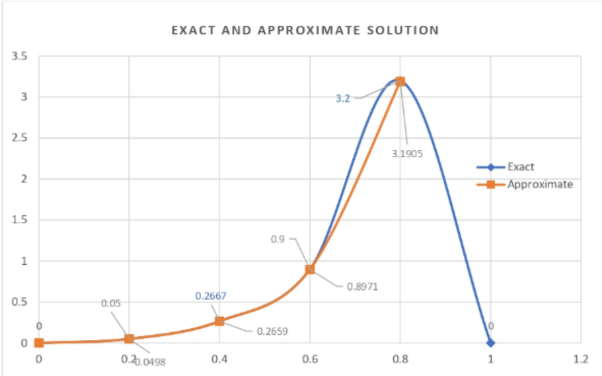
$$\varrho_0(\xi) = 0, \varrho_{i+1}(\xi) = \mathcal{L}\varrho_n(\xi),$$

and compare it with the exact solution $\varrho^*(\xi) = \frac{\xi^2}{1-\xi}$ *(approximated numerically).*

Table 1: Numerical result for example (8)

ξ_i	Exact $\varrho^*(\xi_i)$	Approx. $\varrho_{\text{approx}}(\xi_i)$	Error
0.00	0.00000	0.00000	0.00000
0.20	0.0500	0.0498	0.0002
0.40	0.2667	0.2659	0.0008
0.60	0.9000	0.8971	0.0029
0.80	3.2000	3.1905	0.0095

The following figure illustrates the difference between the exact and approximate solution.



5. Conclusion

This work has extended fixed point theory to neutrosophic $\mathcal{F}$-metric spaces, thereby generalizing classical results to a broader analytical framework. The proposed theorem's applicability has been demonstrated through the modeling of a satellite web coupling problem, with illustrative examples and graphical analyses reinforcing the theoretical findings. These results highlight the effectiveness of neutrosophic metric structures in addressing nonlinear problems under uncertainty and lay the groundwork for further research in this direction.

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