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Analysis of a Fractional Order Enzymatic Reaction Model with Artificial Neural Network Validation

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Abstract. This paper investigates an enzymatic reaction model formulated with the Atangana–Baleanu–Caputo (ABC) fractional derivative, aiming to enhance the classical description of enzyme kinetics. Existence and uniqueness of the solution are established through nonlinear functional analysis, while approximate solutions are obtained via the Laplace Adomian decomposition method (LADM). To validate and complement the numerical scheme, we employ a neural network framework that demonstrates the capability of intelligent computing to approximate fractional biochemical dynamics. Sensitivity analysis is carried out, and numerical simulations are illustrated through 2D and 3D plots. The study highlights how the combination of fractional calculus and neural networks offers new perspectives for modelling enzyme kinetics. The results indicate potential applications in drug development, metabolic engineering, and biochemical process optimization, providing a pathway for more precise control strategies in biochemical systems.

2020 Mathematics Subject Classifications: 26A33, 34A08, 03C65, 92B20

Key Words and Phrases: Enzymatic model, qualitative study, approximate solution, neural network validation, sensitivity analysis

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1. Introduction

The study of enzymatic reactions is central to understanding the biochemical processes that regulate physiological and metabolic activities. Mathematical modelling provides a systematic way to analyze enzyme kinetics, substrate dynamics, and reaction efficiency, [1–3]. Classical models based on integer-order differential equations have been widely used, but they often fail to capture memory effects and hereditary properties inherent in biochemical systems, [4–6]. These limitations motivate the use of generalized frameworks such as fractional calculus, which extends ordinary calculus to derivatives of arbitrary order. Fractional-order models allow the representation of long-range memory and non-local interactions, making them especially suitable for systems with complex temporal dynamics, [7–11].

Recent advances in fractional calculus have introduced several new derivative definitions to improve the accuracy and applicability of mathematical models, [12–14]. Among these, the Atangana–Baleanu–Caputo (ABC) derivative is particularly effective due to its non-singular kernel and exponentially decaying function, which provide numerical stability and a more physically relevant description of dynamical processes. This makes the ABC derivative a natural choice for modelling enzymatic reactions, where both local changes and non-local interactions play significant roles, [15–17].

In this work, we develop an enzymatic reaction model using the ABC derivative to enhance the accuracy of classical kinetic frameworks, [18]. The existence and uniqueness of solutions are established via non-linear functional analysis, while approximate solutions are obtained using the Laplace Adomian decomposition method (LADM), which offers good computational efficiency, [19]. A detailed sensitivity analysis with respect to reaction rate parameters highlights their influence on system dynamics, and comprehensive $2D$ and $3D$ graphical results illustrate the role of fractional differentiation in enzymatic kinetics.

Overall, this study demonstrates that fractional-order modelling provides a refined and flexible framework for enzyme kinetics. By incorporating memory and non-local effects, the ABC-based model yields improved predictive power and control strategies for biochemical and pharmaceutical processes. These findings open new avenues for applications in metabolic engineering, drug development, and biotechnology, and suggest promising directions for future research in biochemical systems.

One can show the basic enzymatic reaction schematically as given as follows in figure 1:

$$\mathrm{S} + \mathrm{E} \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} \mathrm{SE}\,(\mathrm{C}), \quad \mathrm{C} \xrightarrow{\gamma} \mathrm{P} + \mathrm{E}$$

Figure 1: Schematic diagram of basic enzymatic reaction.

A system of differential equations for the preceding reaction is given as follows, [20]:

$$\begin{cases} \dfrac{dS(t)}{dt} = \beta C(t) - \alpha E(t)S(t), \\ \dfrac{dE(t)}{dt} = (\beta+\gamma)C(t) - \alpha E(t)S(t), \\ \dfrac{dC(t)}{dt} = \alpha E(t)S(t) - (\beta+\gamma)C(t), \\ \dfrac{dP(t)}{dt} = \gamma C(t), \\ S(0) = S^0, E(0) = E^0, C(0) = C^0, P(0) = P^0. \end{cases} \tag{1}$$

Where

- $S(t)$ represents substrate concentration.
- $E(t)$ represents enzyme concentration.
- $C(t)$ represents complex enzyme-substrate concentration.
- $P(t)$ represents product concentration,

while α, $\beta\,\gamma$ are constant associated with rates of reaction.

Applying the ABC derivative to the system (1), we obtain our concerned problem:

$$\begin{cases} D^{\theta}_{0,t}S(t) = \beta C(t) - \alpha S(t)E(t), \\ D^{\theta}_{0,t}E(t) = (\beta+\gamma)C(t) - \alpha S(t)E(t), \\ D^{\theta}_{0,t}C(t) = \alpha S(t)E(t) - (\beta+\gamma)C(t), \\ D^{\theta}_{0,t}P(t) = \gamma C(t), \\ S(0) = S^0, E(0) = E^0, C(0) = C^0, P(0) = P^0. \end{cases} \tag{2}$$

The flowchart of the specified model (2), is given as follows:

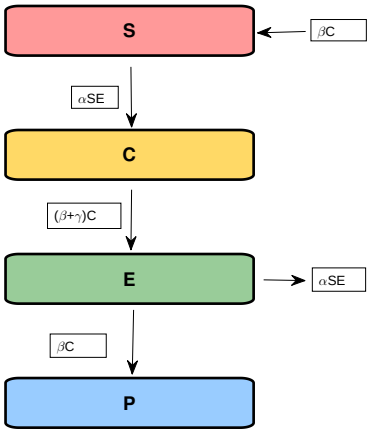


Figure 2: Flowchart representation for model (2).

2. Preliminaries

The present section defines the basic tools that are required for our work. We create a Banach space $(\mathbb{W}, ||.||)$, with norm $\|z\| = \underset{t\in J}{max}\big|z(t)\big|$, where $t \in [0, \eta] = J$. Consequently $\mathbb{W}^4$ is a Banach space with norm $||(S, E, C, P)|| = \underset{t\in J}{max}\{||S||, ||E||, ||C||, ||P||\}$.

Definition 1. *[21] For $\theta \in (0, 1]$, the ABC derivative to a function $z \in H^1(0, \eta)$ is denoted as follows:*

$$D^{\theta}_{0,t}z(t) = \frac{M(\theta)}{(1-\theta)}\int_0^t z'(\kappa)E_\theta\left(\frac{(t-\kappa)^\theta\theta}{\theta-1}\right)d\kappa.$$

Where $M(\theta)$ is normalization and E_θ represent Mittag-Leffler functions.

Definition 2. *[21] Consider for an order $\theta \in (0, 1]$, and $z \in H^1(0, \eta)$, then the fractional integral is expressed as follows*

$$I_{0,t}z(t) = \frac{(1-\theta)z(t)}{M(\theta)} + \frac{\theta}{M(\theta)\Gamma(\theta)}\int_0^t (t-\kappa)^{\theta-1}z(\kappa)d\kappa.$$

Lemma 1. *[21] The solution of the differential equation below, having the condition that $g(t)|_{t=0} = 0$,*

$$\begin{cases} D^{\theta}_{0,t}z(t) = g(t), \\ z(0) = z^0. \end{cases}$$

is given as follows:

$$z(t) = z^0 + \frac{(1-\theta)g(t)}{M(\theta)} + \frac{\theta}{M(\theta)\Gamma(\theta)}\int_0^t (t-\kappa)^{\theta-1} g(\kappa)d\kappa.$$

Definition 3. *[22] The Laplace transform using ABC derivative of a function z is expressed as follows:*

$$\mathbb{L}[D^{\theta}_{0,t} z(t)] = \frac{M(\theta)}{s^{\theta}(1-\theta)+\theta}\left\{s^{\theta}\mathbb{L}\{z(t)\} - s^{\theta-1}z(0)\right\}.$$

Theorem 1. *[22] If $\mathbb{T}_1, \mathbb{T}_2$ are two operators such that the first is contraction and the second is entirely continuous throughout a closed bounded subset $\mathbb{A}$ of a Banach space $\mathbb{W}$, then the operator equation $\mathbb{T}_1 z + \mathbb{T}_2 z = z$ has at least one solution.*

3. Theoretical Results on Existence and Uniqueness

In this section, we investigate the qualitative analysis of the solutions through the theory of non-linear functional analysis. In Theorem 3.1, we present the uniqueness of a solution of the specified model, and in Theorem 3.2, we discuss the existence of the solutions. For this purpose, we use the Krasnoselskii and Banach fixed-point theorems. For further work, we write (2), in the following form:

$$\begin{cases} D^{\theta}_{0,t} S(t) = \Phi_1(S,E,C,P), \\ D^{\theta}_{0,t} E(t) = \Phi_2(S,E,C,P), \\ D^{\theta}_{0,t} C(t) = \Phi_3(S,E,C,P), \\ D^{\theta}_{0,t} P(t) = \Phi_4(S,E,C,P). \end{cases}$$

Where

$$\begin{cases} \Phi_1(S,E,C,P) = \beta C(t) - \alpha S(t)E(t), \\ \Phi_2(S,E,C,P) = (\beta+\gamma)C(t) - \alpha S(t)E(t), \\ \Phi_3(S,E,C,P) = \alpha S(t)E(t) - (\beta+\gamma)C(t), \\ \Phi_4(S,E,C,P) = \gamma C(t). \end{cases}$$

For the rest of the work, we have the following compact form:

$$\begin{cases} D^{\theta}_{0,t} F(t) = G(t, F(t)), \\ F(0) = F^0. \end{cases} \tag{3}$$

Here, $F(t)$, F^0, and $G(t, F(t))$ are defined as follows:

$$F(t) = \begin{Bmatrix} S(t) \\ E(t) \\ C(t) \\ P(t) \end{Bmatrix}, \; F^0 = \begin{Bmatrix} S^0 \\ E^0 \\ C^0 \\ P^0 \end{Bmatrix} \quad \text{and} \quad G(t,F(t)) = \begin{Bmatrix} \Phi_1(t,S,E,C,P) \\ \Phi_2(t,S,E,C,P) \\ \Phi_3(t,S,E,C,P) \\ \Phi_4(t,S,E,C,P) \end{Bmatrix}.$$

According to Lemma 1, the solution of (3), can be written as follows:

$$F(t) = F^0 + \frac{(1-\theta)G(t,F(t))}{M(\theta)} + \frac{\theta}{M(\theta)\Gamma(\theta)}\int_0^t (t-\kappa)^{\theta-1}G(\kappa,F(\kappa))d\kappa.$$

The operator $\mathbb{T}$ is considered for further work:

$$\mathbb{T}F(t) = F^0 + \frac{(1-\theta)G(t,F(t))}{M(\theta)} + \frac{\theta}{M(\theta)\Gamma(\theta)}\int_0^t (t-\kappa)^{\theta-1}G(\kappa,F(\kappa))d\kappa. \tag{4}$$

The following are the assumptions for our upcoming work:

(A_1) For $F_1, F_2 \in \mathbb{W}$, we have $a_G > 0$ such that:

$$\big\|G(F_1) - G(F_2)\big\| \leq a_G\|F_1 - F_2\|.$$

(A_2) For $F \in \mathbb{W}$, we have b_G, $c_G > 0$ such that:

$$\big\|G(F)\big\| \leq b_G\|F\| + c_G.$$

Theorem 2. *Given* (A_1) *and assuming that* $\left(a_G M(\theta)\Gamma(\theta)(1-\theta) + a_G\eta^\theta\right) < M(\theta)\Gamma(\theta)$ *hold. Subsequently, the problem* (3) *has a unique solution.*

Proof. Consider F_1, $F_2 \in \mathbb{W}$, one may write

$$\begin{aligned}
\|\mathbb{T}F_1(t) - \mathbb{T}F_2(t)\| &\leq \max_{t\in J}\Bigg\{(1-\theta)\left|G(t,F_1) - G(t,F_2)\right| + \left|\frac{\theta}{M(\theta)\Gamma(\theta)}\int_0^t (t-\kappa)^{\theta-1}\right| \\
&\qquad \left|G(t,F_1) - G(t,F_2)\right| d\kappa\Bigg\} \\
&\leq \max_{t\in J}\left\{a_G(1-\theta)|F_1 - F_2| + \frac{a_G t^\theta}{M(\theta)\Gamma(\theta)}|F_1 - F_2|\right\} \\
&\leq \left(a_G(1-\theta) + \frac{a_G\eta^\theta}{M(\theta)\Gamma(\theta)}\right)\|F_1 - F_2\| \\
&\leq \left(\frac{a_G M(\theta)\Gamma(\theta)(1-\theta) + a_G\eta^\theta}{M(\theta)\Gamma(\theta)}\right)\|F_1 - F_2\|.
\end{aligned}$$

Obviously $\mathbb{T}$ has a unique solution.

Theorem 3. *If* (A_1, A_2) *and* $a_G(1-\theta) < M(\theta)$ *hold true, then there is at least one solution to the problem* (3)*.*

Proof. Two operators are defined for further work as follows:

$$\mathbb{T}_1F(t) = F^0 + \frac{(1-\theta)G(t,F(t))}{M(\theta)}$$

$$\mathbb{T}_2F(t) = \frac{\theta}{M(\theta)\Gamma(\theta)}\int_0^t (t-\kappa)^{\theta-1}G(\kappa,F(\kappa))d\kappa.$$

Consider $F_1,\ F_2 \in \mathbb{W}$, so one has

$$\begin{aligned}\|\mathbb{T}_1(F_1)-\mathbb{T}_1(F_2)\| &\leq \underset{t\in J}{max}\left|\frac{(1-\theta)G(t,F_1)}{M(\theta)}-\frac{(1-\theta)G(t,F_2)}{M(\theta)}\right| \\ &\leq \frac{a_G(1-\theta)}{M(\theta)}\|F_1-F_2\|.\end{aligned}$$

Therefore, $\mathbb{T}_1$ is a contraction. Moreover, $\mathbb{T}_2$ is a uniformly continuous operator over $\mathbb{A}$ if $\mathbb{A}=\{F\in\mathbb{W}:\|F\|\leq r\}$ is a closed and bounded subset of $\mathbb{W}$, where $\frac{(b_G\ r+c_G)\eta^\theta}{M(\theta)\Gamma(\theta)}\leq r$. $\mathbb{T}_2$ must likewise be continuous as G is continuous. Moreover:

$$\begin{aligned}\|\mathbb{T}_2(F)\| &= \underset{t\in J}{max}\left|\frac{\theta}{M(\theta)}\int_0^t (t-\kappa)^{\theta-1}G(\kappa,F)d\kappa\right| \\ &\leq \frac{\|G(F)\|\eta^\theta}{M(\theta)\Gamma(\theta)} \\ &\leq \frac{(b_G\|F\|+c_G)\,\eta^\theta}{M(\theta)\Gamma(\theta)} \\ &\leq \frac{(b_G\ r+c_G)\,\eta^\theta}{M(\theta)\Gamma(\theta)} \\ &\leq r.\end{aligned}$$

As a result, $\mathbb{T}_2$ is bounded.
Take $t_1<t_2\in J$, therefore, one may write:

$$\begin{aligned}\|\mathbb{T}_2F(t_2)-\mathbb{T}_2F(t_1)\| &= \underset{t\in J}{max}\left\{\frac{\theta}{M(\theta)}\left|\int_0^{t_2}(t_2-\kappa)^{\theta-1}G(\kappa,F)d\kappa-\int_0^{t_1}(t_1-\kappa)^{\theta-1}G(\kappa,F)d\kappa\right|\right\} \\ &= \underset{t\in J}{max}\left\{\frac{\theta}{M(\theta)}\left|\int_0^{t_1}(t_2-\kappa)^{\theta-1}G(\kappa,F)d\kappa+\int_{t_1}^{t_2}(t_2-\kappa)^{\theta-1}G(\kappa,F)d\kappa\right.\right. \\ &- \left.\left.\int_0^{t_1}(t_1-\kappa)^{\theta-1}G(\kappa,F)d\kappa\right|\right\} \\ &\leq \frac{\theta(c_Gr+d_G)\left(t^\theta{}_2-t^\theta{}_1\right)}{M(\theta)}. \qquad (5)\end{aligned}$$

In (5), the right side suggests that $\|\mathbb{T}_2(F(t_2))-\mathbb{T}_2(F(t_1))\|\to 0$. To do this, $t_2\to t_1$ is required. Consequently, the Arzelá - Ascoli theorem holds true in every situation. Krasnoselskii's theorem states that (4) has at least one fixed point. Consequently, we may say that (3) has at least one solution.

3.1. Comparison with Existing Models

To highlight the contribution of our approach, we compare the proposed ABC-fractional enzymatic model with:

(i) **Classical Michaelis–Menten kinetics:** The classical model assumes Markovian dynamics, which neglects memory effects. Our simulations reveal that fractional orders capture substrate delays and enzyme saturation more realistically.

(ii) **Fractional enzymatic models with Caputo derivatives:** Previous studies using Caputo operators introduce memory but lack the flexibility of ABC kernels. In contrast, our model generalizes memory effects through fractional order θ, enabling finer control [23].

This comparison demonstrates that the ABC framework not only generalizes existing models but also aligns better with the complex dynamics expected in biochemical systems.

4. Results and Discussion

In the current portion, we first find the approximate solutions of (2) and then discuss the behaviour of these solutions through different two- and three-dimensional plots. To better understand the current study of the concerned problem, a complete analysis of the specified problem is given under different conditions.
In the first step of finding the approximate solution, we consider the first equation of model (2), under LADM. Applying the Laplace transform to the first equation of the concerned model, we have

$$\mathbb{L}\{S(t)\} = \frac{S(0)}{s} + \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)}\right)\mathbb{L}\left\{\beta C(t) - \alpha S(t)E(t)\right\}. \tag{6}$$

Using the initial condition, the equation (6) becomes

$$\mathbb{L}\{S(t)\} = \frac{S^0}{s} + \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)}\right)\mathbb{L}\left\{\beta C(t) - \alpha S(t)E(t)\right\}. \tag{7}$$

After applying inverse transform to (7), we have:

$$S(t) = S^0 + \mathbb{L}^{-1}\left\{\left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)}\right)\mathbb{L}\left\{\beta C(t) - \alpha S(t)E(t)\right\}\right\}. \tag{8}$$

For further calculations, we consider

$$S = \sum_{n=0}^{\infty} S_n,$$

and

$$SE = \sum_{n=0}^{\infty} \mathbb{A}_n.$$

$\mathbb{A}_n$ is given as follows:

$$\mathbb{A}_n = \frac{1}{\Gamma(n+1)} \frac{d^n}{d\kappa^n} \left[\left(\left(\sum_{j=0}^{n} \kappa^j S_j \right) \left(\sum_{j=0}^{n} \kappa^j E_j \right) \right) \right]_{\kappa=0}.$$

Putting these values in (8), the equation becomes as follows:

$$\sum_{n=0}^{\infty} S_n(t) = S^0 + \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ \beta \sum_{j=0}^{n} C_j - \alpha \sum_{j=0}^{n} \mathbb{A}_j \right\} \right\}.$$

From the preceding equation, we get

$$S_0 = S^0, \quad S_1 = \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ \beta C_0 - \alpha \mathbb{A}_0 \right\} \right\},$$

and so forth. The generic form can be written as:

$$S_{n+1} = \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ \beta C_n - \alpha \mathbb{A}_n \right\} \right\}, \ n \geq 0.$$

To find the required solution, one has to put the values of n in the general formula above using the previous values of S. With this iteration, we may get S_0, S_1, S_2, and so on. We write the general solution as $S = S_0 + S_1 + S_2 + \cdots$. Similarly, applying the same procedure for the remaining equations of the systems, we have

$$E_0 = E^0, \quad E_1 = \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ (\beta+\gamma) C_0 - \alpha \mathbb{A}_0 \right\} \right\},$$

$$E_{n+1} = \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ (\beta+\gamma) C_n - \alpha \mathbb{A}_n \right\} \right\}, \ n \geq 0.$$

$$C_0 = C^0, \quad C_1 = \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ \alpha \mathbb{A}_0 - (\beta+\gamma) C_0 \right\} \right\},$$

$$C_{n+1} = \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ \alpha \mathbb{A}_n - (\beta+\gamma) C_n \right\} \right\}, \ n \geq 0.$$

$$P_0 = P^0, \quad P_1 = \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ \gamma C_0 \right\} \right\},$$

$$P_{n+1} = \mathbb{L}^{-1} \left\{ \left(\frac{s^\theta(1-\theta)+\theta}{s^\theta M(\theta)} \right) \mathbb{L} \left\{ \gamma C_n \right\} \right\}, \ n \geq 0.$$

Now to get a more specific approximate solution from the above general solution. Consider $M(\theta) = 1$, $S^0 = 0.01$, $E^0 = 0.0001$, $C^0 = P^0 = 0$, and applying the above procedure or formulas, one may get the following approximate solution of the specified model (2):

$$S(t) = 0.01 - 0.000001\alpha \left(1 - \theta + \frac{t^\theta}{\Gamma(\theta)} \right),$$

$$
\begin{aligned}
E(t) &= 0.0001 - 0.000001\alpha\left(1-\theta+\frac{t^{\theta}}{\Gamma(\theta)}\right),\\
C(t) &= 0.000001\alpha\left(1-\theta+\frac{t^{\theta}}{\Gamma(\theta)}\right), \qquad (9)\\
P(t) &= 0.000001\alpha\gamma\left((1-\theta)^2+\frac{2(1-\theta)^2t^{\theta}}{\Gamma(\theta)}+\frac{\theta t^{\theta+1}}{(\theta+1)\Gamma(\theta)}\right).
\end{aligned}
$$

In the above solution, we only present the first two or three terms of the solution of each compartment of the specified model. One can get more terms with the help of the above-mentioned procedure, leading to a more precise approximation of the dynamics of the concerned model. To properly understand the physical and geometrical meaning of the enzymatic reaction given in the shape of model (2), we have the following visualizations with complete descriptions.

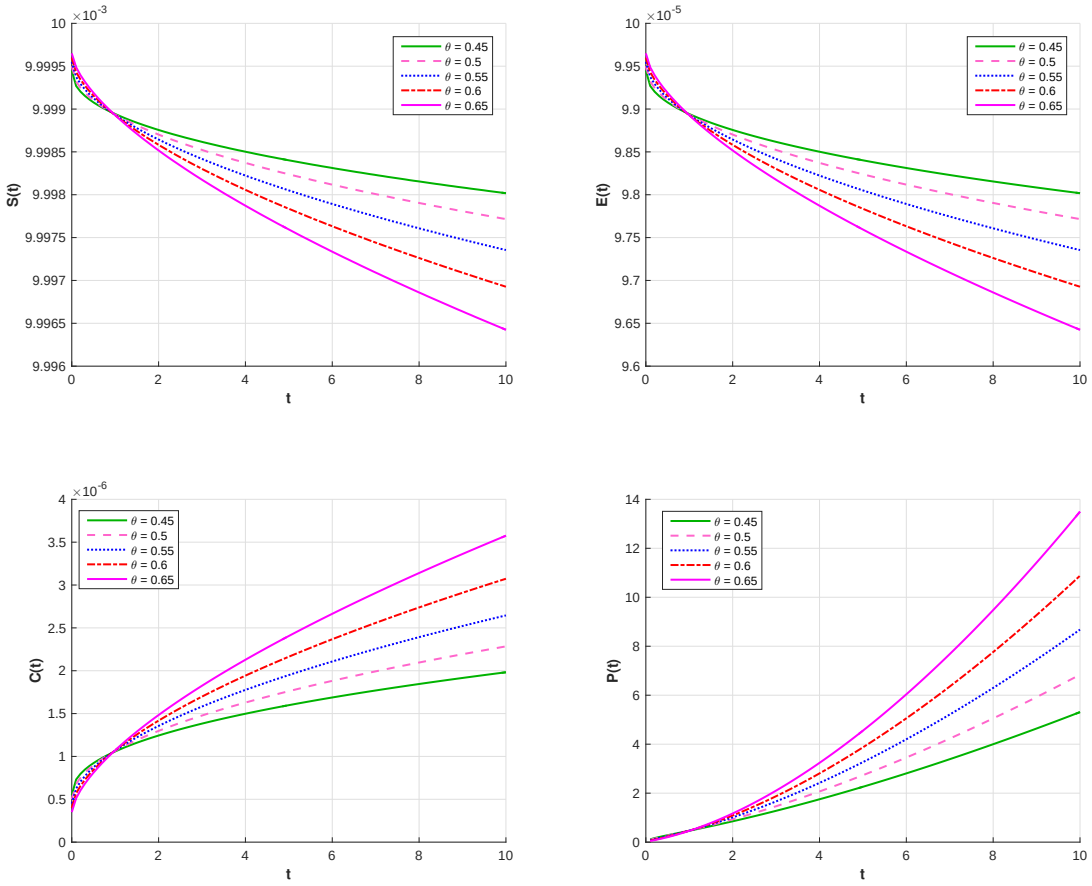


Figure 3: Investigating $S(t), E(t), C(t), P(t)$ visually for various θ values and for fixed values of $\alpha = \gamma = M(\theta) = 1.0$.

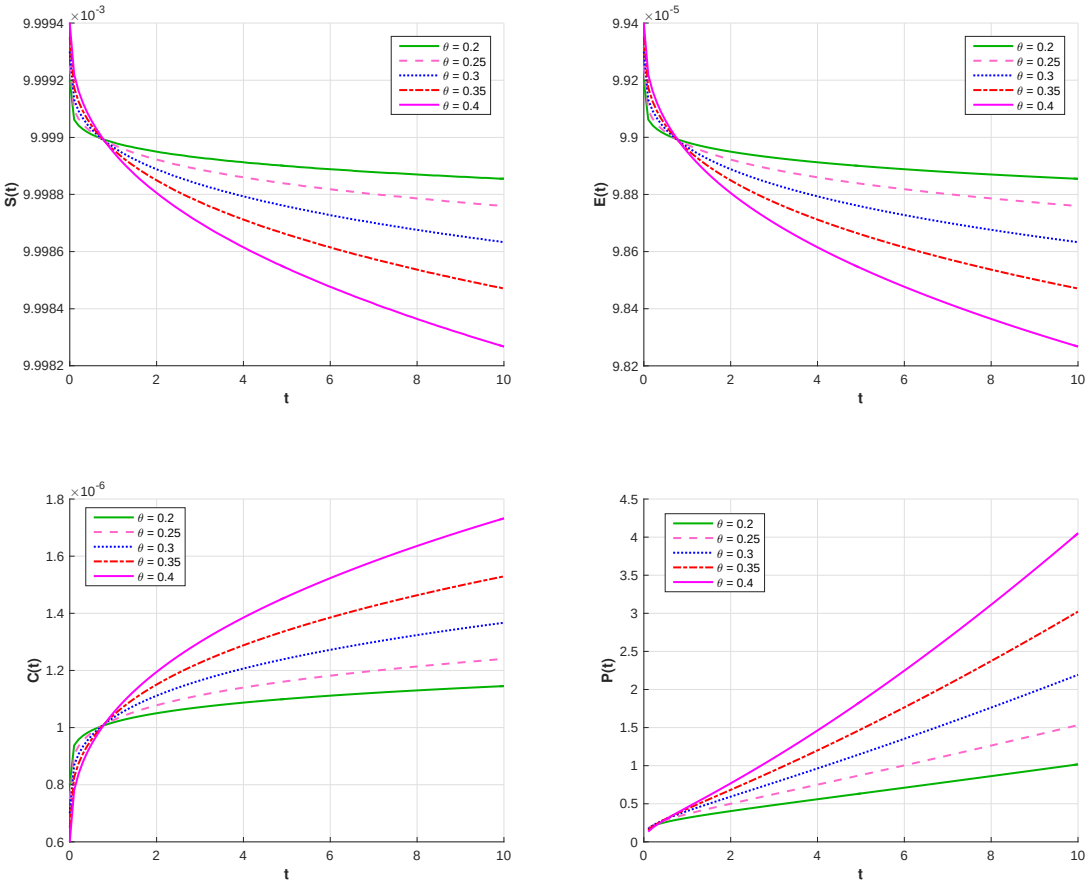
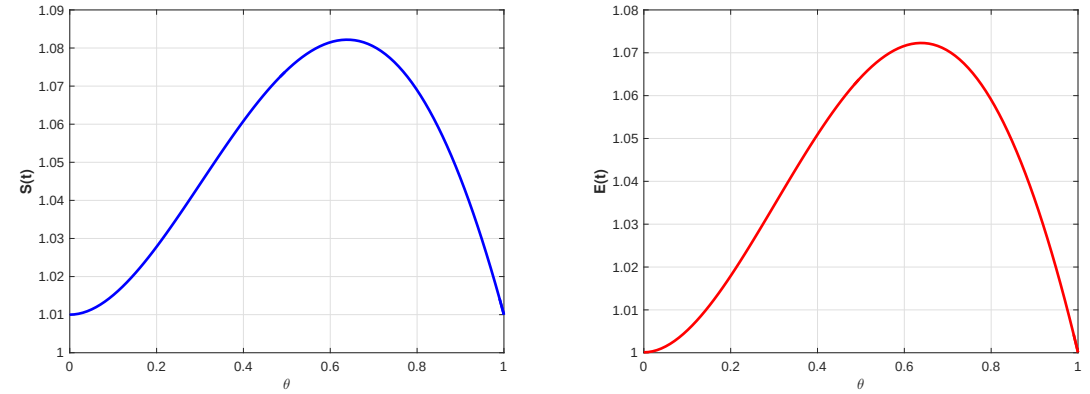


Figure 4: Investigating $S(t), E(t), C(t), P(t)$ visually for various θ values and for fixed values of $\alpha = \gamma = M(\theta) = 1.0$.



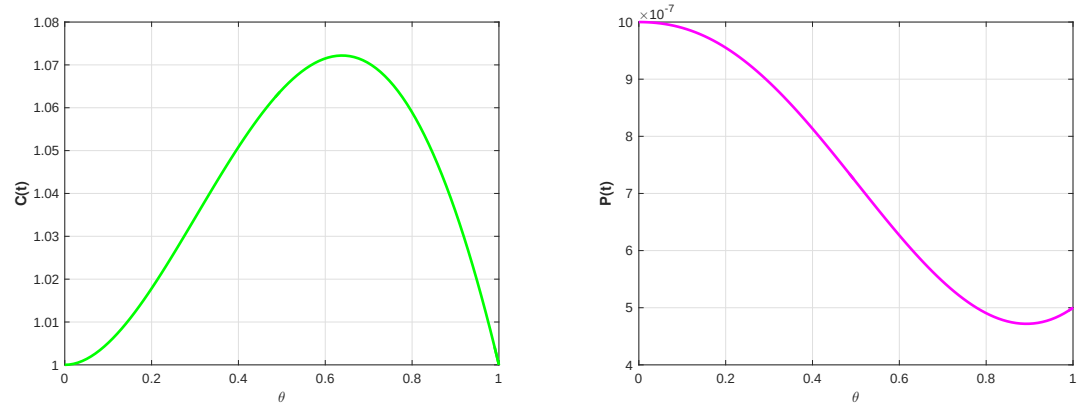


Figure 5: Investigating $S(t), E(t), C(t), P(t)$ visually against θ for fixed values of $t = \alpha = \gamma = M(\theta) = 1.0$.

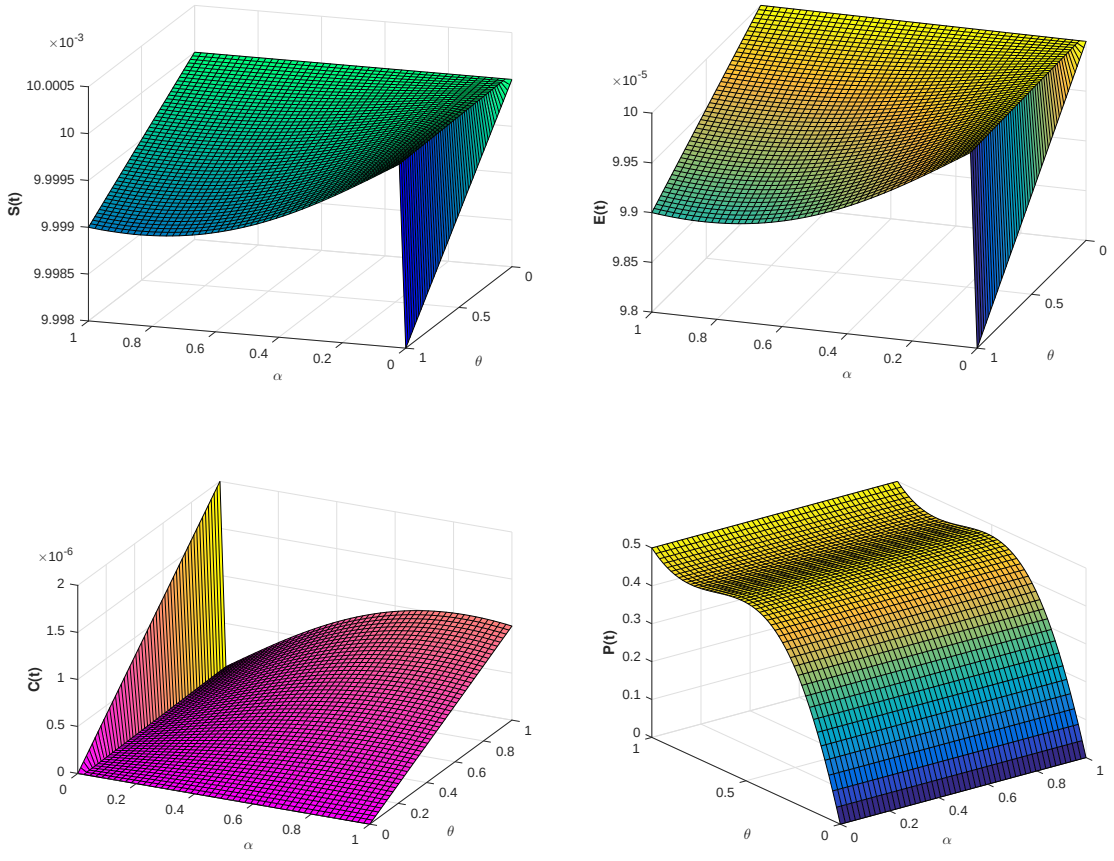


Figure 6: Investigating $S(t), E(t), C(t), P(t)$ visually against α, θ and for fixed values of $t = \gamma = M(\theta) = 1.0$.

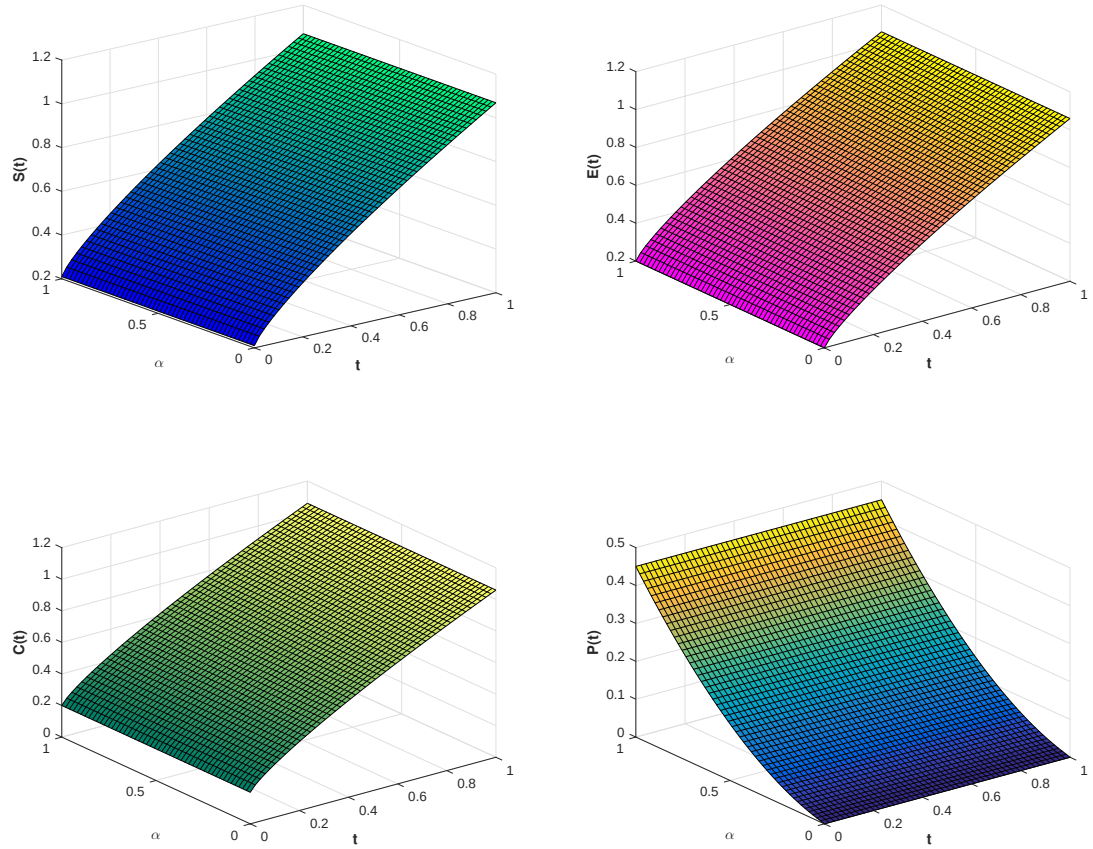
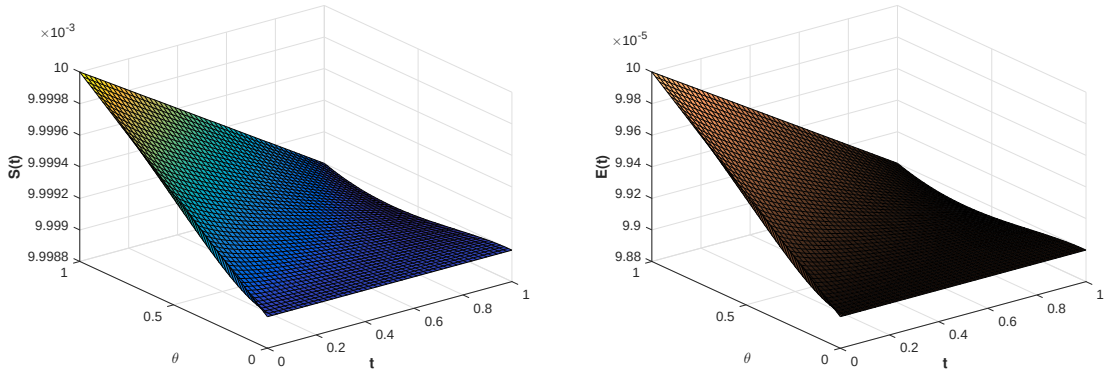


Figure 7: Investigating $S(t), E(t), C(t), P(t)$ visually against t, α and for fixed values of $\theta = 0.8$, $\gamma = M(\theta) = 1.0$.



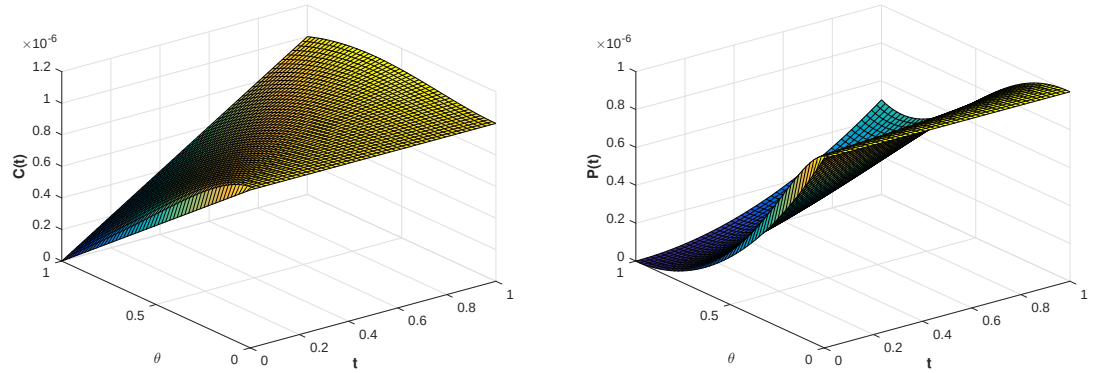


Figure 8: Investigating $S(t), E(t), C(t), P(t)$ visually against t, θ and for fixed values of $\alpha = \gamma = M(\theta) = 1.0$.

From Figures 3–8, the graphical simulations reveal the detailed influence of fractional derivatives on the enzymatic reaction model. Figure 3 presents two-dimensional profiles of the system variables, where the substrate and enzyme concentrations decrease over time, while the product and enzyme-substrate complex increase, reflecting the natural progression of biochemical reactions. The effect of fractional order is evident: smaller values yield smoother and more stable trajectories, indicating the stabilizing role of memory effects. Figure 4 reinforces this observation by showing that decreasing fractional values significantly alters the shape of the curves, driving the system toward a stable state and confirming the positive role of fractional derivatives in system dynamics.

In Figure 4, the relationship between fractional order and solution behavior at a fixed time is displayed. The substrate, enzyme, and complex concentrations follow bell-shaped profiles with maxima near the midpoint of the interval, while the product concentration exhibits a distinct and non-symmetric trend, highlighting its unique contribution to system evolution. Moving to three-dimensional visualizations, Figure 6 illustrates the combined effect of fractional order θ and parameter α on the solutions. The figure makes it clear that these two parameters jointly shape stability and long-term dynamics, emphasizing their central role in fractional modeling.

Figure 7 further explores the solutions with respect to time and parameter α, for a fixed value of θ. The resulting surfaces demonstrate how both time evolution and parameter variation influence the system simultaneously, producing complex but interpretable dynamics. Finally, Figure 8 examines the interaction of time and fractional order, providing a full three-dimensional description of the system behavior. The solutions in this case display distinct patterns compared to the previous figure, highlighting how fractional order fundamentally alters the trajectory of each state variable. Collectively, these results demonstrate that fractional-order derivatives capture memory and long-term dependencies more effectively than classical approaches, offering a richer and more realistic framework for modeling biochemical processes.

5. Sensitivity Analysis

Sensitivity analysis measures how changes in a parameter affect the output of the system. Here, we analyze the sensitivity of the functions $S(t)$, $E(t)$, $C(t)$, and $P(t)$ with respect to the parameter α only. To extend this calculation further to the remaining parameters β and γ, we need more terms in series solutions of $S(t)$, $E(t)$, $C(t)$, and $P(t)$.

Sensitivity of $S(t)$ is given as follows, [24]:

$$S_\alpha(S) = \frac{\alpha}{S(t)}\frac{\partial S(t)}{\partial \alpha} = \frac{-0.000001\left(1-\theta+\frac{t^\theta}{\Gamma(\theta)}\right)\times\alpha}{0.01-0.000001\alpha\left(1-\theta+\frac{t^\theta}{\Gamma(\theta)}\right)}.$$

Sensitivity of $E(t)$ is given as follows:

$$S_\alpha(E) = \frac{-0.000001\left(1-\theta+\frac{t^\theta}{\Gamma(\theta)}\right)\times\alpha}{0.0001-0.000001\alpha\left(1-\theta+\frac{t^\theta}{\Gamma(\theta)}\right)}.$$

Sensitivity of $C(t)$ is given as follows:

$$S_\alpha(C) = 1.$$

Sensitivity of $P(t)$ is given as follows:

$$S_\alpha(P) = 1.$$

The sensitivities of $S(t)$ and $E(t)$ depend on θ and t. Their general expressions indicate a complex relationship with α. Since the sensitivities of $S(t)$ and $E(t)$ are negative, increasing α will decrease both $S(t)$ and $E(t)$. For larger t, the sensitivity increases in magnitude, meaning the impact of α becomes stronger over time. For smaller θ, the term $(1-\theta)$ increases, leading to a greater sensitivity in absolute terms. Since sensitivity is a function of t and θ, exact numerical values will vary, but the overall trend remains: $S(t)$ and $E(t)$ decrease as α increases. The sensitivity of $C(t)$ is exactly 1, meaning it changes proportionally with α. The sensitivity of $P(t)$ is also exactly 1, meaning it scales directly with α.

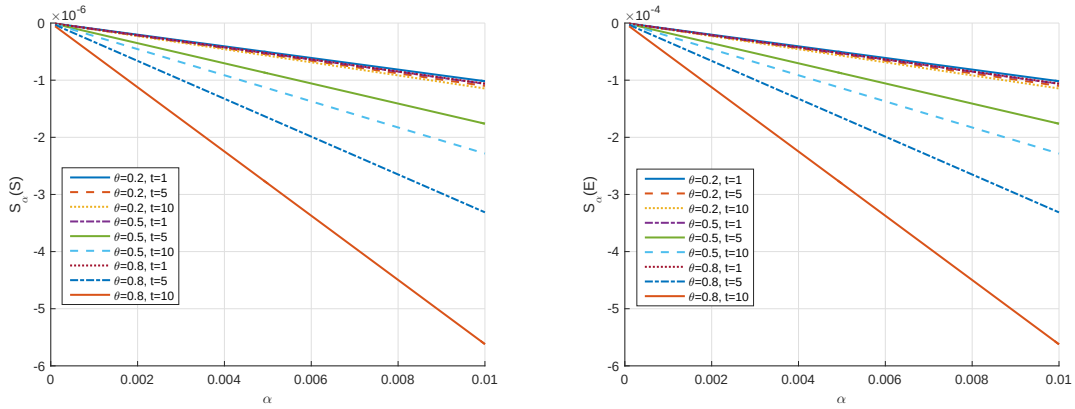


Figure 9: Sensitivity of $S(t)$ and $E(t)$ for different θ and t values.

Figure 9 provides the sensitivity analysis of substrate and enzyme concentrations with respect to α, taking different values of fractional order and time. The negative sensitivity shows an inverse relation between function and parameter, i.e., when sensitivity is negative, it indicates that an increase in the value of the parameter decreases the value of the function. Furthermore, for practical analysis, the choice of fractional order and time is very important in sensitivity analysis.

6. Artificial Neural Network validation

The artificial neural network architecture consists of two hidden layers with 20 and 15 neurons, using the `tansig` activation function. Training employed the Levenberg–Marquardt algorithm with 1000 epochs, a learning goal of 10^{-6}, and an early stopping criterion with validation checks. The dataset was split into training (70%), validation (15%), and testing (15%) subsets, [25]. We consider the fractional order $\theta = 0.5$, initial conditions $S^0 = 0.8$, $E^0 = 0.2$, $C^0 = P^0 = 0$, and the parameter values $\alpha = 0.01$, $\beta = 0.01$, $\gamma = 0.01$.

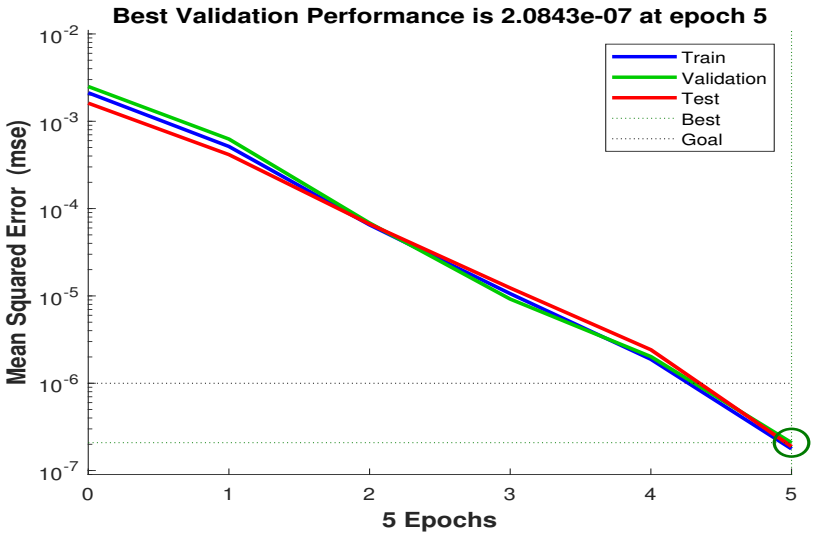


Figure 10: MSE performance of the model at epoch 5.

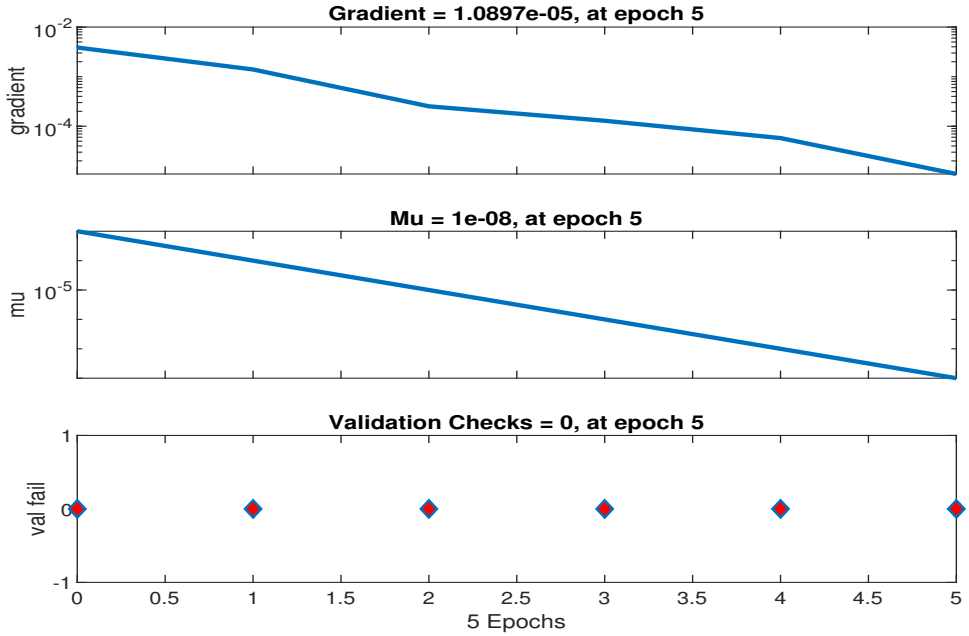


Figure 11: Training test of the model at epoch 5.

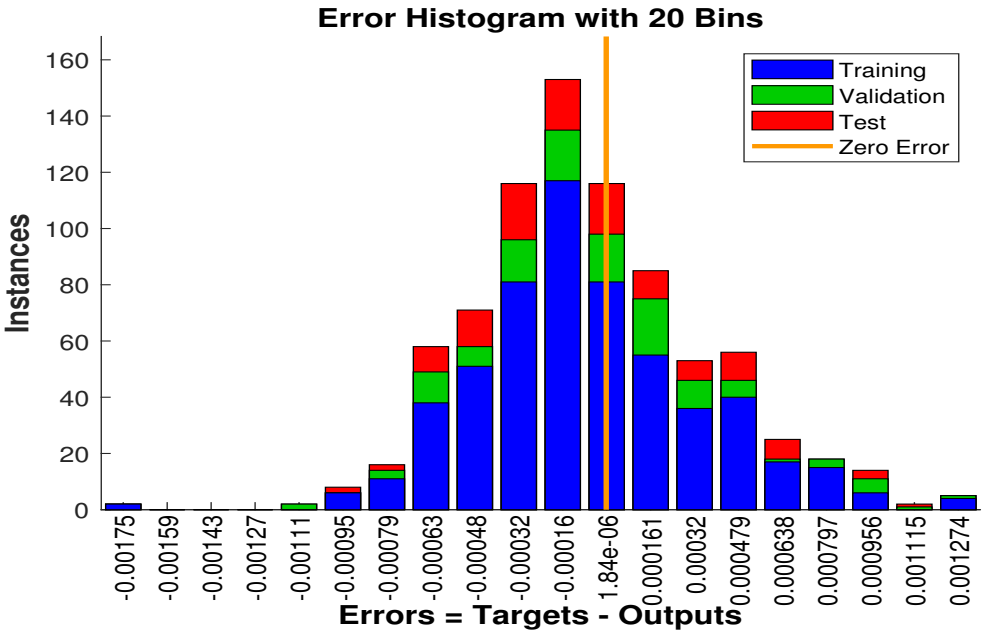


Figure 12: Training histogram of the model at epoch 5.

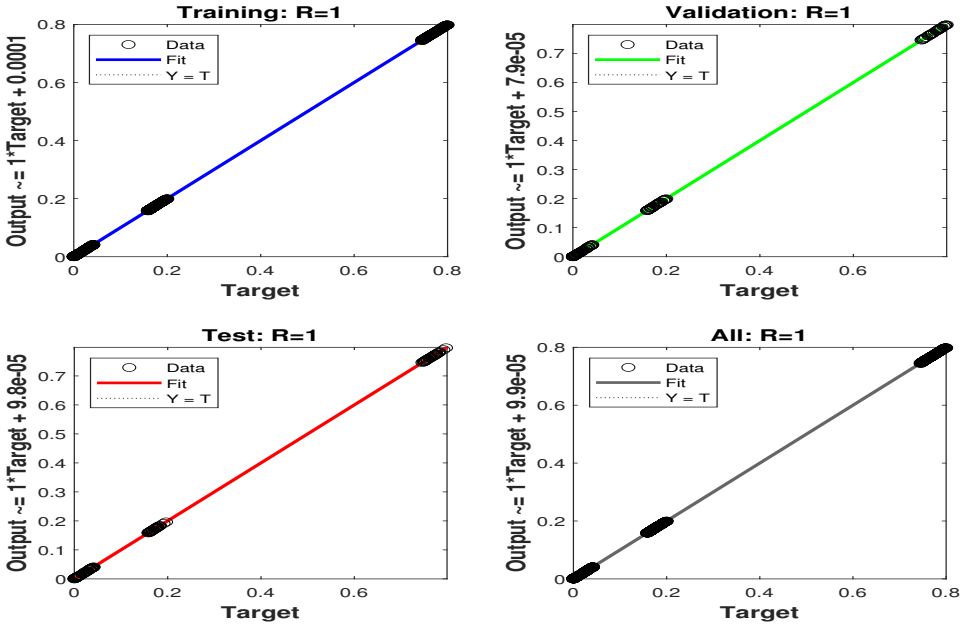


Figure 13: Training regression of the model at epoch 5.

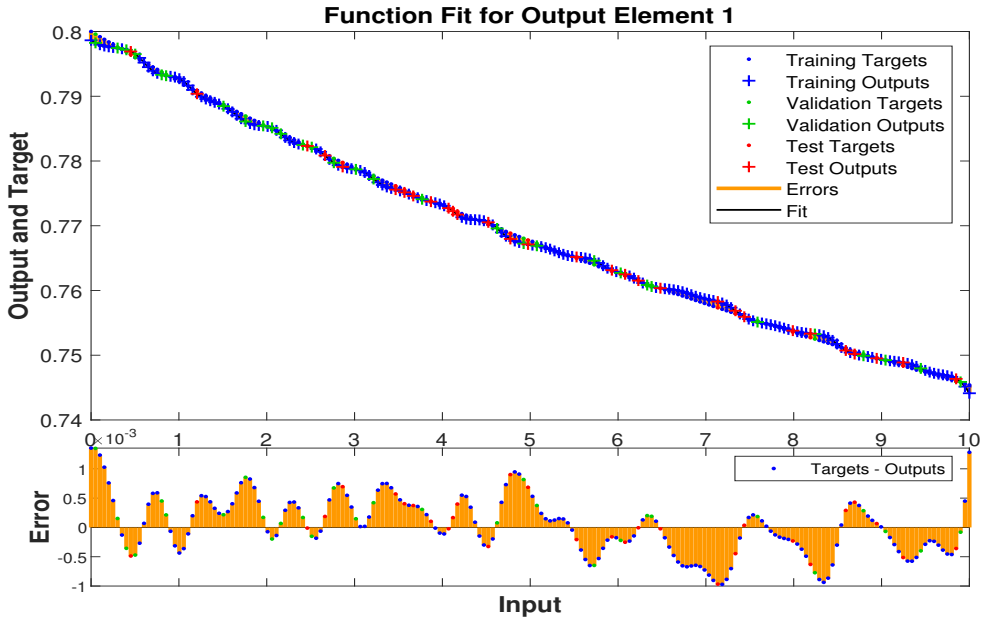


Figure 14: Model fit to the training data at epoch 5.

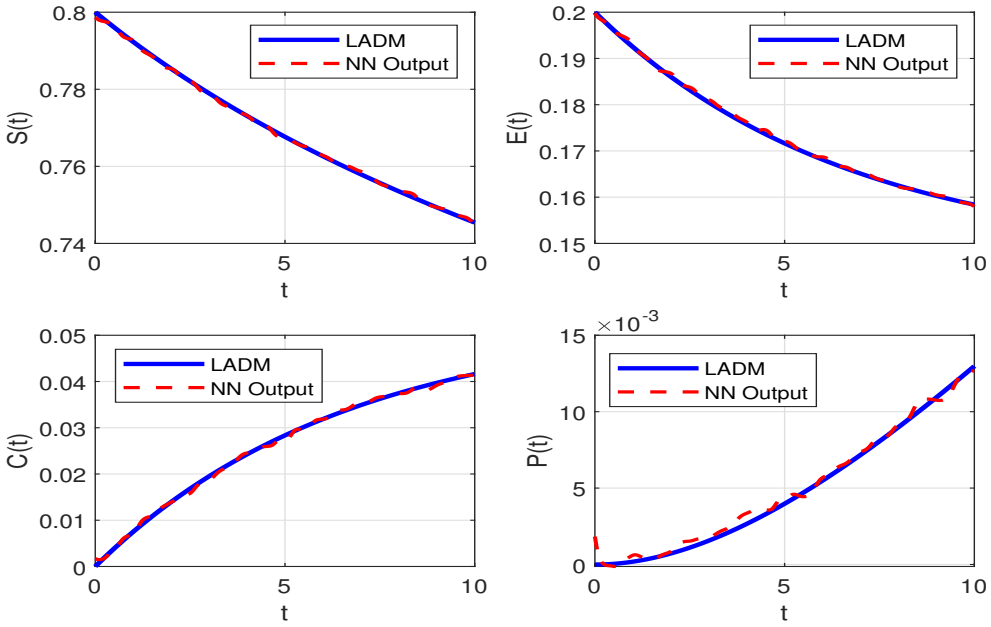


Figure 15: Comparison of NN approximation and LADM solutions.

Figures 10-15 provide illustrations of MSE performance, training tests, and comparison of NN outputs and LADM solutions at $\theta = 0.5$ of (2). In this work, the artificial neural network (ANN) validation has been carried out primarily to confirm the accuracy of the approximate solutions obtained through the LADM. The training, validation, and testing sets were designed to ensure consistency in the results, and the regression values confirmed the reliability of the numerical scheme. The current validation is sufficient for demonstrating the correctness of the proposed methodology. Nevertheless, future studies may incorporate advanced ANN architectures and more detailed performance evaluations to further strengthen the computational validation of fractional enzymatic models.

7. Conclusion

This study has proposed and analyzed a fractional-order enzymatic reaction model using the ABC derivative. The model extends the classical framework by incorporating memory and non-local effects, thereby offering a more realistic description of biochemical processes. Sensitivity analysis with respect to reaction rate parameters has revealed the crucial influence of these rates on system dynamics. These findings suggest that fractional models can serve as powerful tools for refining enzyme kinetics, offering improved interpretability compared to classical integer-order formulations. Furthermore, the framework has potential applications in drug development, metabolic engineering, and biochemical process optimization, where refined control strategies from fractional modelling may lead to more accurate predictions and more effective process design. Future research should address several important extensions of this work. One direction is the validation of the proposed model against experimental or real biochemical datasets, which would strengthen its practical relevance. Secondly, expanding sensitivity analysis to multiple parameters and examining alternative fractional operators could uncover deeper insights into enzymatic dynamics. In addition, integrating advanced deep learning methods with fractional approaches may provide scalable tools for analyzing complex biochemical networks. Taken together, these directions would not only advance the theoretical understanding of fractional enzymatic models but also foster their application in interdisciplinary areas of science and engineering.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data Availability

All data is included in the paper.

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