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Development of Quantum Hermite-Hadamard Type Inequalities Using Green's Function Techniques

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Abstract. In this paper, we investigate the quantum Hermite-Hadamard inequality using the Green's function. This process leads to the derivation of novel quantum identities, which are then employed to establish novel inequalities. Utilizing these identities, we establish novel inequalities. The main results of the paper are derived using various techniques such as q-identities, convexity and Jensen inequality. Furthermore, the study provides numerical validation and graphical representations to support the main results.

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Key Words and Phrases: Quantum integral, Green function, $\mathcal{H}$-$\mathcal{H}$-inequality

1. Introduction

Scientists are very interested in the theory of convexity because of its many uses. Convexity is an important term in the extension and generalization of inequalities. As a result, convexity and inequality theory are closely related. Many inequalities have been motivated by convex functions, which are essential to inequality theory. Therefore, it is evident that the Hermite-Hadamard ($\mathcal{H} - \mathcal{H}$) inequality assumes particular significance in the context of convex functions. The integral mean of any convex function defined within a closed and bounded area, inclusive of the endpoints and midpoints of the function's domain, can

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be estimated through the utilization of upper and lower bounds. This estimation is facilitated by the $\mathcal{H}-\mathcal{H}$ inequality, a geometric-based principle. The aforementioned double inequality can be articulated as follows: Let φ be a convex mapping on $[\omega_1, \omega_2] \subset \mathbb{R}$, where $\omega_1 \neq \omega_2$. Then

$$\varphi\left(\frac{\omega_1+\omega_2}{2}\right) \leq \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \varphi(\varkappa)d\varkappa \leq \frac{\varphi(\omega_1)+\varphi(\omega_2)}{2}.$$

One important finding in convexity theory is the $\mathcal{H}-\mathcal{H}$ inequality. The field has advanced significantly as a result of the efforts of several mathematicians who have concentrated on enhancing and generalizing the inequality. We encourage interested readers to review some of the references and the papers [1–4] as it has been widely researched and used in a variety of settings.

Definition 1. *A function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\varphi\left(\sigma\omega_1 + (1-\sigma)\,\omega_2\right) \leq \sigma\varphi\left(\omega_1\right) + (1-\sigma)\,\varphi\left(\omega_2\right)$$

holds for all $\omega_1, \omega_2 \in I$ and $\sigma \in [0,1]$. If $-\varphi$ is convex, then φ is said to be concave.

Convex function theory plays a crucial role in both pure and applied mathematics. Noteworthy inequalities have been derived using various types of convexity [5–9].

The study of integrals and derivatives requires a solid understanding of calculus, a fundamental branch of mathematics. A new mathematical framework called quantum calculus, or q-calculus, has emerged as a result of the evolution and adaptation of the classical calculus concepts. Quantum calculus, which includes q-integral calculus, q-fractional calculus, and q-transform analysis, is the study of calculus without limits. Numerous mathematical and physical areas have shown how effective these techniques are. In the early 20th century, the first description of quantum calculus was provided by Jackson. The book [10] is recommended for those who wish to investigate deeper into this topic. q-deformation is a key idea in the field of quantum calculus. This procedure includes changing the characteristics of calculus operations, such as differentiation and integration, by adding a parameter q. Similar to ordinary differential equations in classical calculus, q-difference equations are used in q-calculus to define functions and their derivatives.
q-integrals and q-derivatives, which are extensions of their classical counterparts, are introduced in quantum calculus. It is clear that these operators meet several q-analogues of the basic theorem of calculus and the Leibniz rule, which are characteristics of ordinary derivatives and integrals. This research paper's main goal is to investigate the $\mathcal{H}-\mathcal{H}$ inequality related to the quantum integral operator. Several features of the q-integral for a continuous function were defined and shown by Tariboon and Ntouyas in [11] in 2013. However, Kunt and Iscan showed in [12] that the $\mathcal{H}-\mathcal{H}$ inequality derived in [11] is incorrect on the left. The following variation of the $\mathcal{H}-\mathcal{H}$ inequality for the q-integral was then established by Alp *et al.* in [13]:

$$\varphi\left(\frac{q\omega_1+\omega_2}{q+1}\right) \leq \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \varphi(\varkappa_1)_{\omega_1} d_q\varkappa_1 \leq \frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}, \tag{1}$$

where $q \in (0, 1)$ and $\varphi : [\omega_1, \omega_2] \to \mathbb{R}$ is a convex function. The aforementioned inequality (1) is referred to as the quantum $\mathcal{H} - \mathcal{H}$ inequality. In recent years, this inequality has been the focus of research by numerous mathematicians. In [14], Ali *et al.* established an identity related to the quantum $\mathcal{H} - \mathcal{H}$ inequality and the q-integral. Since integral identity and applications of power-mean inequality and Hölder inequality result in the q-integral form of the $\mathcal{H} - \mathcal{H}$ inequality, it is clear that certain conclusions have been established. Previously, the findings were deduced for a certain value of q. Noor *et al.*, in [15], established some novel quantum estimates for $\mathcal{H} - \mathcal{H}$ inequalities via q-differentiable convex functions and q-differentiable quasi-convex functions. In the present study [16], the authors propose a novel definition of convexity (k-harmonically $\gamma-$convex function) and employ this definition to derive new $\mathcal{H} - \mathcal{H}$-type integral inequalities for quantum integrals. A number of authors engaged in research within this field have also studied the symmetric quantum calculus of the $\mathcal{H} - \mathcal{H}$ inequality. Researchers interested in further works may refer to studies [17] and [18]. In [19], Budak and colleagues took into account the class of coordinated convex functions in order to derive the extended form of the quantum $\mathcal{H} - \mathcal{H}$ inequality. In order to further generalize the quantum $\mathcal{H} - \mathcal{H}$ inequality, the double integral identity has been developed. Furthermore, as mentioned in [20] and [21], it has been shown that the above inequality holds for the class of s-convex and r-convex functions, respectively.

This study's main goal is to analyze the quantum $\mathcal{H} - \mathcal{H}$ inequality using a Green function method. Several novel quantum identities were inferred throughout this particular technique. New inequalities have been made possible by the use of these identities. Convexity, Jensen's inequality for convex mappings, and q-identities are among the methods used in the study to arrive at the main results of the work. To support the primary findings, the study offers graphical representations and numerical confirmation.

2. Preliminaries and Definitions of q-calculus

The following discussion will commence with a presentation of these fundamental definitions.

Definition 2. *[11] Let $\varphi : [\omega_1, \omega_2] \to \mathbb{R}$ be a continuous function, and let $c \in [\omega_1, \omega_2]$. Then the expression*

$$_{\omega_1}D_q\varphi(c) = \frac{\varphi(c) - \varphi(qc + (1-q)\omega_1)}{(1-q)(c-\omega_1)}, c \neq \omega_{1,\omega_1} D_q\varphi(\omega_1) = \lim_{c\to b_1} {}_{\omega_1}D_q\varphi(c)$$

is called the q-derivative on $[\omega_1, \omega_2]$ of the function at c.
We call φ q-differentiable on $[\omega_1, \omega_2]$ if $_{\omega_1}D_q\varphi(c)$ exists for all $c \in [\omega_1, \omega_2]$.

The q-Jackson integral, or q-integral ([22]), was found by Jackson in 1910.

Definition 3. *[11] If $\varphi : [\omega_1, \omega_2] \to \mathbb{R}$ is a continuous function, then the q-integral of φ on $[\omega_1, \omega_2]$ is defined as:*

$$\int_{\omega_1}^{c} \varphi(\varkappa)\, _{\omega_1}d_q\varkappa = (c-\omega_1)(1-q)\sum_{k=0}^{\infty} q^k\varphi\left(q^kc + (1-q^k)\omega_1\right),$$

where $0 < q < 1$ *and* $c \in [\omega_1, \omega_2]$.

There are several important properties of q-integral, for example interval addition, linearity, triangular, and monotonicity property.

Theorem 1. *[11] If* $\varphi : [\omega_1, \omega_2] \to \mathbb{R}$ *is a continuous function. Then*

$$\int_{\xi}^{c} {}_{\omega_1}D_q\varphi(c)_{\omega_1}d_q\varkappa = \varphi(c) - \varphi(\xi)$$

where $\xi \in (\omega_1, c)$.

Theorem 2. *[23] If* $\varphi, \Omega : [\omega_1, \omega_2] \to \mathbb{R}$ *are two continuous functions and suppose* $\varphi(\varkappa) \leq \Omega(\varkappa), \forall \varkappa \in [\omega_1, \omega_2]$. *Then we have*

$$\int_{\omega_1}^{c} \varphi(\varkappa)_{\omega_1}d_q\varkappa \leq \int_{\omega_1}^{c} \Omega(\varkappa)\,_{\omega_1}d_q\varkappa.$$

Theorem 3. *[11] If* $\varphi : [\omega_1, \omega_2] \to \mathbb{R}$ *is a continuous function. Then we have*

$$ {}_{\omega_1}D_q \int_{\omega_1}^{c} \varphi(\varkappa)_{\omega_1}d_q\varkappa = \varphi(c);$$
$$\int_{\xi}^{c} {}_{\omega_1}D_q\varphi(\varkappa)_{\omega_1}d_q\varkappa = \varphi(c) - \varphi(\xi)$$

where $\xi \in (\omega_1, c)$.

Theorem 4. *[11] If* $\varphi, \Omega : [\omega_1, \omega_2] \to \mathbb{R}$ *are two continuous functions and suppose* $\kappa \in \mathbb{R}, c \in [\omega_1, \omega_2]$, *and* $\xi \in (\omega_1, c)$. *Then we have*

$$\begin{aligned}
&\int_{\omega_1}^{c} [\varphi(\varkappa) + \Omega(\varkappa)]_{\omega_1}d_q\varkappa = \int_{\omega_1}^{c} \varphi(\varkappa)_{\omega_1}d_q\varkappa + \int_{\omega_1}^{c} \Omega(\varkappa)_{\omega_1}d_q\varkappa;\\
&\int_{\omega_1}^{c} \kappa\varphi(\varkappa)_{\omega_1}d_q\varkappa = \kappa \int_{\omega_1}^{c} \varphi(\varkappa)_{\omega_1}d_q\varkappa;\\
&\int_{\xi}^{c} \varphi(\varkappa)_{\omega_1}D_q\Omega(\varkappa)_{\omega_1}d_q\varkappa\\
&\quad=\varphi(c)\Omega(c) - \varphi(\xi)\Omega(\xi) - \int_{\xi}^{c} \Omega(q\varkappa + (1-q)\omega_1)_{\omega_1}D_q\varphi(\varkappa)_{\omega_1}d_q\varkappa.
\end{aligned}$$

3. Main Results

The fundamental results will be established through the utilization of the following lemma.

Lemma 1. *[24, 25] Let $\mathcal{G}$ be the Green function defined on $[\omega_1, \omega_2] \times [\omega_1, \omega_2]$ by*

$$\mathcal{G}(\varkappa, \ell) = \begin{cases} \omega_1 - \ell, & \omega_1 \leq \ell \leq \varkappa, \\ \omega_1 - \varkappa, & \varkappa \leq \ell \leq \omega_2. \end{cases}$$

Then any $\varphi \in C^2([\omega_1, \omega_2])$ can be expressed as

$$\varphi(\varkappa) = \varphi(\omega_1) + (\varkappa - \omega_1)\varphi'(\omega_2) + \int_{\omega_1}^{\omega_2} \mathcal{G}(\varkappa, \ell)\varphi''(\ell)d\ell. \tag{2}$$

Theorem 5. *Let $\varphi \in C^2[\omega_1, \omega_2]$ such that φ'' is convex and $0 < q < 1$. Then*

$$\begin{aligned}
&\frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \varphi(\varkappa)_{\omega_1} d_q\varkappa - \varphi\left(\frac{q\omega_1 + \omega_2}{q+1}\right) \\
&\quad \leq \frac{1}{\omega_2 - \omega_1}\Bigg[\frac{1}{6}\left(\varphi''(\omega_2) - \varphi''(\omega_1)\right)\left(\frac{(\omega_2 - \omega_1)^3}{(1+q)(1+q^2)}\right) \\
&\quad - \left(\frac{\varphi''(\omega_2) + \varphi''(\omega_1)}{2}\right)\left(\frac{(\omega_2 - \omega_1)}{1+q}\right)\omega_1^2 - \left(\frac{2\varphi''(\omega_2) + \varphi''(\omega_1)}{6}\right)\omega_1^3 \\
&\quad - \left(\frac{\varphi''(\omega_2) + \varphi''(\omega_1)}{6}\right)\left(\frac{(q\omega_1 + \omega_2)}{(1+q)}\right)^3 + \left(\frac{\varphi''(\omega_2)\omega_1 - \varphi''(\omega_1)\omega_2}{2}\right) \\
&\quad \left(\frac{(q\omega_1 + \omega_2)}{(1+q)}\right)^2 + \frac{\omega_1\omega_2\varphi''(\omega_1)(\omega_2 - \omega_1)}{(1+q)} + \varphi''(\omega_1)\omega_1^2\omega_2 \\
&\quad + \frac{1}{2}\left(\frac{(\omega_2 - \omega_1)^3}{1+q+q^2}\right)\varphi''(\omega_1)\Bigg].
\end{aligned} \tag{3}$$

Proof. *If we set $\varkappa = \frac{q\omega_1 + \omega_2}{1+q}$ in (2), then we get*

$$\begin{aligned}
\varphi\left(\frac{q\omega_1 + \omega_2}{q+1}\right) &= \varphi(\omega_1) + \left(\frac{q\omega_1 + \omega_2}{q+1} - \omega_1\right)\varphi^{'}(\omega_2) + \int_{\omega_1}^{\omega_2} \mathcal{G}\left(\frac{q\omega_1 + \omega_2}{q+1}, \ell\right)\varphi^{''} d\ell \\
&= \varphi(\omega_1) + \frac{\omega_2 - \omega_1}{q+1}\varphi'(\omega_2) + \int_{\omega_1}^{\omega_2} \mathcal{G}\left(\frac{q\omega_1 + \omega_2}{q+1}, \ell\right)\varphi''(\ell)d\ell.
\end{aligned} \tag{4}$$

Also from (2), we obtain that

$$\begin{aligned}
&\frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \varphi(\varkappa)_{\omega_1} d_q\varkappa \\
&\quad = \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \left\{\varphi(\omega_1) + (\varkappa - \omega_1)\varphi'(\omega_2) + \int_{\omega_1}^{\omega_2} \mathcal{G}(\varkappa, \ell)\varphi''(\ell)d\ell\right\}_{\omega_1} d_q\varkappa \\
&\quad = \varphi(\omega_1) + \frac{\omega_2 - \omega_1}{q+1}\varphi'(\omega_2) + \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} \mathcal{G}(\varkappa, \ell)\varphi''(\ell)d\ell_{\omega_1} d_q\varkappa.
\end{aligned} \tag{5}$$

Subtracting (4) *from* (5), *we get:*

$$\begin{aligned}\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa-\varphi\left(\frac{q\omega_1+\omega_2}{q+1}\right)&\\=&\varphi(\omega_1)+\frac{\omega_2-\omega_1}{q+1}\varphi'(\omega_2)+\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\int_{\omega_1}^{\omega_2}\mathcal{G}(\varkappa,\ell)\varphi''(\ell)d\ell_{\omega_1}d_q\varkappa\\&-\varphi(\omega_1)-\frac{\omega_2-\omega_1}{q+1}\varphi'(\omega_2)-\int_{\omega_1}^{\omega_2}\mathcal{G}\left(\frac{q\omega_1+\omega_2}{q+1},\ell\right)\varphi''(\ell)d\ell.\\=&\int_{\omega_1}^{\omega_2}\left\{\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathcal{G}(\varkappa,\ell)_{\omega_1}d_q\varkappa-\mathcal{G}\left(\frac{q\omega_1+\omega_2}{q+1},\ell\right)\right\}\varphi''(\ell)d\ell.\end{aligned}\tag{6}$$

$$\text{Let } \gamma(\ell)=\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathcal{G}(\varkappa,\ell)_{\omega_1}d_q\varkappa-\mathcal{G}\left(\frac{q\omega_1+\omega_2}{q+1},\ell\right).$$

Clearly $\gamma(\ell)$ *is the difference of middle and left side of* (1), *for the Green function therefore* $\gamma(\ell)$ *is non-negative.*

$$\text{Let } \ell=\frac{\ell-\omega_1}{\omega_2-\omega_1}\omega_2+\frac{\omega_2-\ell}{\omega_2-\omega_1}\omega_1,\ \text{then from (6) we have}$$

$$\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa-\varphi\left(\frac{q\omega_1+\omega_2}{q+1}\right)=\int_{\omega_1}^{\omega_2}\gamma(\ell)\varphi''\left(\frac{\ell-\omega_1}{\omega_2-\omega_1}\omega_2+\frac{\omega_2-\ell}{\omega_2-\omega_1}\omega_1\right)d\ell.$$

By convexity of φ'', *we obtain*

$$\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa-\varphi\left(\frac{q\omega_1+\omega_2}{q+1}\right)\leq\int_{\omega_1}^{\omega_2}\gamma(\ell)\left\{\frac{\ell-\omega_1}{\omega_2-\omega_1}\varphi''(\omega_2)+\left(\frac{\omega_2-\ell}{\omega_2-\omega_1}\right)\varphi''(\omega_1)\right\}d\ell$$

$$\begin{aligned}\Rightarrow\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa-\varphi\left(\frac{q\omega_1+\omega_2}{q+1}\right)\leq&\frac{1}{(\omega_2-\omega_1)}\left\{\varphi''(\omega_2)\int_{\omega_1}^{\omega_2}\gamma(\ell)(\ell-\omega_1)d\ell+\varphi''(\omega_1)\right.\\&\left.\int_{\omega_1}^{\omega_2}\gamma(\ell)(\omega_2-\ell)d\ell\right\}.\end{aligned}\tag{7}$$

Now we find the integral $\int_{\omega_1}^{\omega_2}\gamma(\ell)(\ell-\omega_1)d\ell$.
If $\varphi(\ell)=\frac{1}{6}\ell^3-\frac{1}{2}\omega_1\ell^2$, *then* $\varphi''(\ell)=\ell-\omega_1$, *using these functions in* (6) *we obtain.*

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)(\ell-\omega_1)d\ell=\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\left(\frac{\varkappa^3}{6}-b_1\frac{\varkappa^2}{2}\right)d_q\varkappa-\frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3+\frac{\omega_1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2.$$

Finding the above integrals, we deduce.

$$\begin{aligned}\int_{\omega_1}^{\omega_2}\gamma(\ell)(\ell-\omega_1)d\ell=&\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}-\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}-\frac{1}{3}\omega_1^3-\frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3\\&+\frac{\omega_1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2.\end{aligned}\tag{8}$$

Now we find the integral $\int_{\omega_1}^{\omega_2}\gamma(\ell)(\omega_2-\ell)d\ell$.
If $\varphi(\ell)=\frac{1}{2}\omega_2\ell^2-\frac{1}{6}\ell^3$, *then* $\varphi^{''}(\ell)=\omega_2-\ell$, *using these functions in* (6) *we obtain.*

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)(\omega_2-\ell)d\ell=\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\left(\frac{\omega_2}{2}\varkappa^2-\frac{1}{6}\varkappa^3\right)\omega_1 d_q\varkappa-\frac{\omega_2}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2-\frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3.$$

Finding the above integrals, we deduce.

$$\begin{aligned}\int_{\omega_1}^{\omega_2}\gamma(\ell)(\omega_2-\ell)d\ell=&\frac{\omega_2}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right)+\frac{\omega_1\omega_2(\omega_2-\omega_1)}{1+q}+{\omega_1}^2\omega_2-\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}\\&-\frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2}-\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}-\frac{1}{6}\omega_1^3-\frac{\omega_2}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2-\frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3.\end{aligned} \tag{9}$$

using (8) *and* (9) *in* (7), *we get*

$$\begin{aligned}\le&\frac{1}{(\omega_2-\omega_1)}\Bigg[\varphi''(\omega_2)\Big\{\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}-\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}-\frac{1}{3}\omega_1^3-\frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3\\&+\frac{\omega_1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2\Big\}+\varphi''(\omega_1)\Big\{\frac{\omega_2}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right)+\frac{\omega_1\omega_2(\omega_2-\omega_1)}{1+q}+{\omega_1}^2\omega_2\\&-\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}-\frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2}-\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}-\frac{1}{6}\omega_1^3-\frac{\omega_2}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2\\&-\frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3\Big\}\Bigg].\end{aligned}$$

$$\begin{aligned}=&\frac{1}{\omega_2-\omega_1}\Bigg[\frac{1}{6}\left(\varphi''(\omega_2)-\varphi''(\omega_1)\right)\left(\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}\right)\\&-\left(\frac{\varphi''(\omega_2)}{2}+\frac{\varphi''(\omega_1)}{2}\right)\left(\frac{(\omega_2-\omega_1)}{1+q}\right)\omega_1^2\\&-\left(\frac{\varphi''(\omega_2)}{3}+\frac{\varphi''(\omega_1)}{6}\right)\omega_1^3-\left(\frac{\varphi''(\omega_2)}{6}+\frac{\varphi''(\omega_1)}{6}\right)\left(\frac{(q\omega_1+\omega_2)}{(1+q)}\right)^3\\&+\left(\frac{\varphi''(\omega_2)\omega_1}{2}-\frac{\varphi''(\omega_1)\omega_2}{2}\right)\left(\frac{(q\omega_1+\omega_2}{(1+q)}\right)^2+\frac{\varphi''(\omega_1)\omega_2}{2}\left(\frac{(\omega_2-\omega_1)^2}{(1+q+q^2)}\right)\\&+\frac{\omega_1\omega_2\varphi''(\omega_1)(\omega_2-\omega_1)}{(1+q)}+\varphi''(\omega_1)\omega_1^2\omega_2-\frac{\omega_1\varphi''(\omega_1)(\omega_2-\omega_1)^2}{2(1+q+q^2)}\Bigg].\end{aligned}$$

$$=\frac{1}{\omega_2-\omega_1}\Bigg[\frac{1}{6}\left(\varphi''(\omega_2)-\varphi''(\omega_1)\right)\left(\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}\right)$$

$$
\begin{aligned}
&-\left(\frac{\varphi''(\omega_2)+\varphi''(\omega_1)}{2}\right)\left(\frac{(\omega_2-\omega_1)}{1+q}\right)\omega_1^2\\
&-\left(\frac{2\varphi''(\omega_2)+\varphi''(\omega_1)}{6}\right)\omega_1^3-\left(\frac{\varphi''(\omega_2)+\varphi''(\omega_1)}{6}\right)\left(\frac{(q\omega_1+\omega_2)}{(1+q)}\right)^3\\
&+\left(\frac{\varphi''(\omega_2)\omega_1-\varphi''(\omega_1)\omega_2}{2}\right)\left(\frac{(q\omega_1+\omega_2)}{(1+q)}\right)^2\\
&+\frac{\omega_1\omega_2\varphi''(\omega_1)(\omega_2-\omega_1)}{(1+q)}+\varphi''(\omega_1)\omega_1^2\omega_2+\frac{1}{2}\left(\frac{(\omega_2-\omega_1)^3}{1+q+q^2}\right)\varphi''(\omega_1)\Bigg]. \qquad (10)
\end{aligned}
$$

(10) *is equivalent to* (3).

Remark 1. *Under the assumptions of Theorem 5 with the limit as* $q \to 1$*, we have the following* $\mathcal{H}-\mathcal{H}$ *inequality:*

$$
\begin{aligned}
&\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)d\varkappa-\varphi\left(\frac{\omega_1+\omega_2}{2}\right)\\
&\leq\frac{1}{\omega_2-\omega_1}\Bigg[\frac{1}{24}\left(\varphi''(\omega_2)-\varphi''(\omega_1)\right)(\omega_2-\omega_1)^3\\
&-\frac{1}{4}\left(\varphi''(\omega_2)+\varphi''(\omega_1)\right)(\omega_2-\omega_1)\,\omega_1^2-\frac{1}{6}\left(2\varphi''(\omega_2)+\varphi''(\omega_1)\right)\omega_1^3\\
&-\frac{1}{48}\left(\varphi''(\omega_2)+\varphi''(\omega_1)\right)(\omega_1+\omega_2)^3+\frac{1}{8}\left(\varphi''(\omega_2)\omega_1-\varphi''(\omega_1)\omega_2\right)\\
&(\omega_1+\omega_2)^2+\frac{1}{2}\left(\omega_1\omega_2\varphi''(\omega_1)(\omega_2-\omega_1)\right)+\varphi''(\omega_1)\omega_1^2\omega_2\\
&+\frac{1}{6}(\omega_2-\omega_1)^3\varphi''(\omega_1)\Bigg].
\end{aligned}
$$

Theorem 6. *Let* $\varphi \in C^2[\omega_1,\omega_2]$ *such that* φ'' *is convex and* $0<q<1$*. Then*

$$
\begin{aligned}
&\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa\\
&\leq\frac{1}{\omega_2-\omega_1}\Bigg[-\frac{1}{6}\left(\varphi''(\omega_2)-\varphi''(\omega_1)\right)\left(\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}\right)\\
&-\left(\frac{\varphi''(\omega_2)-\varphi''(\omega_1)}{2}\right)\left(\frac{(\omega_2-\omega_1)}{1+q}\right)\omega_1^2+\left(\frac{2\varphi''(\omega_2)+\varphi''(\omega_1)}{6}\right)\omega_1^3\\
&-\frac{1}{2}\left(\frac{(\omega_2-\omega_1)^3}{1+q+q^2}\right)\varphi''(\omega_1)-\frac{\omega_1\omega_2\varphi''(\omega_1)(\omega_2-\omega_1)}{(1+q)}-\frac{1}{2}\varphi''(\omega_1)\omega_1^2\omega_2\\
&-\left(\frac{2\varphi''(\omega_2)+\varphi''(\omega_1)}{6}\right)\left(\frac{q\omega_1^3}{1+q}\right)+\left(\frac{\varphi''(\omega_2)+2\varphi''(\omega_1)}{6}\right)\left(\frac{\omega_2^3}{1+q}\right)
\end{aligned}
$$

$$-\left(\frac{\omega_2\varphi''(\omega_2)-q\omega_1\varphi''(\omega_1)}{2}\right)\left(\frac{\omega_1\omega_2}{1+q}\right)\Bigg]. \tag{11}$$

Proof. If we set $\varkappa=\omega_2$ in (2), then we get

$$\varphi(\omega_2)=\varphi(\omega_1)+(\omega_2-\omega_1)\varphi'(\omega_2)+\int_{\omega_1}^{\omega_2}\mathcal{G}(\omega_2,\ell)\varphi''(\ell)d\ell.$$

Adding $q\varphi(\omega_1)$ and divide by $(q+1)$ both sides we get

$$\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}=\varphi(\omega_1)+\frac{\omega_2-\omega_1}{q+1}\varphi'(\omega_2)+\frac{1}{q+1}\int_{\omega_1}^{\omega_2}\mathcal{G}(\omega_2,\ell)\varphi''(\ell)d\ell. \tag{12}$$

Subtracting (5) from (12), we get:

$$\begin{aligned}\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}&-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa\\ &=\int_{\omega_1}^{\omega_2}\left\{\frac{\mathcal{G}(\omega_2,\ell)}{q+1}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathcal{G}(\varkappa,\ell)_{\omega_1}d_q\varkappa\right\}\varphi''(\ell)d\ell. \end{aligned}\tag{13}$$

$$\text{Let } \gamma(\ell)=\frac{\mathcal{G}(\omega_2,\ell)}{q+1}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathcal{G}(\varkappa,\ell)_{\omega_1}d_q\varkappa.$$

Clearly $\gamma(\ell)$ is the difference of right and middle side of (1), for the Green function therefore by $\gamma(\ell)$ is non-negative.

$$\text{Let } \ell=\frac{\ell-\omega_1}{\omega_2-\omega_1}\omega_2+\frac{\omega_2-\ell}{\omega_2-\omega_1}\omega_1. \text{ Then from (13) we have}$$

$$\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa=\int_{\omega_1}^{\omega_2}\gamma(\ell)\varphi''\left(\frac{\ell-\omega_1}{\omega_2-\omega_1}\omega_2+\frac{\omega_2-\ell}{\omega_2-\omega_1}\omega_1\right)d\ell.$$

By convexity of φ'', we obtain

$$\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa\le\int_{\omega_1}^{\omega_2}\gamma(\ell)\left\{\frac{\ell-\omega_1}{\omega_2-\omega_1}\varphi''(\omega_2)+\left(\frac{\omega_2-\ell}{\omega_2-\omega_1}\right)\varphi''(\omega_1)\right\}d\ell.$$

$$\begin{aligned}\Rightarrow\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa\le&\frac{1}{(\omega_2-\omega_1)}\Big\{\varphi''(\omega_2)\int_{\omega_1}^{\omega_2}\gamma(\ell)(\ell-\omega_1)d\ell+\varphi''(\omega_1)\\ &\int_{\omega_1}^{\omega_2}\gamma(\ell)(\omega_2-\ell)d\ell\Big\}.\end{aligned}\tag{14}$$

Now we find the integral $\int_{\omega_1}^{\omega_2}\gamma(\ell)(\ell-\omega_1)d\ell$.
If $\varphi(\ell)=\frac{1}{6}\ell^3-\frac{1}{2}\omega_1\ell^2$, then $\varphi^{''}(\ell)=\ell-\omega_1$, using these functions in (13) we obtain.

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)(\ell-\omega_1)d\ell=\frac{q\left(\frac{\omega_1^3}{6}-\frac{\omega_1\omega_1^2}{2}\right)+\frac{\omega_2^3}{6}-\frac{\omega_1\omega_2^2}{2}}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\left(\frac{\varkappa^3}{6}-b_1\frac{\varkappa^2}{2}\right)_{\omega_1}d_q\varkappa.$$

Finding the above integrals, we deduce.

$$\int_{\omega_1}^{\omega_2} \gamma(\ell)(\ell-\omega_1)d\ell = -\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)} - \frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q} + \frac{1}{3}\omega_1^3 + \frac{-\frac{q\omega_1^3}{3}+\frac{\omega_2^3}{6}-\frac{\omega_1\omega_2^2}{2}}{1+q}. \tag{15}$$

Now we find the integral $\int_{\omega_1}^{\omega_2}\gamma(\ell)(\omega_2-\ell)d\ell$.
If $\varphi(\ell)=\frac{1}{2}\omega_2\ell^2-\frac{1}{6}\ell^3$, *then* $\varphi''(\ell)=\omega_2-\ell$. *using these functions in* (13) *we obtain.*

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)(\omega_2-\ell)d\ell = \frac{q\left(\frac{\omega_2\omega_1^2}{2}-\frac{\omega_1^3}{6}\right)+\frac{\omega_2\omega_2^2}{2}-\frac{\omega_2^3}{6}}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\left(\frac{\omega_2}{2}\varkappa^2-\frac{1}{6}\varkappa^3\right)\omega_1 d_q\varkappa.$$

Finding the above integrals, we deduce.

$$\begin{aligned}\int_{\omega_1}^{\omega_2}\gamma(\ell)(\omega_2-\ell)d\ell =& -\frac{\omega_2}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right) - \frac{\omega_1\omega_2(\omega_2-\omega_1)}{1+q} - \frac{{\omega_1}^2\omega_2}{2} + \frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}\\ &+\frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2}+\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}+\frac{1}{6}\omega_1^3+\frac{\frac{q\omega_1^2\omega_2}{2}-\frac{q\omega_1^3}{6}+\frac{\omega_2^3}{3}}{1+q}.\end{aligned} \tag{16}$$

using (15) *and* (16) *in* (14), *we get*

$$\begin{aligned}\leq&\frac{1}{(\omega_2-\omega_1)}\Bigg[\varphi''(\omega_2)\Big\{-\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}-\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}+\frac{1}{3}\omega_1^3+\frac{-\frac{q\omega_1^3}{3}+\frac{\omega_2^3}{6}-\frac{\omega_1\omega_2^2}{2}}{1+q}\Big\}\\ &+\varphi''(\omega_1)\Big\{-\frac{\omega_2}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right)-\frac{\omega_1\omega_2(\omega_2-\omega_1)}{1+q}-\frac{{\omega_1}^2\omega_2}{2}+\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}\\ &+\frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2}+\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}+\frac{1}{6}\omega_1^3+\frac{\frac{q\omega_1^2\omega_2}{2}-\frac{q\omega_1^3}{6}+\frac{\omega_2^3}{3}}{1+q}\Big\}\Bigg].\\ =&\frac{1}{\omega_2-\omega_1}\Bigg[-\frac{1}{6}\left(\varphi''(\omega_2)-\varphi''(\omega_1)\right)\left(\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}\right)\\ &-\left(\frac{\varphi''(\omega_2)-\varphi''(\omega_1)}{2}\right)\left(\frac{(\omega_2-\omega_1)}{1+q}\right)\omega_1^2+\left(\frac{2\varphi''(\omega_2)+\varphi''(\omega_1)}{6}\right)\omega_1^3\\ &-\frac{1}{2}\left(\frac{(\omega_2-\omega_1)^3}{1+q+q^2}\right)\varphi''(\omega_1)-\frac{\omega_1\omega_2\varphi''(\omega_1)(\omega_2-\omega_1)}{(1+q)}-\frac{1}{2}\varphi''(\omega_1)\omega_1^2\omega_2\\ &-\left(\frac{2\varphi''(\omega_2)+\varphi''(\omega_1)}{6}\right)\left(\frac{q\omega_1^3}{1+q}\right)+\left(\frac{\varphi''(\omega_2)+2\varphi''(\omega_1)}{6}\right)\left(\frac{\omega_2^3}{1+q}\right)\\ &-\left(\frac{\omega_2\varphi''(\omega_2)-q\omega_1\varphi''(\omega_1)}{2}\right)\left(\frac{\omega_1\omega_2}{1+q}\right)\Bigg].\end{aligned} \tag{17}$$

(17) *is equivalent to* (11).

Remark 2. *Under the assumptions of Theorem 6 with the limit as $q \to 1$, we have the following $\mathcal{H}-\mathcal{H}$ inequality:*

$$\begin{aligned}
&\frac{\varphi(\omega_1)+\varphi(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)d\varkappa \\
&\le \frac{1}{\omega_2-\omega_1}\Bigg[-\frac{1}{24}\left(\varphi''(\omega_2)-\varphi''(\omega_1)\right)(\omega_2-\omega_1)^3 \\
&-\frac{1}{4}\left(\varphi''(\omega_2)-\varphi''(\omega_1)\right)(\omega_2-\omega_1)\omega_1^2+\frac{1}{6}\left(2\varphi''(\omega_2)+\varphi''(\omega_1)\right)\omega_1^3 \\
&-\frac{1}{6}(\omega_2-\omega_1)^3\varphi''(\omega_1)-\frac{1}{2}\left(\omega_1\omega_2\varphi''(\omega_1)(\omega_2-\omega_1)\right)-\frac{1}{2}\varphi''(\omega_1)\omega_1^2\omega_2 \\
&-\frac{1}{12}\left(2\varphi''(\omega_2)+\varphi''(\omega_1)\right)\omega_1^3+\frac{1}{12}\left(\varphi''(\omega_2)+2\varphi''(\omega_1)\right)\omega_2^3 \\
&-\frac{1}{4}\left(\omega_2\varphi''(\omega_2)-\omega_1\varphi''(\omega_1)\right)\omega_1\omega_2\Bigg].
\end{aligned}$$

A wide range of inequalities for convex functions have been published in the literature, and Jensen's inequality has a special place among them. The following is the presentation of Jensen's inequality:

Lemma 2. *[26] Let $p:[\omega_1,\omega_2]\to I$ be integrable functions with $p(\varkappa)\ge 0$, $\forall \quad \varkappa\in[\omega_1,\omega_2]$, and $\int_{\omega_1}^{\omega_2}p(\varkappa)d\varkappa>0$. If $\varphi: I\to R$ is convex function, then*

$$\varphi\left(\frac{\int_{\omega_1}^{\omega_2}p(\varkappa)\varkappa d\varkappa}{\int_{\omega_1}^{\omega_2}p(\varkappa)d\varkappa}\right)\le\frac{\int_{\omega_1}^{\omega_2}p(\varkappa)\varphi(\varkappa)\,d\varkappa}{\int_{\omega_1}^{\omega_2}p(\varkappa)d\varkappa}. \tag{18}$$

Theorem 7. *Let $\varphi\in C^2[\omega_1,\omega_2]$ such that φ'' is a convex and $0<q<1$. Then*

$$\begin{aligned}
&\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa-\varphi\left(\frac{q\omega_1+\omega_2}{q+1}\right) \\
&\ge\left(\frac{1}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right)+\frac{\omega_1(\omega_2-\omega_1)}{(1+q)}+\frac{1}{2}\omega_1^2-\frac{1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2\right) \\
&\varphi''\left(\frac{\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)}+\frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2}+\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}+\frac{1}{6}\omega_1^3-\frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3}{\frac{1}{2}\frac{(\omega_2-\omega_1)^2}{1+q+q^2}+\frac{\omega_1(\omega_2-\omega_1)}{1+q}+\frac{1}{2}\omega_1^2-\frac{1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2}\right).
\end{aligned} \tag{19}$$

Proof. From (18), we have

$$\int_{\omega_1}^{\omega_2}p(\varkappa)d\varkappa\varphi\left(\frac{\int_{\omega_1}^{\omega_2}p(\varkappa)\varkappa d\varkappa}{\int_{\omega_1}^{\omega_2}p(\varkappa)d\varkappa}\right)\le\int_{\omega_1}^{\omega_2}p(\varkappa)\varphi(\varkappa)\,d\varkappa. \tag{20}$$

Comparing (20) with (6), we have $p(\varkappa) = \gamma(\ell)$ and $\varphi(\varkappa) = \varphi''(\ell)$.
(20), becomes

$$\int_{\omega_1}^{\omega_2} \gamma(\ell)\varphi''(\ell)\, d\ell \geq \int_{\omega_1}^{\omega_2} \gamma(\ell)d\ell.\varphi''\left(\frac{\int_{\omega_1}^{\omega_2}\gamma(\ell)\ell d\ell}{\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell}\right).$$

By (6), we have

$$\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \varphi(\varkappa)_{\omega_1} d_q\varkappa - \varphi\left(\frac{q\omega_1+\omega_2}{q+1}\right) \geq \int_{\omega_1}^{\omega_2} \gamma(\ell)d\ell.\varphi''\left(\frac{\int_{\omega_1}^{\omega_2}\gamma(\ell)\ell d\ell}{\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell}\right). \tag{21}$$

Now we solve the integral, $\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell$.
If $\varphi(\ell) = \frac{1}{2}\ell^2$, then $\varphi''(\ell) = 1$, using these functions in (6), we obtain.

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell = \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\frac{1}{2}(\ell^2)_{\omega_1}d_q\ell - \frac{1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2.$$

Finding the above integrals, we get.

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell = \frac{1}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right) + \frac{\omega_1(\omega_2-\omega_1)}{(1+q)} + \frac{1}{2}\omega_1^2 - \frac{1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2. \tag{22}$$

Now we solve the integral, $\int_{\omega_1}^{\omega_2}\gamma(\ell)\ell d\ell$.
If $\varphi(\ell) = \frac{1}{6}\ell^3$, then $\varphi''(\ell) = \ell$, using these functions in (6), we obtain.

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)\ell d\ell = \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\frac{1}{6}(\ell^3)_{\omega_1}d_q\ell - \frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3.$$

Finding the above integrals, we get.

$$\begin{aligned}\int_{\omega_1}^{\omega_2}\gamma(\ell)\ell d\ell = \frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)} + \frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2} + \frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q} \\ + \frac{1}{6}\omega_1^3 - \frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3.\end{aligned} \tag{23}$$

Using (22) and (23) in (21), we get

$$\begin{aligned}&\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \varphi(\varkappa)_{\omega_1} d_q\varkappa - \varphi\left(\frac{q\omega_1+\omega_2}{q+1}\right) \\ &\geq \left(\frac{1}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right) + \frac{\omega_1(\omega_2-\omega_1)}{(1+q)} + \frac{1}{2}\omega_1^2 - \frac{1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2\right) \\ &\varphi''\left(\frac{\frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)} + \frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2} + \frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q} + \frac{1}{6}\omega_1^3 - \frac{1}{6}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^3}{\frac{1}{2}\frac{(\omega_2-\omega_1)^2}{1+q+q^2} + \frac{\omega_1(\omega_2-\omega_1)}{1+q} + \frac{1}{2}\omega_1^2 - \frac{1}{2}\left(\frac{q\omega_1+\omega_2}{1+q}\right)^2}\right).\end{aligned} \tag{24}$$

(24) is equivalent to (19).

Remark 3. *Under the assumptions of Theorem 7 with the limit as $q \to 1$, we have the following $\mathcal{H}-\mathcal{H}$ inequality:*

$$\begin{aligned}&\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)d\varkappa-\varphi\left(\frac{\omega_1+\omega_2}{2}\right)\\&\geq\left(\frac{1}{6}(\omega_2-\omega_1)^2+\frac{1}{2}\omega_1(\omega_2-\omega_1)+\frac{1}{2}\omega_1^2-\frac{1}{8}(\omega_1+\omega_2)^2\right)\\&\varphi''\left(\frac{\frac{1}{24}(\omega_2-\omega_1)^3+\frac{1}{6}(\omega_2-\omega_1)^2\omega_1+\frac{1}{4}(\omega_2-\omega_1)\omega_1^2+\frac{1}{6}\omega_1^3-\frac{1}{48}(\omega_1+\omega_2)^3}{\frac{1}{6}(\omega_2-\omega_1)^2+\frac{1}{2}(\omega_2-\omega_1)\omega_1+\frac{1}{2}\omega_1^2-\frac{1}{8}(\omega_1+\omega_2)^2}\right).\end{aligned}$$

Theorem 8. *Let $\varphi\in C^2[\omega_1,\omega_2]$ such that φ'' is convex and $0<q<1$. Then*

$$\begin{aligned}&\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa\\&\geq\left(\frac{1}{2}\left(\frac{q\omega_1^2+\omega_2^2}{1+q}\right)-\frac{1}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right)-\frac{\omega_1(\omega_2-\omega_1)}{(1+q)}-\frac{1}{2}\omega_1^2\right)\\&\varphi''\left(\frac{\frac{1}{6}\left(\frac{q\omega_1^3+\omega_2^3}{1+q}\right)-\frac{1}{6}\frac{(\omega_2-\omega_1)^2}{(1+q)(1+q^2)}-\frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2}-\frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q}-\frac{1}{6}\omega_1^3}{\frac{1}{2}\left(\frac{q\omega_1^2+\omega_2^2}{1+q}\right)-\frac{1}{2}\frac{(\omega_2-\omega_1)^2}{1+q+q^2}-\frac{\omega_1(\omega_2-\omega_1)}{1+q}-\frac{1}{2}\omega_1^2}\right).\end{aligned} \tag{25}$$

Proof. From (18), we have

$$\int_{\omega_1}^{\omega_2}p(\varkappa)d\varkappa\varphi\left(\frac{\int_{\omega_1}^{\omega_2}p(\varkappa)\varkappa d\varkappa}{\int_{\omega_1}^{\omega_2}p(\varkappa)d\varkappa}\right)\leq\int_{\omega_1}^{\omega_2}p(\varkappa)\varphi(\varkappa)\,d\varkappa. \tag{26}$$

Comparing (26) with (13), we have $p(\varkappa)=\gamma(\ell)$ and $\varphi(\varkappa)=\varphi''(\ell)$.
(26), becomes

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)\varphi''(\ell)\,d\ell\geq\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell.\varphi\left(\frac{\int_{\omega_1}^{\omega_2}\gamma(\ell)\ell d\ell}{\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell}\right).$$

By (13), we have

$$\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\varphi(\varkappa)_{\omega_1}d_q\varkappa\geq\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell.\varphi''\left(\frac{\int_{\omega_1}^{\omega_2}\gamma(\ell)\ell d\ell}{\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell}\right). \tag{27}$$

Now we solve the integral, $\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell$.
If $\varphi(\ell)=\frac{1}{2}\ell^2$, then $\varphi''(\ell)=1$, using these functions in (13), we obtain.

$$\int_{\omega_1}^{\omega_2}\gamma(\ell)d\ell=\frac{\frac{1}{2}q\omega_1^2+\frac{1}{2}\omega_2^2}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\frac{1}{2}(\ell^2)_{\omega_1}d_q\ell.$$

Finding the above integrals, we get.

$$\int_{\omega_1}^{\omega_2} \gamma(\ell)d\ell = \frac{1}{2}\left(\frac{q\omega_1^2+\omega_2^2}{1+q}\right) - \frac{1}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right) - \frac{\omega_1(\omega_2-\omega_1)}{(1+q)} - \frac{1}{2}\omega_1^2. \tag{28}$$

Now we solve the integral, $\int_{\omega_1}^{\omega_2} \gamma(\ell)\ell d\ell$.
If $\varphi(\ell) = \frac{1}{6}\ell^3$, then $\varphi''(\ell) = \ell$, using these functions in (13), we obtain.

$$\int_{\omega_1}^{\omega_2} \gamma(\ell)\ell d\ell = \frac{\frac{1}{6}q\omega_1^3 + \frac{1}{6}\omega_2^3}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \frac{1}{6}(\ell^3)_{\omega_1} d_q\ell.$$

Finding the above integrals, we get.

$$\int_{\omega_1}^{\omega_2} \gamma(\ell)\ell d\ell = \frac{1}{6}\left(\frac{q\omega_1^3+\omega_2^3}{1+q}\right) - \frac{1}{6}\frac{(\omega_2-\omega_1)^3}{(1+q)(1+q^2)} - \frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2} - \frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q} - \frac{1}{6}\omega_1^3. \tag{29}$$

Using (28) and (29) in (27), we get

$$\begin{aligned}\frac{q\varphi(\omega_1)+\varphi(\omega_2)}{q+1} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \varphi(\varkappa)_{\omega_1} d_q\varkappa \geq & \left(\frac{1}{2}\left(\frac{q\omega_1^2+\omega_2^2}{1+q}\right) - \frac{1}{2}\left(\frac{(\omega_2-\omega_1)^2}{1+q+q^2}\right) - \frac{\omega_1(\omega_2-\omega_1)}{(1+q)} - \frac{1}{2}\omega_1^2\right) \\ & \varphi\left(\frac{\frac{1}{6}\left(\frac{q\omega_1^3+\omega_2^3}{1+q}\right) - \frac{1}{6}\frac{(\omega_2-\omega_1)^2}{(1+q)(1+q^2)} - \frac{1}{2}\frac{\omega_1(\omega_2-\omega_1)^2}{1+q+q^2} - \frac{1}{2}\frac{\omega_1^2(\omega_2-\omega_1)}{1+q} - \frac{1}{6}\omega_1^3}{\frac{1}{2}\left(\frac{q\omega_1^2+\omega_2^2}{1+q}\right) - \frac{1}{2}\frac{(\omega_2-\omega_1)^2}{1+q+q^2} - \frac{\omega_1(\omega_2-\omega_1)}{1+q} - \frac{1}{2}\omega_1^2}\right).\end{aligned} \tag{30}$$

(30) is equivalent to (25).

Remark 4. *Under the assumptions of Theorem 8 with the limit as $q \to 1$, we have the following $\mathcal{H}-\mathcal{H}$ inequality:*

$$\begin{aligned}\frac{\varphi(\omega_1)+\varphi(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \varphi(\varkappa)d\varkappa \geq & \left(\frac{1}{4}(\omega_1^2+\omega_2^2) - \frac{1}{6}(\omega_2-\omega_1)^2 - \frac{1}{2}(\omega_2-\omega_1)\omega_1 - \frac{1}{2}\omega_1^2\right) \\ & \varphi''\left(\frac{\frac{1}{12}(\omega_1^3+\omega_2^3) - \frac{1}{24}(\omega_2-\omega_1)^2 - \frac{1}{6}(\omega_2-\omega_1)^2\omega_1 - \frac{1}{4}(\omega_2-\omega_1)\omega_1^2 - \frac{1}{6}\omega_1^3}{\frac{1}{4}(\omega_1^2+\omega_2^2) - \frac{1}{6}(\omega_2-\omega_1)^2 - \frac{1}{2}(\omega_2-\omega_1)\omega_1 - \frac{1}{2}\omega_1^2}\right).\end{aligned}$$

4. Numerical Examples and Graphical Analysis

In this section, we present our primary results through numerical examples and graphical representations. Under the assumption of Theorem 5, we take $q \in (0,1)$, $\varphi(\varkappa) = \varkappa^2$, and $[\omega_1, \omega_2] = [6,8]$, as a variable to illustrate a Figure 1 between the left and right-hand sides of Theorem 5.

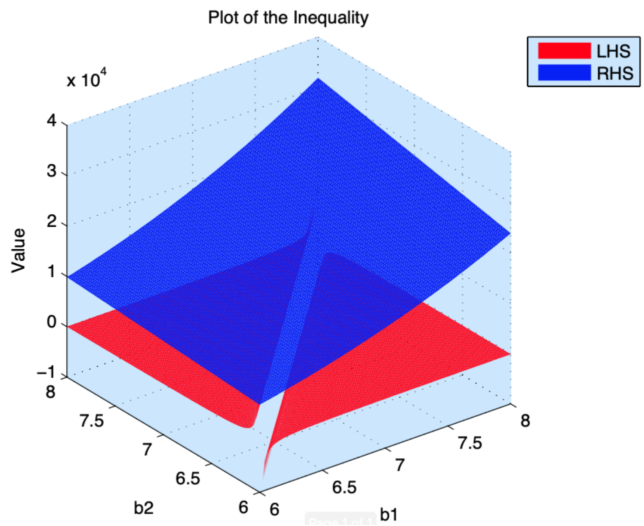

Figure 1: LHS vs RHS of Theorem 5 for $q \in (0,1)$, $\varphi(\varkappa) = \varkappa^2$, on $[\omega_1, \omega_2] = [6,8]$.

Under the assumption of Theorem 6, we take $q \in (0,1)$, $\varphi(\varkappa) = \varkappa^2$, and $[\omega_1, \omega_2] = [2, 2.4]$, as a variable to illustrate a Figure 2 between the left and right-hand sides of Theorem 6.

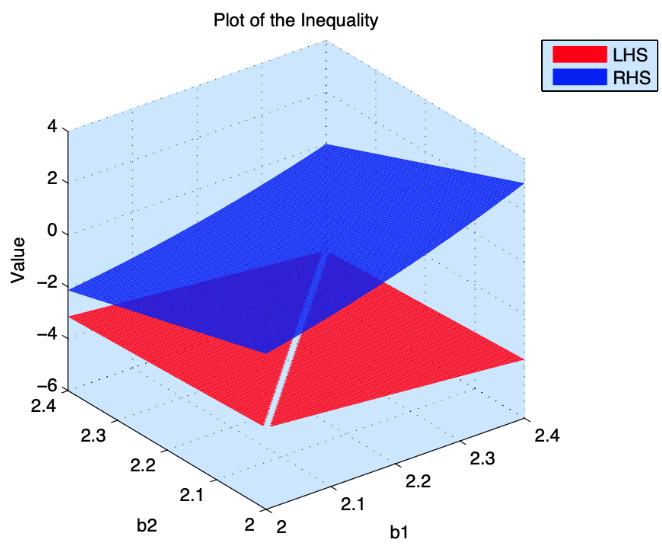

Figure 2: LHS vs RHS of Theorem 6 for $q \in (0,1)$, $\varphi(\varkappa) = \varkappa^2$, on $[\omega_1, \omega_2] = [2, 2.4]$.

Under the assumption of Theorem 7, we take $q \in (0,1)$, $\varphi(\varkappa) = \varkappa^2$, and $[\omega_1, \omega_2] = [0,1]$, as a variable to illustrate a Figure 3 between the left and right-hand sided of Theorem 7.

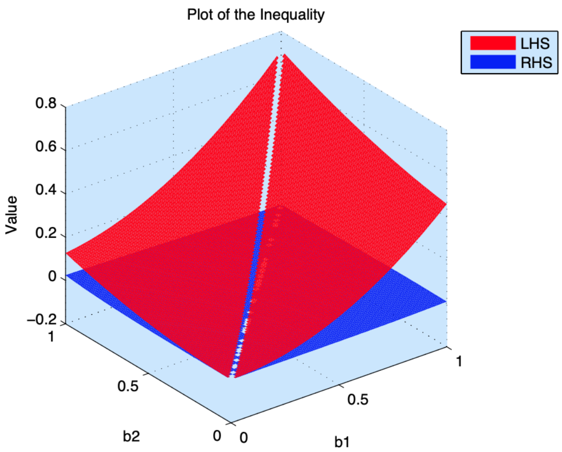

Figure 3: LHS vs RHS of Theorem 7 for $q \in (0,1)$, $\varphi(\varkappa) = \varkappa^2$, on $[\omega_1, \omega_2] = [0,1]$.

Under the assumption of Theorem 8, we take $q \in (0,1)$, $\varphi(\varkappa) = \varkappa^4$, and $[\omega_1, \omega_2] = [7,8]$ as a variable to illustrate a Figure 4 between the left and right hand side of Theorem 8.

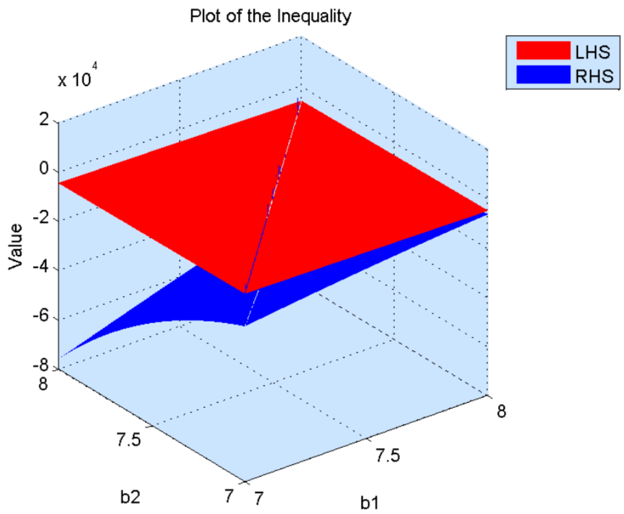

Figure 4: LHS vs RHS of Theorem 8 for $q \in (0,1)$, $\varphi(\varkappa) = \varkappa^4$, on $[\omega_1, \omega_2] = [7,8]$.

5. Conclusion

The present study has employed a Green function technique to analyze the quantum $\mathcal{H} - \mathcal{H}$ inequality. New quantum identities were obtained throughout this procedure and applied to create new inequalities. Jensen's inequality for convex mappings, convexity principles, and q-identities are some of the methods used in this work to arrive at its main conclusions. Furthermore, the major results are supported by graphical representations and numerical validations. In this regard, the presented consequences and methods in this paper may explore further investigation in this area by mathematicians.

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References

[1] S. Khan, M. Adil Khan, and Y.-M. Chu. New converses of jensen inequality via green functions with applications. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 114(3):114, 2020.

[2] M. Adil Khan, Z. Husain, and Y.-M. Chu. New estimates for csiszár divergence and zipf–mandelbrot entropy via jensen–mercer's inequality. *Complexity*, page 8928691, 2020.

[3] S. I. Bradanović, . Pečarić, and J. Pečarić. n-convexity and weighted majorization with applications to f-divergences and zipf–mandelbrot law. *Periodica Mathematica Hungarica*, 90:57–77, 2025.

[4] Ç. Yıldız. New inequalities of the hermite–hadamard type for n-times differentiable functions which are quasi-convex. *Journal of Mathematical Inequalities*, 10(3):703–711, 2016.

[5] S. S. Dragomir and R. Agarwal. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Applied Mathematics Letters*, 11(5):91–95, 1998.

[6] S. S. Dragomir. On some new inequalities of hermite–hadamard type for m-convex functions. *Tamkang Journal of Mathematics*, 33(1):45–56, 2002.

[7] C. P. Niculescu and L. E. Persson. *Convex Functions and Their Applications: A Contemporary Approach*. CMS Books in Mathematics. Springer, 2 edition, 2017.

[8] M. A. Latif and S. S. Dragomir. On some new inequalities for differentiable co-ordinated convex functions. *Journal of Inequalities and Applications*, 2012:28, 2012.

[9] G. Rahman, T. Abdeljawad, F. Jarad, and K. S. Nisar. Bounds of generalized proportional fractional integrals in general form via convex functions and their applications. *Mathematics*, 8(1):113, 2020.

[10] V. Kac and P. Cheung. *Quantum Calculus*. Springer, New York, 2002.

[11] J. Tariboon and S. K. Ntouyas. Quantum calculus on finite intervals and applications to impulsive difference equations. *Advances in Difference Equations*, 2013:28, 2013.

[12] M. Kunt and I. Iscan. Erratum: Quantum integral inequalities on finite intervals, 2016. Available at https://www.researchgate.net/publication/305303595.

[13] N. Alp, M. Z. Sarikaya, M. Kunt, and I. Iscan. q-hermite–hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *Journal of King Saud University - Science*, 30:193–203, 2018.

[14] M. A. Ali, H. Budak, M. Abass, and Y.-M. Chu. Quantum hermite–hadamard-type inequalities for functions with convex absolute values of second q^{ω_2}-derivatives. *Advances in Difference Equations*, 2021:7, 2021.

[15] M. A. Noor, K. I. Noor, and M. U. Awan. Some quantum estimates for hermite–hadamard inequalities. *Applied Mathematics and Computation*, 251:675–679, 2015.

[16] A. F. Shah, S. M. Boulaaras, P. J. Wong, M. S. Saleem, and S. M. Umair. Quantum integral inequalities via different variants of parameterized harmonically convex functions on finite intervals with related applications. *Mathematical Methods in the Applied Sciences*, 48:9194–9206, 2025.

[17] S. I. Butt, M. N. Aftab, H. A. Nabwey, and S. Etemad. Some hermite–hadamard and midpoint type inequalities in symmetric quantum calculus. *AIMS Mathematics*, 9(3):5523–5549, 2024.

[18] S. I. Butt, M. N. Aftab, and Y. Seol. Symmetric quantum inequalities on finite rectangular plane. *Mathematics*, 12(10), 2024.

[19] H. Budak, M. A. Ali, and M. Tarhanaci. Some new quantum hermite–hadamard-like inequalities for coordinated convex functions. *Journal of Optimization Theory and Applications*, 186:899–910, 2020.

[20] L. Ciurdariu and E. Grecu. On q-hermite–hadamard type inequalities via s-convexity and (a, m)-convexity. *Fractal and Fractional*, 8:1–12, 2023.

[21] X. You, H. Kara, H. Budak, and H. Kalsoom. Quantum inequalities of hermite–hadamard type for r-convex functions. *Journal of Mathematics*, pages 1–14, 2021.

[22] F. H. Jackson. On a q-definite integrals. *The Quarterly Journal of Pure and Applied Mathematics*, 41:193–203, 1910.

[23] F. Chen and W. Yang. Some new chebyshev type quantum integral inequalities on finite intervals. *Journal of Applied Analysis and Computation*, 21:417–426, 2016.

[24] Ç. Yıldız and L. I. Cotîrlă. Examining the hermite–hadamard inequalities for k-fractional operators using the green function. *Fractal and Fractional*, 7:161, 2023.

[25] S. I. Bradanović, N. Latif, and J. Pečarić. Generalizations of sherman's inequality via fink's identity and green's function. *Ukrainian Mathematical Journal*, 70(8):1192–1204, 2019.

[26] A. Basir, M. Adil Khan, and J. Pečarić. Majorization type inequalities via 4-convex functions. *Journal of Mathematical Inequalities*, 18(1):51–67, 2024.