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A New Approach in Solving a Class of Control Problems with Fractional Objectives

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Abstract. In this paper, we introduce and investigate a pair of symmetric multi-dimensional variational fractional control problems. To this end, first, we formulate an updated concept of *pseudoinvexity* associated with multiple integral type functionals. Further, we establish a very important connection between the objective functionals of the studied symmetric models.

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1. Introduction

Optimization theory has recently undergone significant development. Many researchers have managed to formulate optimality criteria and conditions that have advanced the current level of knowledge. In this regard, Abdulaleem and Treanţă [1], under the framework of E-differentiability, established sufficient optimality conditions -notably KKT conditions for vector optimization problems where the objective and constraint functions satisfy (generalized) V-E-type I properties. Ahmad [2] extended symmetric duality theory –a form of duality where primal and dual problems have symmetric structures– to the domain of multiobjective fractional variational problems. Specifically, he formulated various duality results under assumptions of generalized invexity. Additionally, he explored the intrinsic relationships between these variational problems and their corresponding multiobjective fractional symmetric dual problems. Antczak et al. [3] studied a type of optimal control

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framework where the objective comprises multiple fractional terms that are not necessarily convex or differentiable. Bagri et al. [4] tackled a multi-dimensional vector fractional variational control problem while explicitly incorporating data uncertainty into the model. The study's core lies in formulating a Wolfe-type dual problem for the uncertain primal control problem and establishing robust duality results, particularly under convexity assumptions on the involved functionals. Bector and Husain [5] extended classical duality theory into the realm of multiobjective variational problems, framing variational calculus techniques within a multiobjective optimization context. Chandra et al. [6] introduced a pair of symmetric dual fractional programming problems, extending classical duality theory into the realm of fractional programming. The authors established appropriate duality theorems for these problems, contributing to the development of duality theory in optimization. In his works, Chen [7, 8] stated a pair of multiple-objective variational mixed integer models over cones and establishes duality theorems under separability and partial-invexity assumptions. The study provided weak, strong, converse, and self-duality results related to efficient solutions. Craven [9] explored the relationship between Lagrange multipliers and quasiduality in optimization problems. He provided conditions under which Lagrange multipliers exist and satisfy certain optimality conditions, contributing to the understanding of quasiduality in optimization theory. Dantzig et al. [10] introduced a symmetric duality framework for nonlinear programming problems, extending classical duality concepts to a broader class of problems. Das et al. [11] extended classical duality theories to set-valued fractional minimax programming problems by employing second-order contingent epi-derivatives. Dorn [12] addressed duality theory specifically for quadratic programming problems, which involve optimizing a quadratic objective function subject to linear constraints. Gulati et al. [13] extended the concept of symmetric duality to minimax variational problems, providing a dynamic generalization of symmetric and self-duality theorems in nonlinear mixed integer programming. Guo et al. [14, 15] introduced symmetric gH-derivative and formulated a duality theory in interval optimization. Jayswal et al. [16] delved into the duality theory of multi-dimensional variational control problems under data uncertainty. The authors formulated robust dual models, including Wolfe-type and Mond-Weir-type dual problems, and established duality results under generalized convexity assumptions. The study provided a comprehensive analysis of robust duality in the context of variational control problems with data uncertainty. Kim and Lee [17] extended the classical duality theory to multiobjective variational problems by introducing a symmetric duality framework under invexity conditions. The authors formulated dual pairs for vector problems and proved duality results under generalized invexity assumptions. These results provided a unified approach to analyzing multiobjective variational problems and contributed to the development of duality theory in optimization. Mond and Hanson [18] derived duality theorems under certain convexity and concavity conditions, providing necessary and sufficient optimality conditions for the primal and dual problems. Recently, Prasad et al. [19] formulated optimality conditions for an interval-valued vector problem. Saeed and Treanţă [20] employed the concept of convex multiple integral functionals, and the notion of robust weak efficient solutions to develop a new mathematical context for stating and proving the duality theorems. Schaible [21–23] presented a unified method for obtaining duality results for concave-convex fractional programs by transforming the original nonconvex programming problem into an equivalent convex program. The papers related known results by several authors and proved additional duality theorems, including converse duality theorems for nondifferentiable and quadratic fractional programs. Shi et al. [24] investigated a Lagrange-type dual theory associated with a fuzzy optimization model involving mixed constraints. Smart and Mond [25] investigated multiobjective variational problems -optimization problems defined over functions- where multiple objective functionals are considered simultaneously. This multiobjective nature required optimization under a partial ordering (e.g., Pareto optimality). Sun et al. [26] introduced a mixed-type robust dual problem for optimization models subject to uncertainty in both objective functions and constraints. This framework generalized classical duality to the robust setting. Treanţă et al. [27, 28] introduced a new class of constrained robust nonlinear optimization problems characterized by: (1) objective functionals defined via path-independent curvilinear integrals, derived from controlled second-order Lagrangians with uncertain data, (2) mixed constraints involving second-order partial derivatives and embedded uncertainty. The paper's core lied in formulating and analyzing three types of robust dual optimization models -Wolfe-type, Mond–Weir-type, and mixed-type- within this novel robust optimization context. Upadhyay et al. [29–31] investigated semi-infinite programming problems on Hadamard manifolds. Yang et al. [32] addressed the symmetric duality framework for a class of nondifferentiable multiobjective fractional programming problems (optimization problems where multiple fractional objectives are optimized without assuming differentiability).

In this paper, the authors introduce and investigate a pair of symmetric multidimensional variational fractional control problems. To this end, first, we formulate an updated concept of *pseudoinvexity* associated with multiple integral type functionals. Further, we establish a very important connection (see Theorem 1) between the objective functionals of the studied symmetric models. The main contributions associated with this article are: (a) the presence of two state and control variables in the considered functionals; (b) employing the multiple integrals as objective functionals; (c) building of suitable symmetric models and constraints; (d) introduction of pseudoinvexity notion for controlled functionals driven by multiple integrals.

The rest of the paper is structured as follows. Next section includes some preliminary ingredients which are useful is establishing the main results. Section 3 contains the main contributions of this paper. The symmetric multi-dimensional multiobjective dual programs are introduced and studied. Last section formulates the conclusions of this study.

2. Preliminaries

Let $K = [x_1, x_2] = [x_1^1, x_2^1] \times \cdots \times [x_1^p, x_2^p]$ be a multi-dimensional interval, with $x_1 = (x_1^1, \cdots, x_1^p),\ x_2 = (x_2^1, \cdots, x_n^p)$ two arbitrary points in $\mathbb{R}^p$. Consider the functions

$$\chi^\delta(x, \lambda(x), \lambda_\gamma(x), \pi(x), \omega(x), \omega_\xi(x), \zeta(x)),$$

$$\Upsilon^{\delta}(x, \lambda(x), \lambda_{\gamma}(x), \pi(x), \omega(x), \omega_{\xi}(x), \zeta(x)),$$

where $\lambda : K \to \mathbb{R}^n, \pi : K \to \mathbb{R}^s, \omega : K \to \mathbb{R}^m, \zeta : K \to \mathbb{R}^l$ (see $\lambda_\gamma := \frac{\partial \lambda}{\partial x^\gamma},\ \gamma = \overline{1,p}$ and $\omega_\xi := \frac{\partial \omega}{\partial x^\xi},\ \xi = \overline{1,p}$, as the partial derivatives for λ and ω with respect to the multi-variable $x = (x^1, \cdots, x^p) \in K \subset \mathbb{R}^p$), are functionals of C^2-class, for $\delta \in \{1, 2, \dots, q\}$. In the paper, $\chi^\delta_\lambda, \chi^\delta_{\lambda_\gamma}, \chi^\delta_\pi, \chi^\delta_\omega, \chi^\delta_{\omega_\xi}$ and χ^δ_ζ represent the gradients associated with the functional

$$\chi^{\delta}(x, \lambda(x), \lambda_{\gamma}(x), \pi(x), \omega(x), \omega_{\xi}(x), \zeta(x))$$

with respect to $\lambda, \lambda_\gamma, \pi, \omega, \omega_\xi$ and ζ, that is

$$\chi^\delta_\lambda = \left(\frac{\partial \chi^\delta}{\partial \lambda^1}, \dots, \frac{\partial \chi^\delta}{\partial \lambda^n}\right)^T, \quad \chi^\delta_{\lambda_\gamma} = \left(\frac{\partial \chi^\delta}{\partial \lambda^1_\gamma}, \dots, \frac{\partial \chi^\delta}{\partial \lambda^n_\gamma}\right)^T,$$

$$\chi^\delta_\omega = \left(\frac{\partial \chi^\delta}{\partial \omega^1}, \dots, \frac{\partial \chi^\delta}{\partial \omega^m}\right)^T, \quad \chi^\delta_{\omega_\xi} = \left(\frac{\partial \chi^\delta}{\partial \omega^1_\xi}, \dots, \frac{\partial \chi^\delta}{\partial \omega^m_\xi}\right)^T,$$

$$\chi^\delta_\pi = \left(\frac{\partial \chi^\delta}{\partial \pi^1}, \dots, \frac{\partial \chi^\delta}{\partial \pi^s}\right)^T, \quad \chi^\delta_\zeta = \left(\frac{\partial \chi^\delta}{\partial \zeta^1}, \dots, \frac{\partial \chi^\delta}{\partial \zeta^l}\right)^T,$$

for $\delta \in \{1, 2, \dots, q\}$. Similarly, $\Upsilon^\delta_\lambda, \Upsilon^\delta_{\lambda_\gamma}, \Upsilon^\delta_\pi, \Upsilon^\delta_\omega, \Upsilon^\delta_{\omega_\xi}$ and Υ^δ_ζ denote the gradient vectors of $\Upsilon^\delta(x, \lambda(x), \lambda_\gamma(x), \pi(x), \omega(x), \omega_\xi(x), \zeta(x))$ with respect to $\lambda, \lambda_\gamma, \pi, \omega, \omega_\xi$ and ζ.

Let $C^1(K, \mathbb{R}^n)$ and $C^1(K, \mathbb{R}^m)$ denote the families of functions λ and ω (*state* variables, continuously differentiable functions), respectively, having the associated norms $\|\lambda\| = \|\lambda\|_\infty + \|\lambda_\gamma\|_\infty$ and $\|\omega\| = \|\omega\|_\infty + \|\omega_\xi\|_\infty$, respectively. Also, let $C^0(K, \mathbb{R}^s)$ and $C^0(K, \mathbb{R}^l)$ denote the classes of functions π and ζ (*control* variables, continuous functions), respectively, with the corresponding uniform norm, as well.

In accordance with Bector and Husain [5], we state the multi-cost variational optimization problem:

$$(\textbf{Problem}) \quad \min_{(\lambda,\pi)} \left(\int_K F^1(x, \lambda(x), \pi(x)) dw, \dots, \int_K F^q(x, \lambda(x), \pi(x)) dw \right)$$

subject to

$$\lambda(x_1) = \alpha = \text{given}, \quad \lambda(x_2) = \beta = \text{given}, \quad (\text{or } \lambda|_{\partial K} = \text{given})$$

$$h(x, \lambda(x), \pi(x)) \leqq 0, \quad x \in K,$$

$$\psi^{\tau,\gamma}(x, \lambda(x), \pi(x)) - \lambda^\tau_\gamma(x) = 0, \quad x \in K,$$

where $F^\delta : C^1(K, \mathbb{R}^n) \times C^0(K, \mathbb{R}^s) \to \mathbb{R}, \delta = \overline{1,q},\ h^\iota : C^1(K, \mathbb{R}^n) \times C^0(K, \mathbb{R}^s) \to \mathbb{R}, \iota = \overline{1,r}$ and $\psi^{\tau,\gamma} : C^1(K, \mathbb{R}^n) \times C^0(K, \mathbb{R}^s) \to \mathbb{R}, \tau = \overline{1,n}, \gamma = \overline{1,p}$, are continuously differentiable functionals, $dw := dx^1 \cdots dx^p$.

Let G be the feasible solution set of (**Problem**),

$$G = \Big\{ (\lambda, \pi) \in C^1(K, \mathbb{R}^n) \times C^0(K, \mathbb{R}^s) \mid \lambda(x_1) = \alpha,\ \lambda(x_2) = \beta,\ h(x, \lambda(x), \pi(x)) \leqq 0,$$

$$\psi^{\tau,\gamma}(x,\lambda(x),\pi(x)) - \lambda^{\tau}_{\gamma}(x) = 0,\ x \in K\Big\}.$$

In accordance with Geoffrion [33], we establish the next useful definitions.

Definition 1. We say that $(\lambda^0, \pi^0) \in G$ is called *efficient solution for* (***Problem***) if the relation

$$\int_K F\left(x, \lambda^0(x), \pi^0(x)\right) dw \leq \int_K F(x, \lambda(x), \pi(x)) dw$$

holds, for all $(\lambda, \pi) \in G$.

Definition 2. The efficient solution $(\lambda^0, \pi^0) \in G$ is called *properly efficient solution for* (***Problem***) if $(\exists)$ $k > 0$ fulfilling

$$\int_K F^{\delta}\left(x, \lambda^0(x), \pi^0(x)\right) dw - \int_K F^{\delta}(x, \lambda(x), \pi(x)) dw$$
$$\leq k \left(\int_K F^{i}(x, \lambda(x), \pi(x)) dw - \int_K F^{i}\left(x, \lambda^0(x), \pi^0(x)\right) dw\right)$$

for $\delta \in \{1, 2, \dots, q\}$ and some i, with

$$\int_K F^{i}(x, \lambda(x), \pi(x)) dw > \int_K F^{i}\left(x, \lambda^0(x), \pi^0(x)\right) dw$$

for $(\lambda, \pi) \in G$, and

$$\int_K F^{\delta}(x, \lambda(x), \pi(x)) dw < \int_K F^{\delta}\left(x, \lambda^0(x), \pi^0(x)\right) dw.$$

Definition 3. The pair $(\lambda^0, \pi^0) \in G$ is called *improperly efficient solution for* (***Problem***) if for every sufficiently large $k > 0$, there exist $(\lambda, \pi) \in G$ and $\delta \in \{1, 2, \dots, q\}$ fulfilling

$$\int_K F^{\delta}(x, \lambda(x), \pi(x)) dw < \int_K F^{\delta}\left(x, \lambda^0(x), \pi^0(x)\right) dw$$

and

$$\int_K F^{\delta}\left(x, \lambda^0(x), \pi^0(x)\right) dw - \int_K F^{\delta}(x, \lambda(x), \pi(x)) dw$$
$$> k \left(\int_K F^{i}(x, \lambda(x), \pi(x)) dw - \int_K F^{i}\left(x, \lambda^0(x), \pi^0(x)\right) dw\right)$$

for $i \in \{1, 2, \dots, q\}$ verifying

$$\int_K F^{i}(x, \lambda(x), \pi(x)) dw > \int_K F^{i}\left(x, \lambda^0(x), \pi^0(x)\right) dw.$$

Definition 4. The pair $(\lambda^0, \pi^0) \in G$ is called *weak efficient solution for* (***Problem***) if there is no $(\lambda, \pi) \in G$ satisfying

$$\int_K F^\delta\left(x, \lambda^0(x), \pi^0(x)\right) dw > \int_K F^\delta(x, \lambda(x), \pi(x)) dw, \quad \text{for all } \delta \in \{1, 2, \ldots, q\}.$$

Remark 1. We can remark that if $(\lambda^0, \pi^0) \in G$ is an efficient solution for (**Problem**), then it is a weak efficient solution for (**Problem**).

Definition 5. We say that the multiple integral type functional

$$E(\bar{\lambda}, \omega) = \int_K \chi(x, \bar{\lambda}(x), \bar{\lambda}_\gamma(x), \bar{\pi}(x), \omega(x), \omega_\xi(x), \zeta(x)) dw$$

is called *pseudoinvex at* λ, λ_γ *and* π if there exist $z \in \mathbb{R}^n$, with

$$z(x, \lambda, \lambda_\gamma, \pi, \lambda, \lambda_\gamma, \pi) = z|_{x=x_1, x=x_2} = 0, \text{ (vanishes on every face of } \partial K)$$

and $\mu \in \mathbb{R}^s$, with

$$\mu(x, \lambda, \lambda_\gamma, \pi, \lambda, \lambda_\gamma, \pi) = 0,$$

such that, for each ω, ω_ξ and ζ, we have

$$\int_K \Big[z^T \chi_{\bar{\lambda}}(x, \lambda, \lambda_\gamma, \pi, \omega, \omega_\xi, \zeta) + \mu^T \chi_{\bar{\pi}}(x, \lambda, \lambda_\gamma, \pi, \omega, \omega_\xi, \zeta)$$

$$+ \frac{dz^T}{dx^\gamma} \chi_{\bar{\lambda}_\gamma}(x, \lambda, \lambda_\gamma, \pi, \omega, \omega_\xi, \zeta) \Big] dw \geq 0$$

$$\Rightarrow \int_K \chi(x, \bar{\lambda}, \bar{\lambda}_\gamma, \bar{\pi}, \omega, \omega_\xi, \zeta) dw \geq \int_K \chi(x, \lambda, \lambda_\gamma, \pi, \omega, \omega_\xi, \zeta) dw,$$

for all $\bar{\lambda}, \bar{\lambda}_\gamma$ and $\bar{\pi}$.

Definition 6. We say that the multiple integral type functional

$$-\int_K \chi(x, \lambda(x), \lambda_\gamma(x), \pi(x), \bar{\omega}(x), \bar{\omega}_\xi(x), \bar{\zeta}(x)) dw$$

is called *pseudoinvex at* ω, ω_ξ *and* ζ if there exist $\eta \in \mathbb{R}^m$, with

$$\eta(x, \omega, \omega_\xi, \zeta, \omega, \omega_\xi, \zeta) = \eta|_{x=x_1, x=x_2} = 0, \text{ (vanishes on every face of } \partial K)$$

and $\nu \in \mathbb{R}^l$, with

$$\nu(x, \omega, \omega_\xi, \zeta, \omega, \omega_\xi, \zeta) = 0,$$

such that, for each λ, λ_γ and π, we have

$$\int_K \Big[\eta^T \chi_{\bar{\omega}}(x, \lambda, \lambda_\gamma, \pi, \omega, \omega_\xi, \zeta) + \nu^T \chi_{\bar{\zeta}}(x, \lambda, \lambda_\gamma, \pi, \omega, \omega_\xi, \zeta)$$

$$+\frac{d\eta^T}{dx^\xi}\chi_{\bar{\omega}_\xi}(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)\Big]dw \le 0$$

$$\Rightarrow \int_K \chi(x,\lambda,\lambda_\gamma,\pi,\bar{\omega},\bar{\omega}_\xi,\bar{\zeta})dw \le \int_K \chi(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw,$$

for all $\bar{\omega},\bar{\omega}_\xi$ and $\bar{\zeta}$.

Remark 2. In the following, we briefly write $z(x,\lambda,\omega)$ for $z(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)$ and $\eta(x,\lambda,\omega)$ for $\eta(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)$. In addition, we use (λ,ω) instead of $(\lambda,\pi,\omega,\zeta)$.

3. Main result

In this section, according to Weir [34], we introduce the following symmetric multidimensional multiobjective dual programs:

$$\text{(P)} \quad \min_{(\lambda,\omega)} \left(\frac{\int_K \chi^1(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw}{\int_K \Upsilon^1(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw}, \ldots, \frac{\int_K \chi^q(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw}{\int_K \Upsilon^q(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw} \right)$$

subject to

$$\lambda(x_1) = 0 = \lambda(x_2), \quad \omega(x_1) = 0 = \omega(x_2),$$

$$\lambda_\gamma(x_1) = 0 = \lambda_\gamma(x_2), \quad \omega_\xi(x_1) = 0 = \omega_\xi(x_2),$$

$$\sum_{\delta=1}^q \Omega^\delta \left\{ I^\delta(\lambda,\omega)\left(\chi^\delta_\omega - \frac{d}{dx^\xi}\chi^\delta_{\omega_\xi}\right) - E^\delta(\lambda,\omega)\left(\Upsilon^\delta_\omega - \frac{d}{dx^\xi}\Upsilon^\delta_{\omega_\xi}\right)\right\} \leqq 0, \ x \in K,$$

$$\sum_{\delta=1}^q \Omega^\delta \left\{ I^\delta(\lambda,\omega)\left(\chi^\delta_\zeta - 0\right) - E^\delta(\lambda,\omega)\left(\Upsilon^\delta_\zeta - 0\right)\right\} \leqq 0, \ x \in K,$$

$$\omega^T \sum_{\delta=1}^q \Omega^\delta \left\{ I^\delta(\lambda,\omega)\left(\chi^\delta_\omega - \frac{d}{dx^\xi}\chi^\delta_{\omega_\xi}\right) - E^\delta(\lambda,\omega)\left(\Upsilon^\delta_\omega - \frac{d}{dx^\xi}\Upsilon^\delta_{\omega_\xi}\right)\right\} \ge 0, \ x \in K,$$

$$\zeta^T \sum_{\delta=1}^q \Omega^\delta \left\{ I^\delta(\lambda,\omega)\left(\chi^\delta_\zeta - 0\right) - E^\delta(\lambda,\omega)\left(\Upsilon^\delta_\zeta - 0\right)\right\} \ge 0, \ x \in K,$$

$$\Omega > 0,$$

and

$$\text{(D)} \quad \max_{(b,v)} \left(\frac{\int_K \chi^1(x,b,b_\gamma,\rho,v,v_\xi,\varrho)dw}{\int_K \Upsilon^1(x,b,b_\gamma,\rho,v,v_\xi,\varrho)dw}, \ldots, \frac{\int_K \chi^q(x,b,b_\gamma,\rho,v,v_\xi,\varrho)dw}{\int_K \Upsilon^q(x,b,b_\gamma,\rho,v,v_\xi,\varrho)dw} \right)$$

subject to

$$b(x_1) = 0 = b(x_2), \quad v(x_1) = 0 = v(x_2),$$

$$b_\gamma(x_1) = 0 = b_\gamma(x_2), \quad v_\xi(x_1) = 0 = v_\xi(x_2),$$

$$\sum_{\delta=1}^{q} \Omega^\delta \left\{ I^\delta(b,v) \left(\chi^\delta_\lambda - \frac{d}{dx^\gamma}\chi^\delta_{\lambda_\gamma} \right) - E^\delta(b,v) \left(\Upsilon^\delta_\lambda - \frac{d}{dx^\gamma}\Upsilon^\delta_{\lambda_\gamma} \right) \right\} \geqq 0,\ x \in K,$$

$$\sum_{\delta=1}^{q} \Omega^\delta \left\{ I^\delta(b,v) \left(\chi^\delta_\pi - 0 \right) - E^\delta(b,v) \left(\Upsilon^\delta_\pi - 0 \right) \right\} \geqq 0,\ x \in K,$$

$$b^T \sum_{\delta=1}^{q} \Omega^\delta \left\{ I^\delta(b,v) \left(\chi^\delta_\lambda - \frac{d}{dx^\gamma}\chi^\delta_{\lambda_\gamma} \right) - E^\delta(b,v) \left(\Upsilon^\delta_\lambda - \frac{d}{dx^\gamma}\Upsilon^\delta_{\lambda_\gamma} \right) \right\} \leq 0,\ x \in K,$$

$$\rho^T \sum_{\delta=1}^{q} \Omega^\delta \left\{ I^\delta(b,v) \left(\chi^\delta_\pi - 0 \right) - E^\delta(b,v) \left(\Upsilon^\delta_\pi - 0 \right) \right\} \leq 0,\ x \in K,$$

$$\Omega > 0,$$

where, for $\delta = 1, 2, \ldots, q, \chi^\delta : K \times \mathbb{R}^{2n} \times \mathbb{R}^s \times \mathbb{R}^{2m} \times \mathbb{R}^l \to \mathbb{R}_+$, and $\Upsilon^\delta : K \times \mathbb{R}^{2n} \times \mathbb{R}^s \times \mathbb{R}^{2m} \times \mathbb{R}^l \to \mathbb{R}_+ \backslash \{0\}$ are functionals of C^2-class and

$$E^\delta(\lambda,\omega) = \int_K \chi^\delta(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw,\ I^\delta(\lambda,\omega) = \int_K \Upsilon^\delta(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw.$$

Remark 3. If we consider $I^\delta(\lambda,\omega) = 1,\ \delta = 1, 2, \ldots, q$, and remove control functions, the considered symmetric programs (P) and (D) are converted into the optimization models investigated by Gulati et al. [35]. Also, for $q = 1$ and removing the control variables, the symmetric problems (P) and (D) are transformed in the variational problems considered by Gulati et al. [36]. If K is a classical real interval, then the present study is investigated by Treanţă et al. [37]. In addition, let us note that we do not impose (see Kim et al. [38]) the constraint $\Omega^T e = 1, e = (1, 1, \cdots, 1) \in \mathbb{R}^q$, in (P) and (D) since it does not matter in establishing the main result (see Theorem 1).

Further, we reformulate the considered symmetric programs (P) and (D) as follows:

$$\text{(P')} \quad \min_{(\lambda,\omega)} \left(Y^1, Y^2, \ldots, Y^q \right)$$

subject to

$$\lambda(x_1) = 0 = \lambda(x_2), \quad \omega(x_1) = 0 = \omega(x_2), \tag{1}$$

$$\lambda_\gamma(x_1) = 0 = \lambda_\gamma(x_2), \quad \omega_\xi(x_1) = 0 = \omega_\xi(x_2), \tag{2}$$

$$\int_K \chi^\delta(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw - Y^\delta \int_K \Upsilon^\delta(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)dw = 0,\ \delta = 1, 2, \ldots, q, \tag{3}$$

$$\sum_{\delta=1}^{q} \Omega^\delta \left\{ \left(\chi^\delta_\omega - \frac{d}{dx^\xi}\chi^\delta_{\omega_\xi} \right) - Y^\delta \left(\Upsilon^\delta_\omega - \frac{d}{dx^\xi}\Upsilon^\delta_{\omega_\xi} \right) \right\} \leqq 0, \text{ for almost every } x \in K, \tag{4}$$

$$\sum_{\delta=1}^{q} \Omega^{\delta} \left\{ \left(\chi_{\zeta}^{\delta} - 0 \right) - Y^{\delta} \left(\Upsilon_{\zeta}^{\delta} - 0 \right) \right\} \leqq 0, \text{ for almost every } x \in K, \tag{4'}$$

$$\omega^T \sum_{\delta=1}^{q} \Omega^{\delta} \left\{ \left(\chi_{\omega}^{\delta} - \frac{d}{dx^{\xi}} \chi_{\omega_{\xi}}^{\delta} \right) - Y^{\delta} \left(\Upsilon_{\omega}^{\delta} - \frac{d}{dx^{\xi}} \Upsilon_{\omega_{\xi}}^{\delta} \right) \right\} \geq 0, \text{ for almost every } x \in K, \tag{5}$$

$$\zeta^T \sum_{\delta=1}^{q} \Omega^{\delta} \left\{ \left(\chi_{\zeta}^{\delta} - 0 \right) - Y^{\delta} \left(\Upsilon_{\zeta}^{\delta} - 0 \right) \right\} \geq 0, \text{ for almost every } x \in K, \tag{5'}$$

$$\Omega > 0, \tag{6}$$

and

$$\text{(D')} \quad \max_{(b,v)} \left(X^1, X^2, \ldots, X^q \right)$$

subject to

$$b(x_1) = 0 = b(x_2), \quad v(x_1) = 0 = v(x_2), \tag{7}$$

$$b_{\gamma}(x_1) = 0 = b_{\gamma}(x_2), \quad v_{\xi}(x_1) = 0 = v_{\xi}(x_2), \tag{8}$$

$$\int_K \chi^{\delta}(x, b, b_{\gamma}, \rho, v, v_{\xi}, \varrho) dw - X^{\delta} \int_K \Upsilon^{\delta}(x, b, b_{\gamma}, \rho, v, v_{\xi}, \varrho) dw = 0, \ \delta = 1, 2, \ldots, q, \tag{9}$$

$$\sum_{\delta=1}^{q} \Omega^{\delta} \left\{ \left(\chi_{\lambda}^{\delta} - \frac{d}{dx^{\gamma}} \chi_{\lambda_{\gamma}}^{\delta} \right) - X^{\delta} \left(\Upsilon_{\lambda}^{\delta} - \frac{d}{dx^{\gamma}} \Upsilon_{\lambda_{\gamma}}^{\delta} \right) \right\} \geqq 0, \text{ for almost every } x \in K, \tag{10}$$

$$\sum_{\delta=1}^{q} \Omega^{\delta} \left\{ \left(\chi_{\pi}^{\delta} - 0 \right) - X^{\delta} \left(\Upsilon_{\pi}^{\delta} - 0 \right) \right\} \geqq 0, \text{ for almost every } x \in K, \tag{10'}$$

$$b^T \sum_{\delta=1}^{q} \Omega^{\delta} \left\{ \left(\chi_{\lambda}^{\delta} - \frac{d}{dx^{\gamma}} \chi_{\lambda_{\gamma}}^{\delta} \right) - X^{\delta} \left(\Upsilon_{\lambda}^{\delta} - \frac{d}{dx^{\gamma}} \Upsilon_{\lambda_{\gamma}}^{\delta} \right) \right\} \leq 0, \text{ for almost every } x \in K, \tag{11}$$

$$\rho^T \sum_{\delta=1}^{q} \Omega^{\delta} \left\{ \left(\chi_{\pi}^{\delta} - 0 \right) - X^{\delta} \left(\Upsilon_{\pi}^{\delta} - 0 \right) \right\} \leq 0, \text{ for almost every } x \in K, \tag{11'}$$

$$\Omega > 0. \tag{12}$$

Next, denote by A and B the set of feasible solutions associated with the symmetric models (P) and (D), respectively.

The main result given in the following is formulated in terms of (P') and (D'). Of course, this result is equally valid to (P) and (D).

Theorem 1. *If* $(\lambda(x), \omega(x), \Omega, Y) \in A$ *and* $(b(x), v(x), \Omega, X) \in B$ *are some feasible solutions for the considered symmetric models (P') and (D'), respectively, and the next assumptions are satisfied:*

(i) $\sum_{\delta=1}^{q} \Omega^{\delta} \int_K \left\{ \chi^{\delta}(x,(\cdot),(\cdot),(\cdot),v,v_{\xi},\varrho) - X^{\delta}\Upsilon^{\delta}(x,(\cdot),(\cdot),(\cdot),v,v_{\xi},\varrho) \right\} dw$ *is pseudoinvex at* b, b_{γ} *and* ρ*, with* $z(x,\lambda,b)+b(x) \geqq 0, \mu(x,\lambda,b)+\rho(x) \geqq 0,\ x \in K$*;*

(ii) $-\sum_{\delta=1}^{q} \Omega^{\delta} \int_K \left\{ \chi^{\delta}(x,b,b_{\gamma},\rho,(\cdot),(\cdot),(\cdot)) - Y^{\delta}\Upsilon^{\delta}(x,b,b_{\gamma},\rho,(\cdot),(\cdot),(\cdot)) \right\} dw$ *is pseudoinvex at* v, v_{ξ} *and* ϱ*, with* $\eta(x,v,\omega)+\omega(x) \geqq 0, \nu(x,v,\omega)+\zeta(x) \geqq 0,\ x \in K$*;*

then the connection $Y \not\leq X$ *is true between the corresponding objective functionals of (P') and (D').*

Proof. By considering (10) and (10'), together with $z(x,\lambda,b)+b(x) \geqq 0, \mu(x,\lambda,b)+\rho(x) \geqq 0, x \in K$, we get

$$[z(x,\lambda,b)+b(x)]^T \left[\Omega \left\{ \left(\chi_{\lambda} - \frac{d}{dx^{\gamma}} \chi_{\lambda_{\gamma}} \right) - X \left(\Upsilon_{\lambda} - \frac{d}{dx^{\gamma}} \Upsilon_{\lambda_{\gamma}} \right) \right\} \right] \geq 0,$$

$$[\mu(x,\lambda,b)+\rho(x)]^T \left[\Omega \left\{ (\chi_{\pi} - 0) - X(\Upsilon_{\pi} - 0) \right\} \right] \geq 0,$$

and, by using (11) and (11'), we get

$$(z(x,\lambda,b))^T \left[\Omega \left\{ \left(\chi_{\lambda} - \frac{d}{dx^{\gamma}} \chi_{\lambda_{\gamma}} \right) - X \left(\Upsilon_{\lambda} - \frac{d}{dx^{\gamma}} \Upsilon_{\lambda_{\gamma}} \right) \right\} \right] \geq 0,\ x \in K,$$

$$(\mu(x,\lambda,b))^T \left[\Omega \left\{ (\chi_{\pi} - 0) - X(\Upsilon_{\pi} - 0) \right\} \right] \geq 0,\ x \in K,$$

which imply

$$\begin{aligned} 0 &\leq \int_K z(x,\lambda,b)^T \left[\Omega \left\{ (\chi_{\lambda} - X\Upsilon_{\lambda}) - \frac{d}{dx^{\gamma}} \left(\chi_{\lambda_{\gamma}} - X\Upsilon_{\lambda_{\gamma}} \right) \right\} \right] dw \\ &= \int_K \left[z(x,\lambda,b)^T \Omega(\chi_{\lambda} - X\Upsilon_{\lambda}) + \frac{dz(x,\lambda,b)^T}{dx^{\gamma}} \Omega \left(\chi_{\lambda_{\gamma}} - X\Upsilon_{\lambda_{\gamma}} \right) \right] dw \\ &- z(x,\lambda,b)^T \Omega \left(\chi_{\lambda_{\gamma}} - X\Upsilon_{\lambda_{\gamma}} \right) \Big|_{x=x_1}^{x=x_2}, \end{aligned}$$

and

$$\int_K \left[\mu(x,\lambda,b)^T \Omega(\chi_{\pi} - X\Upsilon_{\pi}) + \frac{d\mu(x,\lambda,b)^T}{dx^{\gamma}} \Omega(0 - X \cdot 0) \right] dw \geq 0.$$

Since $z(x,\lambda,b) = 0$, at $x = x_1$ and $x = x_2$, it follows

$$\int_K \left[z(x,\lambda,b)^T \Omega(\chi_{\lambda} - X\Upsilon_{\lambda}) + \frac{dz(x,\lambda,b)^T}{dx^{\gamma}} \Omega \left(\chi_{\lambda_{\gamma}} - X\Upsilon_{\lambda_{\gamma}} \right) \right] dw \geq 0$$

and

$$\int_K \left[\mu(x,\lambda,b)^T \Omega(\chi_{\pi} - X\Upsilon_{\pi}) + \frac{d\mu(x,\lambda,b)^T}{dx^{\gamma}} \Omega(0 - X \cdot 0) \right] dw \geq 0,$$

involving

$$\int_K \left[z(x,\lambda,b)^T \Omega(\chi_{\lambda} - X\Upsilon_{\lambda}) + \mu(x,\lambda,b)^T \Omega(\chi_{\pi} - X\Upsilon_{\pi}) \right.$$

$$+\frac{dz(x,\lambda,b)^T}{dx^\gamma}\Omega\left(\chi_{\lambda_\gamma}-X\Upsilon_{\lambda_\gamma}\right)\Big]dw\geq 0.$$

Now, by using the pseudoinvexity assumption given in (i), we obtain

$$\Omega\int_K\{\chi(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho)-X\Upsilon(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho)\}\,dw$$
$$\geq\Omega\int_K\{\chi(x,b,b_\gamma,\rho,v,v_\xi,\varrho)-X\Upsilon(x,b,b_\gamma,\rho,v,v_\xi,\varrho)\}\,dw.$$

By (9), the inequality given above involves

$$\Omega\int_K\{\chi(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho)-X\Upsilon(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho)\}\,dw\geq 0. \tag{13}$$

On the other hand, relations given in (4) and (4'), together with $\eta(x,v,\omega)+\omega(x)\geqq 0,\nu(x,v,\omega)+\zeta(x)\geqq 0,x\in K$, implies

$$[\eta(x,v,\omega)+\omega(x)]^T\left[\Omega\left\{\left(\chi_\omega-\frac{d}{dx^\xi}\chi_{\omega_\xi}\right)-Y\left(\Upsilon_\omega-\frac{d}{dx^\xi}\Upsilon_{\omega_\xi}\right)\right\}\right]\leq 0,$$

$$[\nu(x,v,\omega)+\zeta(x)]^T\left[\Omega\left\{(\chi_\zeta-0)-Y\left(\Upsilon_\zeta-0\right)\right\}\right]\leq 0,$$

and, by using (5) and (5'), we get

$$(\eta(x,v,\omega))^T\left[\Omega\left\{\left(\chi_\omega-\frac{d}{dx^\xi}\chi_{\omega_\xi}\right)-Y\left(\Upsilon_\omega-\frac{d}{dx^\xi}\Upsilon_{\omega_\xi}\right)\right\}\right]\leq 0,\ x\in K,$$

$$(\nu(x,v,\omega))^T\left[\Omega\left\{(\chi_\zeta-0)-Y\left(\Upsilon_\zeta-0\right)\right\}\right]\leq 0,\ x\in K,$$

which imply

$$0\geq\int_K\eta(x,v,\omega)^T\left[\Omega\left\{(\chi_\omega-Y\Upsilon_\omega)-\frac{d}{dx^\xi}\left(\chi_{\omega_\xi}-Y\Upsilon_{\omega_\xi}\right)\right\}\right]dw$$
$$=\int_K\left[\eta(x,v,\omega)^T\Omega(\chi_\omega-Y\Upsilon_\omega)+\frac{d\eta(x,v,\omega)^T}{dx^\xi}\Omega\left(\chi_{\omega_\xi}-Y\Upsilon_{\omega_\xi}\right)\right]dw$$
$$-\ \eta(x,v,\omega)^T\Omega\left(\chi_{\omega_\xi}-Y\Upsilon_{\omega_\xi}\right)\Big|_{x=x_1}^{x=x_2},$$

and

$$\int_K\left[\nu(x,v,\omega)^T\Omega(\chi_\zeta-Y\Upsilon_\zeta)+\frac{d\nu(x,v,\omega)^T}{dx^\xi}\Omega\left(0-Y\cdot 0\right)\right]dw\leq 0.$$

Since $\eta(x,v,\omega)=0$, at $x=x_1$ and $x=x_2$, it follows

$$\int_K\left[\eta(x,v,\omega)^T\Omega(\chi_\omega-Y\Upsilon_\omega)+\frac{d\eta(x,v,\omega)^T}{dx^\xi}\Omega\left(\chi_{\omega_\xi}-Y\Upsilon_{\omega_\xi}\right)\right]dw\leq 0$$

and

$$\int_K\left[\nu(x,v,\omega)^T\Omega(\chi_\zeta-Y\Upsilon_\zeta)+\frac{d\nu(x,v,\omega)^T}{dx^\xi}\Omega\left(0-Y\cdot 0\right)\right]dw\leq 0,$$

involving

$$\int_K \Big[\eta(x,v,\omega)^T\Omega(\chi_\omega - Y\Upsilon_\omega) + \nu(x,v,\omega)^T\Omega(\chi_\zeta - Y\Upsilon_\zeta)$$
$$+\frac{d\eta(x,v,\omega)^T}{dx^\xi}\Omega\left(\chi_{\omega_\xi} - Y\Upsilon_{\omega_\xi}\right)\Big]dw \le 0.$$

Now, by using the pseudoinvexity assumption given in (ii), we obtain

$$\Omega\int_K \{\chi(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho) - Y\Upsilon(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho)\}\,dw$$
$$\le \Omega\int_K \{\chi(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta) - Y\Upsilon(x,\lambda,\lambda_\gamma,\pi,\omega,\omega_\xi,\zeta)\}\,dw.$$

By (3), the inequality given above involves

$$\Omega\int_K \{\chi(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho) - Y\Upsilon(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho)\}\,dw \le 0.$$

The above inequality along with (13) yields

$$\sum_{\delta=1}^{q}\Omega^\delta\left(Y^\delta - X^\delta\right)\int_K \Upsilon^\delta(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho)dw \ge 0. \tag{14}$$

If, for some $\delta \in \{1,2,\ldots,q\}$, we have $Y^\delta < X^\delta$, and, for $i \in \{1,2,\ldots,q\}$, with $i \neq \delta$, we have $Y^i \le X^i$, then since $\int_K \Upsilon^\delta(x,\lambda,\lambda_\gamma,\pi,v,v_\xi,\varrho)dw > 0$ and $\Omega > 0$, we are in contradiction with (14). □

4. Conclusions

In this paper, we have introduced a pair of symmetric multi-dimensional variational fractional control problems. By using the updated concept of *pseudoinvexity* associated with multiple integral type functionals, we have established a very important connection between the objective functionals of the studied symmetric models. This approach (duality technique) is useful in solving complex optimization problems that arise in various research and practical areas. Therefore, this paper offered a solid and original contribution to the theory of variational fractional control problems and duality.

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