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Fixed Point Results in Complex-Valued Neutrosophic Metric Spaces with Application to Integral Equations

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Abstract. This work presents the notion of complex-valued neutrosophic metric spaces (CVNMSs) and provides a fresh mathematical framework extending conventional fuzzy and intuitionistic fuzzy metric spaces (IFMSs). This new method is especially appropriate for studying complicated mathematical structures since it helps to better depict uncertainty and imprecision by including neutrosophic sets. We determine the existence and uniqueness of fixed points under several contractive mappings in this newly defined metric space. Our results extend the classical Banach contraction ideas and modify them for the neutrosophic environment. We demonstrate several fixed-point theorems, expanding the applicability of current fixed-point results in non-classical metric spaces. We use our results to solve Fredholm integral equations and show the efficiency of CVNMSs in tackling real-world mathematical problems by illustrating the pragmatic relevance of our results. Furthermore, comprehensive cases are included to show the relevance of our findings. This work also extends the fixed-point theory in neutrosophic environments and provides fresh directions for investigation in integral equations and contractive mappings.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Fixed point, neutrosophic sets, existence and uniqueness, metric spaces, contraction mappings

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1. Introduction

A neutrosophic metric space expands on the traditional notion of metric spaces by including the concept of neutrosophy, which deals with uncertainty and indeterminacy. The distances between points in a typical metric space are well defined and accurate; however, in a neutrosophic metric space, the distance between points is specified by three distinct functions that indicate the degree of truth, indeterminacy, and falsity. This approach enables more sophisticated modeling of complicated, uncertain, or ambiguous processes in which information may be inadequate or inconsistent. Essentially, it provides a mathematical structure for analyzing environments where traditional metrics fall short, allowing more adaptable and resilient approaches to problems in fields such as fuzzy logic, decision making, and applied research. Neutrosophic metric spaces allow for a more in-depth examination of phenomena that classical approaches cannot fully represent.

In mathematical analysis, fixed-point theory is a powerful tool with wide-ranging applications. The most applicable fixed point theorem in metric spaces is the Banach contraction mapping theorem, which was introduced in [1]. It has been generalized into several versions of fixed-point theorems and has been advanced through numerous methodologies. Mathematical models based on classical set theory cannot always capture ambiguous circumstances in nature or real-world challenges. To overcome this challenge, Zadeh [2] introduced fuzzy sets to represent an element's membership by designating an element's inclusion in a set by allocating a value within the interval $[0, 1]$. Afterward, Atanassov [3] Presented intuitionistic fuzzy sets, which facilitates the representation of level of ambiguity when determining whether an element in a set is a member or not.

Kramosil and Michalek [4] proposed fuzzy metric spaces (FMSs), which extend probabilistic metric spaces. Grabiec [5] was the first to research the ideology of fuzzy metric fixed-point theory (FMFPT). By introducing G-completeness and G-Cauchy sequences, he established a fuzzy counterpart to the Banach contraction principle on FMSs inspired by [4]. In 1994, George and Veeramani [6] developed a Hausdorff topology for FMSs. Additionally, they showed several fixed-point outcomes on the modified spaces and suggested changes to Grabiec's Cauchy sequence concept. In 2004, Park [7] proposed the framework of IFMSs, which increased the scope of fuzzy metrics. Researchers are still delving deeper into FMFPT. The study mostly goes in two different directions: expanding the scope of FMSs (comprehensive analysis provided in [8–13]) and examining the existence of fixed points for mappings adhering to various contractive conditions (for comprehensive information, refer to [14–17]. Azam et al. [18] introduced complex-valued metric spaces to metric fixed-point theory in 2011. Instead of using non-negative real numbers, they used ordered complex numbers to provide fixed-point results for translations that meet logical inequality criteria. Shukla et al. [19] used this notion in FMFPT. The authors defined complex-valued fuzzy metric spaces (CVFMSs) and identified fixed-point transformations that meet contractive conditions. Umar et al. [20] worked on Some Common Fixed-Point Theorems in Neutrosophic metric spaces (NMSs). Current research focuses on analyzing fixed-point translations using CVFMSs. Examples of relevant research include publications by [21, 22] and Humaira et al. [23–27], which provide practical applications. Kiri¸sci and Simsek [28] proposed NMSs for membership, nonmembership, and naturalness functions. The authors in [29, 30] and Sowndrarajan et al. [31] demonstrated fixed point findings in NMSs. Umar et al. [32] worked on fixed point results in orthogonal NMSs. See [33, 34] for applications and research direction.

This paper introduces CVNMSs (CVNMSs) as a new type of fuzzy metric space. This novel idea encompasses both CVFMSs by [19] and IFMSs by [7]. We give some fixed-point outcomes for transformations with contractive constraints in freshly formed spaces. We apply Banach's fuzzy variation to intuitionistic fuzzy spaces, resulting in common fixed-point outcomes in CVFMSs. Our findings are demonstrated through practical examples and applications.

2. Preliminaries

This section summarizes key concepts in CVFMSs, as defined in previous work by [19]. Throughout the paper, we denote the set of positive integers by $\mathbb{M}$ and the set of nonnegative integers by $\mathbb{M}_0$. We represent any complex number $z = c + id$ by (c, d). Assume that $\mathbb{S} = \{(c, d) : 0 \leq c < \infty, 0 \leq d < \infty\} \subset \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. We express (0,0),(1,1) and (3,1) in $\mathbb{C}$ as $\varnothing, \Im'$ and $\Im$ respectively. We denote the closed unit complex interval by $\mathbb{T} = \{(c, d) : 0 \leq c \leq 1, 0 \leq d \leq 1\}$, the open unit complex interval by $\mathbb{T}_0 = \{(c, d) : 0 < c < 1, 0 < d < 1\}$, and $\mathbb{S}_0 = \{(c, d) : 0 < c < \infty, 0 < d < \infty\}$.

Assume that $\preceq$ is a partial order in $\mathbb{C}$, $e_2 - e_1 \in \mathbb{S}$ if and only if $e_1 \preceq e_2$, where $e_1, e_2 \in \mathbb{C}$. We write $e_1 \prec e_2$ to express $Re(e_2) > Re(e_1)$ and $Im(e_2) > Im(e_1)$. It is obvious that $e_1 \prec e_2$ if and only if $e_2 - e_1 \in \mathbb{S}_0$. Assume that $\{e_n\}$ is a sequence in $\mathbb{C}$. If for each $n \in \mathbb{M}$, $c_n + 1 \preceq e_n$ or $e_n \preceq e_n + 1$ hold, then $\{e_n\}$ is called a monotonic sequence in relation to $\preceq$.

Remark 1. *[19] Let $e_n \in \mathbb{S}$ for every $n \in \mathbb{M}$, all of the following propositions are true:*

(1) If $\{e_n\}$ is a monotonic sequence with respect to $\preceq$ and for some $\acute{x}, \acute{y} \in \mathbb{S}$ fulfil $\acute{x} \preceq e_n \preceq \acute{y}$ for each $n \in \mathbb{M}$, then there exists $e \in \mathbb{S}$ such that $\lim_{n\to\infty} e_n = e$.

(2) $\preceq$ creates a lattice structure on set of complex numbers $\mathbb{C}$, but does not create a total ordering on $\mathbb{C}$.

(3) If for each $k \in \mathbb{L}$ satisfies $\acute{x} \preceq k \preceq \acute{y}$ for some $\acute{x}, \acute{y} \in \mathbb{C}$, then infimum of $\mathbb{L}$ and supremum of $\mathbb{L}$ are exists.

Remark 2. *[19] Considering that the following criteria hold for each $n \in \mathbb{M}$, $e_n, e'_n \in \mathbb{S}$,*

(1) If for each $n \in \mathbb{M}$, we have $e_n \preceq e'_n \preceq \Im'$ in addition to $e_n \to \Im'$ as n approaches to infinity , it follows that $e'_n = \Im'$.

(2) If every element e_n in a sequence satisfies $e_n \preceq z$, and if the sequence e_n has a limit $e \in \mathbb{S}$, then the limit e also satisfies $e \preceq z$.

(3) If every element e_n in a sequence satisfies $z \preceq e_n$, and if the sequence e_n has a limit $e \in \mathbb{S}$, then the limit e also satisfies $z \preceq e$.

Definition 1. *[19] Assume that $\mathbb{V}$ is a set that is not empty. Complex fuzzy set (CFS) $\mathcal{E}$ is defined as the mapping from $\mathbb{V}$ to the closed unit complex interval $\mathbb{T}$.*

Definition 2. *[19] A binary operation $\star : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is called a complex-valued t-norm if it meets the following conditions:*

(1) $\varnothing \star e = \varnothing, \Im' \star e = e$ *for every* $e \in \mathbb{T}$;

(2) $\star$ *is associative and commutative*

(3) $e_3 \star e_4 \succeq e_2 \star e_1$ *given that* $e_3 \succeq e_1, e_4 \succeq e_2$ *for each* $e_1, e_2, e_3, e_4 \in \mathbb{T}$.

Example 1. *[19] Suppose that $e_i = (\tau_i, \xi_i) \in \mathbb{T}$ for $i = 1, 2$, binary operations $\star_x, \star_y, \star_z : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ are defined below:*

(1) $e_1 \star_x e_2 = (\tau_1 \tau_2, \xi_1 \xi_2)$;

(2) $e_1 \star_y e_2 = (\min\{\tau_1, \tau_2\}, \min\{\xi_1, \xi_2\})$;

(3) $e_1 \star_z e_2 = (\max\{\tau_1 + \tau_2 - 1, 0\}, \max\{\xi_1 + \xi_2 - 1, 0\})$.

Therefore, $\star_x, \star_y, \star_z$ are complex-valued t-norms.

Definition 3. *[19] Let $\mathbb{V} \neq 0, \star$ be a continuous complex-valued t-norms and $\mathcal{E}$ be a CFS defined on $\mathbb{V}^2 \times \mathbb{S}_0$ whereby the criteria following hold:*

(1) $\mathcal{E}(\varrho^o, \nu, e) \succ \varnothing$;

(2) $\mathcal{E}(\varrho^o, \nu, e) = \Im'$ *for each* $e \in \mathbb{S}_0$ *if and only if* $\varrho^o = \nu$;

(3) $\mathcal{E}(\varrho^o, \nu, e) = \mathcal{E}(\nu, \varrho^o, e)$;

(4) $\mathcal{E}(\varrho^o, \varsigma, e + e') \succeq \mathcal{E}(\varrho^o, \nu, e) \star \mathcal{E}(\nu, \varsigma, e')$;

(5) $\mathcal{E}(\varrho^o, \nu, \cdot) : \mathbb{S}_0 \to \mathbb{T}$ *is continuous;*

for every $\varrho^o, \nu, \varsigma \in \mathbb{V}$ and $e, e' \in \mathbb{S}_0$.

Then $\mathcal{E}$ is called a complex-valued fuzzy metric on $\mathbb{V}$, and $(\mathbb{V}, \mathcal{E}, \star)$ is called a complex-valued fuzzy metric space. A CVFM $\mathcal{E}$ characterizes the degree of nearness between two points of the set $\mathbb{V}$ relative to a complex factor $e \in \mathbb{S}_0$.

Definition 4. *A binary operation $\triangle : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is called complex-valued t-conorm if it meets the following conditions:*

(1) $e \triangle \varnothing = e, e \triangle \Im' = \Im'$ *for every* $e \in \mathbb{T}$;

(2) $\triangle$ *is associative and commutative;*

(3) $e_3 \triangle e_4 \succeq e_1 \triangle e_2$ *given that* $e_3 \succeq e_1, e_4 \succeq e_2$ *for each* $e_1, e_2, e_3, e_4 \in \mathbb{T}$.

Example 2. *Suppose that* $e_i = (\tau_i, \xi_i) \in \mathbb{T}$ *for* $i = 1, 2$, *binary operations* $\triangle_x, \triangle_y, \triangle_z : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ *are defined as below:*

(1) $e_1 \triangle_x e_2 = (\tau_1 + \tau_2, \xi_1 + \xi_2) - (\tau_1\tau_2, \xi_1\xi_2)$;

(2) $e_1 \triangle_y e_2 = (\max\{\tau_1, \tau_2\}, \max\{\xi_1, \xi_2\})$;

(3) $e_1 \triangle_z e_2 = (\min\{\tau_1 + \tau_2, 1\}, \min\{\xi_1 + \xi_2, 1\})$.

Therefore, $\triangle_x, \triangle_y$ *and* $\triangle_z$ *are complex-valued triangular-conorms.*

Remark 3. *Both binary operations, triangular-norm, and triangular-conorm, are frequently employed in fuzzy set theory, especially within the contexts of* $[0, 1]$ *and lattices. The former denotes the shared area between two fuzzy sets, sometimes expressed as a conjunction in fuzzy logic. The t-conorm, which is the dual of the t-norm, is seen as the area where two fuzzy sets meet, or, in other words, a disjunction in fuzzy logic. For additional insights into the ideas of t-norm and t-conorm are encouraged to refer to [35] and [36].*

Definition 5. *Assume that* $\mathbb{V} \neq 0$, $\star$ *and* $\triangle$ *are continuous complex-valued t-norm and t-conorm, respectively, and* $\mathcal{E}, \mathcal{G}$ *are complex fuzzy sets (CFSs) defined on* $\mathbb{V}^2 \times \mathbb{S}_0$ *in which the following conditions are satisfied:*

(1) $\mathcal{E}(\varrho^o, \nu, e) + \mathcal{G}(\varrho^o, \nu, e) \preceq \Im'$;

(2) $\mathcal{E}(\varrho^o, \nu, e) \succ \varnothing$;

(3) $\mathcal{E}(\varrho^o, \nu, e) = \Im'$ *for each* $e \in \mathbb{S}_0$ *if and only if* $\varrho^o = \nu$;

(4) $\mathcal{E}(\varrho^o, \nu, e) = \mathcal{E}(\nu, \varrho^o, e)$;

(5) $\mathcal{E}(\varrho^o, \varsigma, e + e') \succeq \mathcal{E}(\varrho^o, \nu, e) \star \mathcal{E}(\nu, \varsigma, e')$;

(6) $\mathcal{E}(\varrho^o, \nu, \cdot) : \mathbb{S}_0 \to \mathbb{T}$ *is continuous;*

(7) $\mathcal{G}(\varrho^o, \nu, e) \prec \Im'$;

(8) $\mathcal{G}(\varrho^o, \nu, e) = \varnothing$ *for each* $e \in \mathbb{S}_0$ *if and only if* $\varrho^o = \nu$;

(9) $\mathcal{G}(\varrho^o, \nu, e) = \mathcal{G}(\nu, \varrho^o, e)$;

(10) $\mathcal{G}(\varrho^o, \varsigma, e + e') \preceq \mathcal{G}(\varrho^o, \nu, e) \triangle \mathcal{G}(\nu, \varsigma, e')$;

(11) $\mathcal{G}(\varrho^o, \nu, \cdot) : \mathbb{S}_0 \to \mathbb{T}$ *is continuous;*

for every $\varrho^o, \nu, \varsigma \in \mathbb{V}$ *and* $e, e' \in \mathbb{S}_0$.

The pair $(\mathcal{E}, \mathcal{G})$ is called complex-valued intuitionistic fuzzy metric on $\mathbb{V}$ and $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \star, \triangle)$ is called complex-valued intuitionistic fuzzy metric space. The pair $(\mathcal{E}, \mathcal{G})$ denotes the degree of nearness and the degree of non-nearness between two points of the set $\mathbb{V}$ with respect to a complex parameter $e \in \mathbb{S}_0$.

3. Main Results

In this section, we introduce the concept of complex-valued neutrosophic metric space (CVNMS).

Definition 6. *[28] Assume that $\mathbb{V} \neq 0$, and let $\star$ and $\triangle$ represent continuous triangular norm and continuous triangular conorm, respectively, and $\mathcal{E}, \mathcal{G}, \mathcal{H}$ are the neutrosophic sets defined on $\mathbb{V} \times \mathbb{V} \times (0, \infty)$ in which the following conditions are satisfied:*

(1) $\mathcal{E}(\varrho^o, \nu, e) + \mathcal{G}(\varrho^o, \nu, e) + \mathcal{H}(\varrho^o, \nu, e) \leq 3$;

(2) $\mathcal{E}(\varrho^o, \nu, e) > 0$;

(3) $\mathcal{E}(\varrho^o, \nu, e) = 1$ *if and only if* $\varrho^o = \nu$;

(4) $\mathcal{E}(\varrho^o, \nu, e) = \mathcal{E}(\nu, \varrho^o, e)$;

(5) $\mathcal{E}(\varrho^o, \varsigma, e + e') \geq \mathcal{E}(\varrho^o, \nu, e) \star \mathcal{E}(\nu, \varsigma, e')$;

(6) $\mathcal{E}(\varrho^o, \nu, \triangle)$ *is a non decreasing function of* $\mathbb{R}^+$ *and* $\lim_{e\to\infty} \mathcal{E}(\varrho^o, \nu, e) = 1$;

(7) $\mathcal{G}(\varrho^o, \nu, e) < 1$;

(8) $\mathcal{G}(\varrho^o, \nu, e) = 0$ *if and only if* $\varrho^o = \nu$;

(9) $\mathcal{G}(\varrho^o, \nu, e) = \mathcal{G}(\nu, \varrho^o, e)$;

(10) $\mathcal{G}(\varrho^o, \varsigma, e + e') \leq \mathcal{G}(\varrho^o, \nu, e) \triangle \mathcal{G}(\nu, \varsigma, e')$;

(11) $\mathcal{G}(\varrho^o, \nu, \triangle)$ *is a non increasing function of* $\mathbb{R}^+$ *and* $\lim_{e\to\infty} \mathcal{G}(\varrho^o, \nu, e) = 0$;

(12) $\mathcal{H}(\varrho^o, \nu, e) < 1$;

(13) $\mathcal{H}(\varrho^o, \nu, e) = 0$ *if and only if* $\varrho^o = \nu$;

(14) $\mathcal{H}(\varrho^o, \nu, e) = \mathcal{H}(\nu, \varrho^o, e)$;

(15) $\mathcal{H}(\varrho^o, \varsigma, e + e') \preceq \mathcal{H}(\varrho^o, \nu, c) \triangle \mathcal{H}(\nu, \varsigma, e')$;

(16) $\mathcal{H}(\varrho^o, \nu, \triangle)$ *is a non increasing function of* $\mathbb{R}^+$ *and* $\lim_{e\to\infty} \mathcal{H}(\varrho^o, \nu, e) = 0$;

(17) *If* $e \leq 0$ *then* $\mathcal{E}(\varrho^o, \nu, e) = 0, \mathcal{G}(\varrho^o, \nu, e) = 1$ *and* $\mathcal{H}(\varrho^o, \nu, e) = 1$;

for every $\varrho^o, \nu, \varsigma \in \mathbb{V}$ *and* $e, e' \in (0, \infty)$.

The triplet $(\mathcal{E}, \mathcal{G}, \mathcal{H})$ is called neutrosophic metric on $\mathbb{V}$ and $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ is called CVNMS. The triplet $(\mathcal{E}, \mathcal{G}, \mathcal{H})$ indicates the closeness degree, the non-closeness degree, and the neutralness degree between two points of the set $\mathbb{V}$ with respect to a parameter $e \in (0, \infty)$.

Definition 7. *Assume that $\mathbb{V} \neq 0$, $\star$ and $\triangle$ represent continuous complex-valued triangular norm and continuous complex-valued triangular conorm, respectively, and $\mathcal{E}, \mathcal{G}, \mathcal{H}$ are CFSs defined on $\mathbb{V}^2 \times \mathbb{S}_0$ in which the following conditions are satisfied:*

(1) $\mathcal{E}(\varrho^o, \nu, e) + \mathcal{G}(\varrho^o, \nu, e) + \mathcal{H}(\varrho^o, \nu, e) \preceq \Im$;

(2) $\mathcal{E}(\varrho^o, \nu, e) \succ \varnothing$;

(3) $\mathcal{E}(\varrho^o, \nu, e) = \Im$ *for each* $e \in \mathbb{S}_0$ *if and only if* $\varrho^o = \nu$;

(4) $\mathcal{E}(\varrho^o, \nu, e) = \mathcal{E}(\nu, \varrho^o, e)$;

(5) $\mathcal{E}(\varrho^o, \varsigma, e + e') \succeq \mathcal{E}(\varrho^o, \nu, e) \star \mathcal{E}(\nu, \varsigma, c')$;

(6) $\mathcal{E}(\varrho^o, \nu, \cdot) : \mathbb{S}_0 \to \mathbb{T}$ *is continuous;*

(7) $\mathcal{G}(\varrho^o, \nu, e) \prec \Im$;

(8) $\mathcal{G}(\varrho^o, \nu, e) = \varnothing$ *for each* $e \in \mathbb{S}_0$ *if and only if* $\varrho^o = \nu$;

(9) $\mathcal{G}(\varrho^o, \nu, e) = \mathcal{G}(\nu, \varrho^o, e)$;

(10) $\mathcal{G}(\varrho^o, \varsigma, e + e') \preceq \mathcal{G}(\varrho^o, \nu, e) \triangle \mathcal{G}(\nu, \varsigma, e')$;

(11) $\mathcal{G}(\varrho^o, \nu, \cdot) : \mathbb{S}_0 \to \mathbb{T}$ *is continuous;*

(12) $\mathcal{H}(\varrho^o, \nu, e) \prec \Im$;

(13) $\mathcal{H}(\varrho^o, \nu, e) = \varnothing$ *for each* $e \in \mathbb{S}_0$ *if and only if* $\varrho^o = \nu$;

(14) $\mathcal{H}(\varrho^o, \nu, e) = \mathcal{H}(\nu, \varrho^o, e)$;

(15) $\mathcal{H}(\varrho^o, \varsigma, e + e') \preceq \mathcal{H}(\varrho^o, \nu, e) \triangle \mathcal{H}(\nu, \varsigma, e')$;

(16) $\mathcal{H}(\varrho^o, \nu, \cdot) : \mathbb{S}_0 \to \mathbb{T}$ *is continuous;*

for every $\varrho^o, \nu, \varsigma \in \mathbb{V}$ *and* $e, e' \in \mathbb{S}_0$.

Then the triplet $(\mathcal{E}, \mathcal{G}, \mathcal{H})$ is called complex-valued neutrosophic metric on $\mathbb{V}$ and $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ is called CVNMS. The triplet $(\mathcal{E}, \mathcal{G}, \mathcal{H})$ characterizes the degree of nearness, the degree of non-nearness , and the neutralness degree between two points of the set $\mathbb{V}$ relative to a complex parameter $e \in \mathbb{S}_0$.

Example 3. *Consider the metric space $(\mathbb{V}, d)$. Two binary operations, $\star_y$ and $\triangle_y$, are defined for $e_i = (\tau_i, \xi_i) \in \mathbb{T}$, where i=1,2, as follows:*

$$e_1 \star_y e_2 = (\min\{\tau_1, \tau_2\}, \min\{\xi_1, \xi_2\}) \text{ and } e_1 \triangle_y e_2 = (\max\{\tau_1, \tau_2\}, \max\{\xi_1, \xi_2\}).$$

The CFSs $\mathcal{E}, \mathcal{G}$, and $\mathcal{H}$ are defined as follows:

$$\mathcal{E}(\varrho^o, \nu, e) = \frac{\tau + \xi}{\tau + \xi + d(\varrho^o, \nu)}\Im, \quad \mathcal{G}(\varrho^o, \nu, e) = \frac{d(\varrho^o, \nu)}{\tau + \xi + d(\varrho^o, \nu)}\Im, \quad \mathcal{H}(\varrho^o, \nu, e) = \frac{d(\varrho^o, \nu)}{\tau + \xi}\Im$$

for each $\varrho^o, \nu \in \mathbb{V}$ and $e = (\tau, \xi) \in \mathbb{S}_0$. Consequently, $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star_y, \triangle_y)$ is a CVNMS.

Lemma 1. *Given that* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *is a CVNMS,* $\mathcal{E}(\varrho^o, \nu, \cdot)$ *is non-decreasing,* $\mathcal{G}(\varrho^o, \nu, \cdot)$ *is non-increasing, and* $\mathcal{H}(\varrho^o, \nu, \cdot)$ *is non-increasing, that is, for any* $e, e' \in \mathbb{S}_0$ *with* $e \prec e'$*, it follows that* $\mathcal{E}(\varrho^o, \nu, e) \preceq \mathcal{E}(\varrho^o, \nu, e'), \mathcal{G}(\varrho^o, \nu, e) \succeq \mathcal{G}(\varrho^o, \nu, e')$ *and* $\mathcal{H}(\varrho^o, \nu, e) \succeq \mathcal{H}(\varrho^o, \nu, e')$ *for each* $\varrho^o, \nu \in \mathbb{V}$.

Proof. Consider $e, e' \in \mathbb{S}_0$ *where* $e \prec e'$*, this implies that* $e' - e \in \mathbb{S}_0$*. By using condition (5) from Definition 7, we obtain*

$$\begin{aligned}\mathcal{E}(\varrho^o, \nu, e') &= \mathcal{E}(\varrho^o, \nu, e' - e + e)\\ &\succeq \mathcal{E}(\varrho^o, \varrho^o, e' - e) \star \mathcal{E}(\varrho^o, \nu, e)\\ &= \Im \star \mathcal{E}(\varrho^o, \nu, e)\\ &= \mathcal{E}(\varrho^o, \nu, e).\end{aligned}$$

Hence $\mathcal{E}(\varrho^o, \nu, e') \succeq \mathcal{E}(\varrho^o, \nu, e)$*. Alternatively by using condition (10) from Definition 7, we have*

$$\begin{aligned}\mathcal{G}(\varrho^o, \nu, e') &= \mathcal{G}(\varrho^o, \nu, e' - e + e)\\ &\preceq \mathcal{G}(\varrho^o, \varrho^o, e' - e) \triangle \mathcal{G}(\varrho^o, \nu, e)\\ &= \varnothing \triangle \mathcal{G}(\varrho^o, \nu, e)\\ &= \mathcal{G}(\varrho^o, \nu, e).\end{aligned}$$

Hence $\mathcal{G}(\varrho^o, \nu, e') \preceq \mathcal{G}(\varrho^o, \nu, e)$*. Similarly by using condition (15) from Definition 7, we have*

$$\begin{aligned}\mathcal{H}(\varrho^o, \nu, e') &= \mathcal{H}(\varrho^o, \nu, e' - e + e)\\ &\preceq \mathcal{H}(\varrho^o, \varrho^o, e' - e) \triangle \mathcal{H}(\varrho^o, \nu, e)\\ &= \varnothing \triangle \mathcal{H}(\varrho^o, \nu, e)\\ &= \mathcal{H}(\varrho^o, \nu, e).\end{aligned}$$

Hence $\mathcal{H}(\varrho^o, \nu, e') \preceq \mathcal{H}(\varrho^o, \nu, e)$.

Definition 8. *Let* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *be a CVNMS. A sequence* $\{\varrho^o_n\}$ *in* $\mathbb{V}$ *converges to* $\varrho^o \in \mathbb{V}$ *provided that all* $r \in \mathbb{T}_0$ *as well as* $e \in \mathbb{S}_0$*, for some* $n_0 \in \mathbb{M}$ *if it meets the following criteria:*

$$\mathcal{E}(\varrho^o_n, \varrho^o, e) \succ \Im - r, \;\; \mathcal{G}(\varrho^o_n, \varrho^o, e) \prec r \text{ and } \mathcal{H}(\varrho^o_n, \varrho^o, e) \prec r \;\; \text{for each } \; n > n_0.$$

Definition 9. *Let* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *be a CVNMS. A cauchy sequence is a sequence* $\{\varrho^o_n\}$ *in* $\mathbb{V}$ *that satisfies the following criteria:*

$$\begin{aligned}\lim_{n\to\infty} \inf_{m>n} \mathcal{E}(\varrho^o_n, \varrho^o_m, e) = \Im,\\ \lim_{n\to\infty} \sup_{m>n} \mathcal{G}(\varrho^o_n, \varrho^o_m, e) = \varnothing,\\ \lim_{n\to\infty} \sup_{m>n} \mathcal{H}(\varrho^o_n, \varrho^o_m, e) = \varnothing,\end{aligned}$$

for each $e \in \mathbb{S}_0$.

A CVNMS is said to be complete if every Cauchy sequence in $\mathbb{V}$ converges.
The examples below help to clarify the ideas covered in Definitions 8 and 9.

Example 4. *Observe the CVNMS, denoted as $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star_y, \triangle_y)$ in example 3. Furthermore, set $\mathbb{V} = [4, 5]$ and define d as $d(\varrho^o, \nu) = |\varrho^o - \nu|$ for all $\varrho^o, \nu \in \mathbb{V}$. Let the sequence $\{\varrho^o_n\} = \{4 + \frac{1}{n}\}$ and $\varrho^o = 4$.*
Now we verify that $\mathcal{E}(\varrho^o_n, \varrho^o, e) \succ \Im - s$ for each $s = (s_1, s_2) \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$. For the real part,

$$\begin{aligned} Re(\mathcal{E}(\varrho^o_n, \varrho^o, e) - \Im + s) &= \frac{\tau + \xi}{\tau + \xi + d(\varrho^o_n, \varrho^o)} - 1 + s_1 \\ &= \frac{\tau + \xi}{\tau + \xi + |4 + \frac{1}{n} - 4|} - 1 + s_1 \\ &= \frac{\tau + \xi}{\tau + \xi + \frac{1}{n}} - 1 + s_1. \end{aligned}$$

As $n \to \infty$, then $Re(\mathcal{E}(\varrho^o_n, \varrho^o, e) - \Im + s) \to s_1$. Therefore, for each $s \in \mathbb{T}_0$ together with $e \in \mathbb{S}_0$, there is always an $M_1 \in \mathbb{M}$ in which $Re(\mathcal{E}(\varrho^o_n, \varrho^o, e) - \Im + s) > 0$ holds for all $n > M_1$. The approach for determining the imaginary part is the same, leading to $Im(\mathcal{E}(\varrho^o_n, \varrho^o, e) - \Im + s) \to s_2$ as $n \to \infty$. Therefore, for every $s \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$, always there is a $M_2 \in \mathbb{M}$ such that $Im(\mathcal{E}(\varrho^o_n, \varrho^o, e)) > 0$ holds for all $n > M_2$. Therefore, for every $s \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$, by taking $n_0 = \max\{M_1, M_2\}$, we establish $\mathcal{E}(\varrho^o_n, \varrho^o, e) > \Im - s$ foe each $n > n_0$.
Now we verify that $\mathcal{G}(\varrho^o_n, \varrho^o, e) < s$ for every $(s_1, s_2) \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$. For the real part,

$$\begin{aligned} Re(s - \mathcal{G}(\varrho^o_n, \varrho^o, e)) &= s_1 - \frac{d(\varrho^o_n, \varrho^o)}{\tau + \xi + d(\varrho^o_n, \varrho^o)} \\ &= s_1 - \frac{|4 + \frac{1}{n} - 4|}{\tau + \xi + |4 + \frac{1}{n} - 4|} \\ &= s_1 - \frac{\frac{1}{n}}{\tau + \xi + \frac{1}{n}}. \end{aligned}$$

As $n \to \infty$, then $Re(s - \mathcal{G}(\varrho^o_n, \varrho^o, e)) \to s_1$. Therefore, for each $s \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$, there is always an $M_1 \in \mathbb{M}$ in which $Re(\mathcal{G}(\varrho^o_n, \varrho^o, e) - \Im + s) > 0$ holds for each $n > M_1$. The approach for determining the imaginary part is the same, this leads to $Im(s - \mathcal{G}(\varrho^o_n, \varrho^o, e)) \to s_2$ as $n \to \infty$. Therefore, for each $r \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$, there is always an $M_2 \in \mathbb{M}$ in which $Im(s - \mathcal{G}(\varrho^o_n, \varrho^o, e)) > 0$ holds for all $n > M_2$. Therefore, for every $s \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$, by taking $n_0 = \max\{M_1, M_2\}$, we establish $\mathcal{G}(\varrho^o_n, \varrho^o, e) < s$ for each $n > n_0$.

Now we verify that $\mathcal{H}(\varrho^o_n, \varrho^o, e) < s$ for every $(s_1, s_2) \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$. For the real part,

$$Re(s - \mathcal{H}(\varrho^o_n, \varrho^o, e)) = s_1 - \frac{d(\varrho^o_n, \varrho^o)}{\tau + \xi}$$

$$= s_1 - \frac{|4 + \frac{1}{n} - 4|}{\tau + \xi}$$
$$= s_1 - \frac{\frac{1}{n}}{\tau + \xi}.$$

As $n \to \infty$, then $Re(s - \mathcal{H}(\varrho_n^o, \varrho^o, e)) \to s_1$. Therefore, for each $s \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$, there is always an $M_1 \in \mathbb{M}$ in which $Re(\mathcal{H}(\varrho_n^o, \varrho^o, e) - \Im + s) > 0$ holds for each $n > M_1$. The approach for determining the imaginary part is the same, this leads to $Im(s - \mathcal{H}(\varrho_n^o, \varrho^o, e)) \to s_2$ as $n \to \infty$. Therefore, for each $s \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$, there is always an $M_2 \in \mathbb{M}$ in which $Im(s - \mathcal{H}(\varrho_n^o, \varrho^o, e)) > 0$ holds for all $n > M_2$. Therefore, for every $s \in \mathbb{T}_0$ and $e \in \mathbb{S}_0$, by taking $n_0 = \max\{M_1, M_2\}$, we establish $\mathcal{H}(\varrho_n^o, \varrho^o, e) < s$ for each $n > n_0$. Every requirement of Definition 8 is fulfilled. We may therefore say that $\{4 + \frac{1}{n}\}$ converges to 4.

Example 5. *Using the same conditions as the previous example, we will demonstrate that $\{4 + \frac{1}{n}\}$ is a Cauchy sequence. For all $e \in \mathbb{S}$ for any $n, m \in \mathbb{M}$ where $m > n$,*

$$\mathcal{E}(\varrho_n^o, \varrho_m^o, e) = \frac{\tau + \xi}{\tau + \xi + d(\varrho_n^o, \varrho_m^o)}\Im$$
$$= \frac{\tau + \xi}{\tau + \xi + |4 + \frac{1}{n} - (4 + \frac{1}{m})|}\Im$$
$$= \frac{\tau + \xi}{\tau + \xi + |\frac{1}{n} - \frac{1}{m}|}\Im,$$

$$\mathcal{G}(\varrho_n^o, \varrho_m^o, e) = \frac{d(\varrho_n^o, \varrho_m^o)}{\tau + \xi + d(\varrho_n^o, \varrho_m^o)}\Im$$
$$= \frac{|4 + \frac{1}{n} - (4 + \frac{1}{m})|}{\tau + \xi + |4 + \frac{1}{n} - (4 + \frac{1}{m})|}\Im$$
$$= \frac{|\frac{1}{n} - \frac{1}{m}|}{\tau + \xi + |\frac{1}{n} - \frac{1}{m}|}\Im$$

And

$$\mathcal{H}(\varrho_n^o, \varrho_m^o, e) = \frac{d(\varrho_n^o, \varrho_m^o)}{\tau + \xi}\Im$$
$$= \frac{|4 + \frac{1}{n} - (4 + \frac{1}{m})|}{\tau + \xi}\Im$$
$$= \frac{|\frac{1}{n} - \frac{1}{m}|}{\tau + \xi}\Im.$$

As $m, n \to \infty$, we observe that $\mathcal{E}(\varrho_n^o, \varrho_m^o, e) \to \Im, \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \to \varnothing$ and $\mathcal{H}(\varrho_n^o, \varrho_m^o, e) \to \varnothing$, which leads to $\lim_{n\to\infty} \inf_{m>n} \mathcal{E}(\varrho_n^o, \varrho_m^o, e) = \Im, \lim_{n\to\infty} \sup_{m>n} \mathcal{G}(\varrho_n^o, \varrho_m^o, e) = \varnothing$ and $\lim_{n\to\infty} \sup_{m>n} \mathcal{H}(\varrho_n^o, \varrho_m^o, e) = \varnothing$. Hence show that $4 + \frac{1}{n}$ is a Cauchy sequence.

Lemma 2. *Let* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *be a CVNMS. The sequence* $\{\varrho_n^o\} \in \mathbb{V}$ *converges to* $\varrho^o \in \mathbb{V}$ *if and only if* $\lim_{n\to\infty} \mathcal{E}(\varrho_n^o, \varrho^o, e) = \Im, \lim_{n\to\infty} \mathcal{G}(\varrho_n^o, \varrho^o, e) = \varnothing$ *and* $\lim_{n\to\infty} \mathcal{H}(\varrho_n^o, \varrho^o, e) = \varnothing$ *are satisfied for each* $e \in \mathbb{S}_0$.

Proof. *Assume that* $\lim_{n\to\infty} \mathcal{E}(\varrho_n^o, \varrho^o, e) = \Im, \lim_{n\to\infty} \mathcal{G}(\varrho_n^o, \varrho^o, e) = \varnothing$ *and* $\lim_{n\to\infty} \mathcal{H}(\varrho_n^o, \varrho^o, e) = \varnothing$ *for every* $e \in \mathbb{S}_0$. *Assume that* e *be a fixed element from* $\mathbb{S}_0$. *It is feasible to identify a real number* $\epsilon > 0$ *such that* $z \prec r$ *for all* $z \in \mathbb{C}$ *and* $|z| < \epsilon$ *for any* $r \in \mathbb{T}_0$. *Considering this particular* ϵ, *we may find a* $n_0 \in \mathbb{M}$ *such that*

$$|\Im - \mathcal{E}(\varrho_n^o, \varrho^o, e)| < \epsilon, |\mathcal{G}(\varrho_n^o, \varrho^o, e)| < \epsilon \text{ and} |\mathcal{H}(\varrho_n^o, \varrho^o, e)| < \epsilon \quad \text{for each} \quad n \in n_0.$$

These two inequalities indicate that

$$\Im - \mathcal{E}(\varrho_n^o, \varrho^o, e) \prec s$$

$$-\mathcal{E}(\varrho_n^o, \varrho^o, e) \prec s - \Im$$

$$\mathcal{E}(\varrho_n^o, \varrho^o, e) \succ \Im - s$$

as well as

$$\mathcal{G}(\varrho_n^o, \varrho^o, e) \prec s$$

also

$$\mathcal{H}(\varrho_n^o, \varrho^o, e) \prec s$$

for all $n > n_0$ *respectively. Consequently,* $\{\varrho_n^o\}$ *is convergent to* $\varrho^o \in \mathbb{V}$. *Conversely, assume that* $e \in \mathbb{S}_0$ *is fixed and a real value* $\epsilon > 0$ *is specified. Assume that a sequence* $\{\varrho_n^o\}$ *is converges to* $\varrho^o \in \mathbb{V}$ *that is, For each element* s *in the set* $\mathbb{T}_0$ *and every element* n_o *in the set* $\mathbb{M}$, *it is possible to select a value such that* $\mathcal{E}(\varrho_n^o, \varrho^o, e) > \Im - s, \mathcal{G}(\varrho_n^o, \varrho^o, e) < s$ *and* $\mathcal{H}(\varrho_n^o, \varrho^o, e) < s$ *for each* $n > n_0$. *A complex number* s *is selected from the set* $\mathbb{T}_0$ *such that* $|s| < \epsilon$. *Therefore*

$$|\Im - \mathcal{E}(\varrho_n^o, \varrho^o, e)| < |s|, \; |\mathcal{G}(\varrho_n^o, \varrho^o, e)| < |s| < \epsilon \text{ and } |\mathcal{H}(\varrho_n^o, \varrho^o, e)| < |s| < \epsilon \text{ for any } n > n_0.$$

Therefore, $\lim_{n\to\infty} \mathcal{E}(\varrho_n^o, \varrho^o, e) = \Im, \; \lim_{n\to\infty} \mathcal{G}(\varrho_n^o, \varrho^o, e) = \varnothing$ *and* $\lim_{n\to\infty} \mathcal{H}(\varrho_n^o, \varrho^o, e) = \varnothing$ *is satisfied for every* $e \in \mathbb{S}_0$.

Lemma 3. *Let* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *be a CVNMS. A sequence* $\{\varrho_n^o\} \in \mathbb{V}$ *is classified as a cauchy sequence if and only if for every* $s \in \mathbb{T}_0$ *and* $e \in \mathbb{S}_0$, *one can find* $n_0 \in \mathbb{M}$ *satisfying*

$$\mathcal{E}(\varrho_n^o, \varrho^o, e) \succ \Im - s, \; \mathcal{G}(\varrho_n^o, \varrho^o, e) \prec s \text{ and } \mathcal{H}(\varrho_n^o, \varrho^o, e) \prec s \quad \text{for each} \quad n, m > n_0.$$

Proof. *Assume that* $\{\varrho_n^o\}$ *is Cauchy sequence. Suppose* e *is a fixed element from* $\mathbb{S}_0$, *Then, for each* $s \in \mathbb{T}_0$, *it is possible to identify* $n_0 \in \mathbb{M}$ *that satisfies the conditions* $\Im - \inf_{m>n} \mathcal{E}(\varrho_n^o, \varrho_m^o, e) \prec s, \; \sup_{m>n} \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec s,$ *and* $\sup_{m>n} \mathcal{H}(\varrho_n^o, \varrho_m^o, e) \prec s$ *for each* $n > n_0$. *Here, we look at three situations. For the situation where* $m > n > n_0$, *this leads to* $\Im - s \prec \inf_{m>n} \mathcal{E}(\varrho_n^o, \varrho_m^o, e) \prec \mathcal{E}(\varrho_n^o, \varrho_m^o, e), \; \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec \sup_{m>n} \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec s$ *and* $\mathcal{H}(\varrho_n^o, \varrho_m^o, e) \prec \sup_{m>n} \mathcal{H}(\varrho_n^o, \varrho_m^o, e) \prec$. *Now if* $m = n > n_0$, *then* $\Im - s \prec \Im = \mathcal{E}(\varrho_n^o, \varrho_m^o, e), \; \mathcal{G}(\varrho_n^o, \varrho_m^o, e) = \varnothing \prec s$ *and* $\mathcal{H}(\varrho_n^o, \varrho_m^o, e) = \varnothing \prec s$. *Finally, but not least, given* *the situation when $n > m > n_0$, it follows that $\Im - s \prec \inf_{n>m} \mathcal{E}(\varrho_m^o, \varrho_n^o, e) \preceq \mathcal{E}(\varrho_m^o, \varrho_n^o, e) = \mathcal{E}(\varrho_n^o, \varrho_m^o, e)$, $\mathcal{G}(\varrho_m^o, \varrho_n^o, e) = \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \preceq \sup_{n>m} \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec s$ and $\mathcal{H}(\varrho_m^o, \varrho_n^o, e) = \mathcal{H}(\varrho_n^o, \varrho_m^o, e) \preceq \sup_{n>m} \mathcal{H}(\varrho_n^o, \varrho_m^o, e) \prec s$. Thus, we deduce that $\mathcal{E}(\varrho_n^o, \varrho_m^o, e) \succ \Im - s$, $\mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec s$ as well as $\mathcal{H}(\varrho_n^o, \varrho_m^o, e) \prec s$ for any $n, m > n_0$. Conversely, let $e \in \mathbb{S}$ be fixed, and a real number $\epsilon > 0$ be given. Suppose that for each $s \in \mathbb{T}_0$, one can identify an $n_0 \in \mathbb{M}$ in which $\mathcal{E}(\varrho_n^o, \varrho_m^o, e) > \Im - s$, $\mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec s$ and $\mathcal{H}(\varrho_n^o, \varrho_m^o, e) \prec s$ for each $n > m > n_0$. Consequently*

$$\Im - 2s \prec \Im - s \preceq \inf_{m>n} \mathcal{E}(\varrho_n^o, \varrho_m^o, e)$$

$$\sup_{m>n} \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \preceq s \prec 2s$$

and

$$\sup_{m>n} \mathcal{H}(\varrho_n^o, \varrho_m^o, e) \preceq s \prec 2s$$

for every $n > n_0$. Select a complex number $s \in \mathbb{T}_0$ that fulfils $|s| < \frac{\epsilon}{2}$, then we get

$$|\Im - \inf_{m>n} \mathcal{E}(\varrho_n^o, \varrho_m^o, e)| < 2|s| < \epsilon, \ |\sup_{m>n} \mathcal{G}(\varrho_n^o, \varrho_m^o, e)| < 2|s| < \epsilon \text{ and } |\sup_{m>n} \mathcal{H}(\varrho_n^o, \varrho_m^o, e)| < 2|s| < \epsilon \text{ for each } n >$$

Thus, we have $\lim_{n\to\infty} \inf_{m>n} \mathcal{E}(\varrho_n^o, \varrho_m^o, e) = \Im$, $\lim_{n\to\infty} \sup_{m>n} \mathcal{G}(\varrho_n^o, \varrho_m^o, e) = \varnothing$, and $\lim_{n\to\infty} \sup_{m>n} \mathcal{H}(\varrho_n^o, \varrho_m^o, e) = \varnothing$, This demonstrates that the sequence $\{\varrho^o\}_n$ is Cauchy.

4. Fixed-point Results

We will now examine the existence and uniqueness of fixed points for self-mappings that satisfy the specified contractive requirements in CVNMS. Let $\{e_n\}$ be a sequence from $\mathbb{C}$. It is said that $\lim_{n\to\infty} e_n = \infty = (\infty, \infty)$ if for every $c \in \mathbb{C}$, there is a $n_0 \in \mathbb{M}$ such that $e_n \succeq e$ for all $n > n_0$

Theorem 1. *Let $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ be a complete CVNMS with the characteristic that any sequence $\{e_n\} \in \mathbb{S}_0$ fulfills $\lim_{n\to\infty} e_n = \infty$ implies*

$$\lim_{n\to\infty} \inf_{\nu\in\mathbb{V}} \mathcal{E}(\varrho^o, \nu, e_n) = \Im, \ \lim_{n\to\infty} \sup_{\nu\in\mathbb{V}} \mathcal{G}(\varrho^o, \nu, e_n) = \varnothing, \ \lim_{n\to\infty} \sup_{\nu\in\mathbb{V}} \mathcal{H}(\varrho^o, \nu, e_n) = \varnothing,$$

for any $\varrho^o \in \mathbb{V}$. Consider a self-mapping $h : \mathbb{V} \to \mathbb{V}$ meets the following condition:

$$\mathcal{E}(h\varrho^o, h\nu, ke) \succeq \mathcal{E}(\varrho^o, \nu, e), \ \mathcal{G}(h\varrho^o, h\nu, ke) \preceq \mathcal{G}(\varrho^o, \nu, e) \text{ and } \mathcal{H}(h\varrho^o, h\nu, ke) \preceq \mathcal{H}(\varrho^o, \nu, e) \tag{1}$$

for each $\varrho^o, \nu \in \mathbb{V}$ and $e \in \mathbb{S}_0$, where $k \in (0, 1)$. Then, there is a unique fixed point of the mapping h that is located in $\mathbb{V}$.

Proof. Let $\varrho^o \in \mathbb{V}$ be an arbitrarily chosen point. In $\mathbb{V}$, a sequence $\{\varrho_n^o\}$ is defined by $\varrho_n^o = h{\varrho^o}_{n-1}$ for all $n \in \mathbb{M}$. The existence of a $n_0 \in \mathbb{M}$ such that ${\varrho^o}_{n_0} = {\varrho^o}_{n_0-1}$ guarantees that ${\varrho^o}_{n_0}$ is a fixed point of h. To prove that the sequence $\{\varrho_n^o\}$ is Cauchy, we see that $\varrho_n^o \neq {\varrho^o}_{n-1}$ for each $n \in \mathbb{M}$.*

For every $n \in \mathbb{M}$ and a fixed $e \in \mathbb{S}_0$, we define

$$\mathbb{A}_n := \{\mathcal{E}(\varrho_n^o, \varrho_m^o, e) : m > n\} \subset \mathbb{T},$$

$$\mathbb{B}_n := \{\mathcal{G}(\varrho_n^o, \varrho_m^o, e) : m > n\} \subset \mathbb{T},$$

$$\mathbb{C}_n := \{\mathcal{H}(\varrho_n^o, \varrho_m^o, e) : m > n\} \subset \mathbb{T}.$$

As $\varnothing < \mathcal{E}(\varrho_n^o, \varrho_m^o, e) \preceq \Im$ for every $n \in \mathbb{M}$ and $n < m$, subsequent to the remarks 1, the greatest lower bound of the set $\mathbb{A}_n$, denoted as $\inf \mathbb{A}_n = \acute{x}_n$, exist for every $n \in \mathbb{M}$. In the same way, since $\varnothing \preceq \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec \Im$ for all $n \in \mathbb{M}$ such that $n < m$, subsequent to the remarks 1, the least upper bound of the set $\mathbb{B}_n$, denoted as $\sup \mathbb{B}_n = \acute{y}_n$, exists for every $n \in \mathbb{M}$. and also, since $\varnothing \preceq \mathcal{H}(\varrho_n^o, \varrho_m^o, c) \prec \Im$ for all $n \in \mathbb{M}$ where $n < m$, subsequent to the remarks 1, The supremum of the set $\mathbb{C}_n$, denoted as $\sup \mathbb{C}_n = \acute{c}_n$, exists for every n belonging to the set $\mathbb{M}$. For $c \in \mathbb{S}_0$ and $n, m \in \mathbb{M}$ where $m > n$, using equation (1), we obtain

$$\mathcal{E}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \mathcal{E}(h{\varrho^o}_{n+1}, h\varrho_m^o, e) \succeq \mathcal{E}\left(\varrho_n^o, \varrho_m^o, \frac{e}{k}\right) \tag{2}$$

$$\mathcal{G}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \mathcal{G}(h\varrho_n^o, h\varrho_m^o, e) \preceq \mathcal{G}\left(\varrho_n^o, \varrho_m^o, \frac{e}{k}\right) \tag{3}$$

and

$$\mathcal{H}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \mathcal{H}(h\varrho_n^o, h\varrho_m^o, e) \preceq \mathcal{H}\left(\varrho_n^o, \varrho_m^o, \frac{e}{k}\right) \tag{4}$$

Since $k \in (0,1)$, according to Lemma 1, this implies that

$$\mathcal{E}\left(\varrho_n^o, \varrho_m^o, \frac{e}{k}\right) \succeq \mathcal{E}(\varrho_n^o, \varrho_m^o, e),\ \ \mathcal{G}\left(\varrho_n^o, \varrho_m^o, \frac{e}{k}\right) \preceq \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \text{ and } \mathcal{H}\left(\varrho_n^o, \varrho_m^o, \frac{e}{k}\right) \preceq \mathcal{H}(\varrho_n^o, \varrho_m^o, e).$$

This ultimately results in

$$\mathcal{E}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \succeq \mathcal{E}(\varrho_n^o, \varrho_m^o, e)$$

$$\mathcal{G}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \preceq \mathcal{G}(\varrho_n^o, \varrho_m^o, e)$$

and

$$\mathcal{H}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \preceq \mathcal{H}(\varrho_n^o, \varrho_m^o, e)$$

for every $n, m \in \mathbb{M}$ and $m > n$.After verifying the $inf(\mathcal{E})$, $sup(\mathcal{G})$ and $sup(\mathcal{H})$ above, we may conclude that

$$\varnothing \preceq \acute{x}_n \preceq \acute{x}_{n+1} \preceq \Im, \quad \varnothing \preceq \acute{y}_{n+1} \preceq \acute{y}_n \preceq \Im, \quad \varnothing \preceq \acute{c}_{n+1} \preceq \acute{c}_n \preceq \Im$$

for any $n \in \mathbb{M}$. Thus $\{\acute{x}_n\}, \{\acute{y}_n\}$ and $\{\acute{c}_n\}$ are monotonic sequences in $\mathbb{S}$. According to Remarks 1, there are complex numbers $\acute{x}, \acute{y}, \acute{c} \in \mathbb{S}$ satisfying $\lim_{n\to\infty} \acute{x}_n = \acute{x}, \lim_{n\to\infty} \acute{y}_n = \acute{y}$ and $\lim_{n\to\infty} \acute{c}_n = \acute{c}$. By using equations (2), (3), and (4), we have

$$\acute{x}_{n+1} = \inf_{m>n} \mathcal{E}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \succeq \inf_{m>n} \mathcal{E}\left(\varrho_n^o, \varrho_m^o, \frac{e}{k}\right)$$

$$\acute{y}_{n+1} = \sup_{m>n} \mathcal{G}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \preceq \sup_{m>n} \mathcal{G}\left(\varrho^o_n, \varrho^o_m, \frac{e}{k}\right)$$

and

$$\acute{c}_{n+1} = \sup_{m>n} \mathcal{H}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \preceq \sup_{m>n} \mathcal{H}\left(\varrho^o_n, \varrho^o_m, \frac{e}{k}\right)$$

for $c \in \mathbb{S}_0$ *and* $n \in \mathbb{M}$. *By successively applying equation* (1) *to the inequalities mentioned above, we obtain*

$$\begin{aligned}
\acute{x}_{n+1} &\succeq \inf_{m>n} \mathcal{E}(\varrho^o_n, \varrho^o_m, \frac{e}{k}) \\
&\succeq \inf_{m>n} \mathcal{E}({\varrho^o}_{n-1}, {\varrho^o}_{m-1}, \frac{e}{k^2}) \\
&\succeq \inf_{m>n} \mathcal{E}({\varrho^o}_{n-2}, {\varrho^o}_{m-2}, \frac{e}{k^3}) \\
&. \\
&. \\
&. \\
&\succeq \inf_{m>n} \mathcal{E}({\varrho^o}_{0}, {\varrho^o}_{m-n}, \frac{e}{k^{n+1}})
\end{aligned}$$

$$\begin{aligned}
\acute{y}_{n+1} &\preceq \sup_{m>n} \mathcal{G}(\varrho^o_n, \varrho^o_m, \frac{e}{k}) \\
&\preceq \sup_{m>n} \mathcal{G}({\varrho^o}_{n-1}, {\varrho^o}_{m-1}, \frac{e}{k^2}) \\
&\preceq \sup_{m>n} \mathcal{G}({\varrho^o}_{n-2}, {\varrho^o}_{m-2}, \frac{e}{k^3}) \\
&. \\
&. \\
&. \\
&\preceq \sup_{m>n} \mathcal{G}({\varrho^o}_{0}, {\varrho^o}_{m-n}, \frac{e}{k^{n+1}})
\end{aligned}$$

and

$$\begin{aligned}
\acute{c}_{n+1} &\preceq \sup_{m>n} \mathcal{H}(\varrho^o_n, \varrho^o_m, \frac{e}{k}) \\
&\preceq \sup_{m>n} \mathcal{H}({\varrho^o}_{n-1}, {\varrho^o}_{m-1}, \frac{e}{k^2}) \\
&\preceq \sup_{m>n} \mathcal{H}({\varrho^o}_{n-2}, {\varrho^o}_{m-2}, \frac{e}{k^3}) \\
&. \\
&. \\
&.
\end{aligned}$$

$$\preceq \sup_{m>n} \mathcal{H}({\varrho^o}_0, {\varrho^o}_{m-n}, \frac{e}{k^{n+1}})$$

for any $e \in \mathbb{S}_0$ *and* $n \in \mathbb{M}$. *In addition, we obtain*

$$\acute{x}_{n+1} \succeq \inf_{m>n} \mathcal{E}\left({\varrho^o}_0, {\varrho^o}_{m-n}, \frac{e}{k^{n+1}}\right) \succeq \inf_{\nu \in \mathbb{V}} \mathcal{E}\left({\varrho^o}_0, \nu, \frac{e}{k^{n+1}}\right)$$

$$\acute{y}_{n+1} \preceq \sup_{m>n} \mathcal{G}\left({\varrho^o}_0, {\varrho^o}_{m-n}, \frac{e}{k^{n+1}}\right) \preceq \sup_{\nu \in \mathbb{V}} \mathcal{G}\left({\varrho^o}_0, \nu, \frac{e}{k^{n+1}}\right)$$

and

$$\acute{c}_{n+1} \preceq \sup_{m>n} \mathcal{H}\left({\varrho^o}_0, {\varrho^o}_{m-n}, \frac{e}{k^{n+1}}\right) \preceq \sup_{\nu \in \mathbb{V}} \mathcal{H}\left({\varrho^o}_0, \nu, \frac{e}{k^{n+1}}\right)$$

for each $e \in \mathbb{S}_0$ *and* $n \in \mathbb{M}$. *As* $n \to \infty$ *on both sides of the given inequality, the hypothesis yields*

$$\acute{x} = \lim_{n\to\infty} \acute{x}_{n+1} \succeq \lim_{n\to\infty} \inf_{\nu \in \mathbb{V}} \mathcal{E}\left({\varrho^o}_0, \nu, \frac{e}{k^{n+1}}\right) = \Im$$

$$\acute{y} = \lim_{n\to\infty} \acute{y}_{n+1} \preceq \lim_{n\to\infty} \sup_{\nu \in \mathbb{V}} \mathcal{G}\left({\varrho^o}_0, \nu, \frac{e}{k^{n+1}}\right) = \varnothing$$

and

$$\acute{c} = \lim_{n\to\infty} \acute{c}_{n+1} \preceq \lim_{n\to\infty} \sup_{\nu \in \mathbb{V}} \mathcal{H}\left({\varrho^o}_0, \nu, \frac{e}{k^{n+1}}\right) = \varnothing$$

Which imply $\acute{x} = \Im, \acute{y} = \varnothing$ *and* $\acute{c} = \varnothing$. *Thus,*

$$\lim_{n\to\infty} \inf_{m>n} \mathcal{E}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \lim_{n\to\infty} \acute{x}_n = \Im,$$

$$\lim_{n\to\infty} \sup_{m>n} \mathcal{G}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \lim_{n\to\infty} \acute{y}_n = \varnothing,$$

$$\lim_{n\to\infty} \sup_{m>n} \mathcal{H}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \lim_{n\to\infty} \acute{c}_n = \varnothing,$$

for all $e \in \mathbb{S}_0$ *which show sequence* $\{\varrho^o_n\}$ *is Cauchy. Given that* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *is complete,* 2 *implies the existence of* $\varrho^o \in \mathbb{V}$ *satisfying*

$$\lim_{n\to\infty} \mathcal{E}(\varrho^o_n, \varrho^o, e) = \Im, \ \lim_{n\to\infty} \mathcal{G}(\varrho^o_n, \varrho^o_m, e) = \varnothing \text{ and } \lim_{n\to\infty} \mathcal{H}(\varrho^o_n, \varrho^o_m, e) = \varnothing \ \text{ for any } \ e \in \mathbb{S}_0 \tag{5}$$

As a result of equation (1), *and conditions* (5), (10), (15) *of Definition* 7 *for any* $e \in \mathbb{S}_0$, *lead us to the conclusion that*

$$\begin{aligned}\mathcal{E}(\varrho^o, h\varrho^o, e) &\succeq \mathcal{E}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) * \mathcal{E}\left({\varrho^o}_{n+1}, h\varrho^o, \frac{e}{2}\right) \\ &= \mathcal{E}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{c}{2}\right) * \mathcal{E}\left(h\varrho^o_n, h\varrho^o, \frac{c}{2}\right) \\ &\succeq \mathcal{E}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) * \mathcal{E}\left(\varrho^o_n, \varrho^o, \frac{e}{2k}\right),\end{aligned}$$

$$\mathcal{G}(\varrho^o, h\varrho^o, e) \preceq \mathcal{G}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle\, \mathcal{G}\left({\varrho^o}_{n+1}, h\varrho^o, \frac{e}{2}\right)$$

$$= \mathcal{G}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{G}\left(h\varrho^o_n, h\varrho^o, \frac{e}{2}\right)$$
$$\preceq \mathcal{G}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{G}\left(\varrho^o_n, \varrho^o, \frac{e}{2k}\right),$$

and

$$\begin{aligned}\mathcal{H}(\varrho^o, h\varrho^o, e) &\preceq \mathcal{H}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{H}\left({\varrho^o}_{n+1}, h\varrho^o, \frac{e}{2}\right)\\ &= \mathcal{H}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{H}\left(h\varrho^o_n, h\varrho^o, \frac{e}{2}\right)\\ &\preceq \mathcal{H}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{H}\left(\varrho^o_n, \varrho^o, \frac{e}{2k}\right).\end{aligned}$$

Now considering the limit as $n \to \infty$ for both given inequalities, utilizing equation (5) *together with Remarks* 2*, for any $e \in \mathbb{S}_0$, it follows that*

$$\mathcal{E}(\varrho^o, h\varrho^o, e) = \Im,\ \mathcal{G}(\varrho^o, h\varrho^o, e) = \varnothing \text{ and } \mathcal{H}(\varrho^o, h\varrho^o, e) = \varnothing.$$

By conditions (3), (8), and (11) of Definition 7*, it can be concluded that ϱ^o is equal to $h\varrho^o$, indicating that ϱ^o is a fixed point of h. To demonstrate uniqueness, let us assume that z and ϱ^o are two distinct fixed points of h. This indicates that there are some elements $e' \in \mathbb{S}_0$ for which $\mathcal{E}(\varrho^o, z, e') \neq \Im, \mathcal{G}(\varrho^o, z, e') \neq \varnothing$ and $\mathcal{H}(\varrho^o, z, e') \neq \varnothing$. By iteratively applying equation* (1)*, for every $n \in \mathbb{M}$ we obtain*

$$\begin{aligned}\mathcal{E}(\varrho^o, z, e') = \mathcal{E}(h\varrho^o, hz, e') &\succeq \mathcal{E}(\varrho^o, z, \frac{e'}{k})\\ &\succeq \mathcal{E}(\varrho^o, z, \frac{e'}{k^2})\\ &.\\ &.\\ &.\\ &\succeq \mathcal{E}(\varrho^o, z, \frac{e'}{k^n})\end{aligned}$$

$$\begin{aligned}\mathcal{G}(\varrho^o, z, e') = \mathcal{G}(h\varrho^o, hz, e') &\preceq \mathcal{G}(\varrho^o, z, \frac{e'}{k})\\ &\preceq \mathcal{G}(\varrho^o, z, \frac{e'}{k^2})\\ &.\\ &.\\ &.\\ &\preceq \mathcal{G}(\varrho^o, z, \frac{e'}{k^n})\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(\varrho^o, z, e') = \acute{w}(h\varrho^o, hz, e') &\preceq \mathcal{H}(\varrho^o, z, \frac{e'}{k})\\ &\preceq \mathcal{H}(\varrho^o, z, \frac{e'}{k^2})\\ &.\\ &.\\ &.\\ &\preceq \mathcal{H}(\varrho^o, z, \frac{e'}{k^n}).\end{aligned}$$

Consequently, we derive that

$$\mathcal{E}(\varrho^o, z, e') \succeq \mathcal{E}(\varrho^o, z, \frac{e'}{k^n}) \succeq \inf_{\nu\in\mathbb{V}} \mathcal{E}(\varrho^o, z, \frac{e'}{k^n})$$

$$\mathcal{G}(\varrho^o, z, e') \preceq \mathcal{G}(\varrho^o, z, \frac{e'}{k^n}) \preceq \sup_{\nu\in\mathbb{V}} \mathcal{G}(\varrho^o, z, \frac{e'}{k^n})$$

and

$$\mathcal{H}(\varrho^o, z, e') \preceq \mathcal{H}(\varrho^o, z, \frac{e'}{k^n}) \preceq \sup_{\nu\in\mathbb{V}} \mathcal{H}(\varrho^o, z, \frac{e'}{k^n}).$$

As $k \in (0, 1)$, it is obvious that $\lim_{n\to\infty} \frac{e'}{k^n} = \infty$. So by assuming the limit as n approaches infinity for the above inequalities, it implies

$$\mathcal{E}(\varrho^o, z, e') = \Im,\ \mathcal{G}(\varrho^o, z, e') = \varnothing \text{ and } \mathcal{H}(\varrho^o, z, e') = \varnothing$$

which is a contradiction. Thus $\mathcal{E}(\varrho^o, z, e) = \Im, \mathcal{G}(\varrho^o, z, e) = \varnothing$ and $\mathcal{H}(\varrho^o, z, e) = \varnothing$ for each $e \in \mathbb{S}_0$. The uniqueness of the fixed points of h is confirmed by the fact that $\varrho^o = z$, as deduced from the conditions (3), (8), and (11) of Definitions 7.

Remark 4. *The proof of Theorem 1 remains the same by substituting equation (1) with the subsequently contractive condition of mapping h:*

$$\mathcal{E}(h\varrho^o, h\nu, \kappa(e)e) \succeq \mathcal{E}(\varrho^o, \nu, e),\ \mathcal{G}(h\varrho^o, h\nu, \kappa(e)e) \preceq \mathcal{G}(\varrho^o, \nu, e) \text{ and } \mathcal{H}(h\varrho^o, h\nu, \kappa(e)e) \preceq \mathcal{H}(\varrho^o, \nu, e).$$

For each $e \in \mathbb{S}_0$ and $\varrho^o, \nu \in \mathbb{V}$, κ denotes a mapping from $\mathbb{S}_0$ to $(0, 1)$.

Example 6. *Assume that $(\mathbb{V}, d)$ is a metric space and $\mathbb{V} = [0, 1]$ combined with $d(\varrho^o, \nu) = |\varrho^o - \nu|$ for each $\varrho^o, \nu \in \mathbb{V}$. Define the complex-valued t-norm $*$ and the complex-valued t-conorm $\triangle$ by $\acute{w}_1 * \acute{w}_2 = (\mu_1\mu_2, \varkappa_1\varkappa_2)$ and $\acute{w}_1 \triangle \acute{w}_2 = (\max\{\mu_1, \mu_2\}, \max\{\varkappa_1, \varkappa_2\})$ for all $\acute{w}_1 = (\mu_1, \varkappa_1), \acute{w}_2 = (\mu_2, \varkappa_2) \in \mathbb{T}$ respectively. Define complex-valued fuzzy sets $\mathcal{E}$, $\mathcal{G}$, and $\mathcal{H}$ as:*

$$\mathcal{E}(\varrho^o, \nu, e) = \frac{\tau\xi}{\tau\xi + d(\varrho^o, \nu)}\Im,\quad \mathcal{G}(\varrho^o, \nu, e) = \frac{d(\varrho^o, \nu)}{\tau\xi + d(\varrho^o, \nu)}\Im,\quad \mathcal{H}(\varrho^o, \nu, e) = \frac{d(\varrho^o, \nu)}{\tau\xi}\Im,$$

for each $\varrho^o, \nu \in \mathbb{V}$ and $e = (\tau, \xi) \in \mathbb{S}_0$. It is simple to prove that $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ is a complete CVNMS induced by metric d. Let us consider a sequence $\{e_n\} \in \mathbb{S}_0$ where each element $e_n = (\tau_n, \xi_n)$ for $n \in \mathbb{M}$ is arbitrary, Along with the fact that for all $\nu \in \mathbb{V}$, $0 \leq d(\varrho^o, \nu) \leq 1$, this means that

$$
\begin{aligned}
\Im &\succeq \inf_{\nu \in \mathbb{V}} \mathcal{E}(\varrho^o, \nu, e_n) \\
&= \inf_{\nu \in \mathbb{V}} \frac{\tau_n \xi_n}{\tau_n \xi_n + d(\varrho^o, \nu)} \Im \\
&= \frac{\tau_n \xi_n}{\tau_n \xi_n + \sup_{\nu \in \mathbb{V}} d(\varrho^o, \nu)} \Im \\
&\succeq \frac{\tau_n \xi_n}{\tau_n \xi_n + 1} \Im.
\end{aligned}
$$

As n approaches infinity, we have

$$\Im \succeq \lim_{n \to \infty} \inf_{\nu \in \mathbb{V}} \mathcal{E}(\varrho^o, \nu, e_n) \succeq \lim_{n \in \infty} \frac{\tau_n \xi_n}{\tau_n \xi_n + 1} \Im = \Im$$

the given expression implies that $\lim_{n \to \infty} \inf_{\nu \in \mathbb{V}} \mathcal{E}(\varrho^o, \nu, e_n) = \Im$. Furthermore, we obtain

$$
\begin{aligned}
\varnothing &\preceq \sup_{\nu \in \mathbb{V}} \mathcal{G}(\varrho^o, \nu, e_n) \\
&= \sup_{\nu \in \mathbb{V}} \frac{d(\varrho^o, \nu)}{\tau_n \xi_n + d(\varrho^o, \nu)} \Im \\
&= \frac{\sup_{\nu \in \mathbb{V}} d(\varrho^o, \nu)}{\tau_n \xi_n + \inf_{\nu \in \mathbb{V}} d(\varrho^o, \nu)} \Im \\
&\preceq \frac{1}{\tau_n \xi_n} \Im.
\end{aligned}
$$

As n approaches infinity, we have

$$\varnothing \preceq \lim_{n \to \infty} \sup_{\nu \in \mathbb{V}} \mathcal{G}(\varrho^o, \nu, e_n) \preceq \lim_{n \in \infty} \frac{1}{\tau_n \xi_n} \Im = \varnothing$$

the given expression implies that $\lim_{n \to \infty} \sup_{\nu \in \mathbb{V}} \mathcal{G}(\varrho^o, \nu, e_n) = \varnothing$. In a similar way, we have

$$
\begin{aligned}
\varnothing &\preceq \sup_{\nu \in \mathbb{V}} \mathcal{H}(\varrho^o, \nu, e_n) \\
&= \sup_{\nu \in \mathbb{V}} \frac{d(\varrho^o, \nu)}{\tau_n \xi_n} \Im \\
&= \frac{\sup_{\nu \in \mathbb{V}} d(\varrho^o, \nu)}{\tau_n \xi_n} \Im \\
&\preceq \frac{1}{\tau_n \xi_n} \Im.
\end{aligned}
$$

As n approaches infinity, we have

$$\varnothing \preceq \lim_{n\to\infty} \sup_{\nu\in\mathbb{V}} \mathcal{H}(\varrho^o,\nu,e_n) \preceq \lim_{n\in\infty} \frac{1}{\tau_n\xi_n}\Im = \varnothing$$

the given expression implies that $\lim_{n\to\infty}\sup_{\nu\in\mathbb{V}}\mathcal{H}(\varrho^o,\nu,e_n) = \varnothing$. *Let $\mathcal{K}$ be a mapping from $\mathbb{V}$ to $\mathbb{V}$ which defined as $\mathcal{K}_{\varrho^o} = \frac{\varrho^o}{2}$ for each $\varrho^o \in \mathbb{V}$. If we choose a number $k \in [1/2,1) \subset (0,1)$, then for all $\varrho^o,\nu \in \mathbb{V}$ and $e \in \mathbb{S}_0$, fulfills equation (1). However, since $2k > 1$, we have*

$$\begin{aligned}
\mathcal{E}(\mathcal{K}\varrho^o,\mathcal{K}\nu,ke) &= \frac{k\tau\xi}{k\tau\xi + d(\mathcal{K}\varrho^o,\mathcal{K}\nu)}\Im \\
&= \frac{k\tau\xi}{k\tau\xi + |\frac{\varrho^o}{2} - \frac{\nu}{2}|}\Im \\
&= \frac{k\tau\xi}{k\tau\xi + \frac{1}{2}|\varrho^o - \nu|}\Im \\
&= \frac{2k\tau\xi}{2k\tau\xi + |\varrho^o - \nu|}\Im \\
&\succeq \frac{\tau\xi}{\tau\xi + |\varrho^o - \nu|}\Im \\
&= \mathcal{E}(\varrho^o,\nu,e)
\end{aligned}$$

for any $\varrho^o,\nu \in \mathbb{V}$ and $e = (\tau,\xi) \in \mathbb{S}_0$.

$$\begin{aligned}
\mathcal{G}(\mathcal{K}\varrho^o,\mathcal{K}\nu,ke) &= \frac{d(\mathcal{K}\varrho^o,\mathcal{K}\nu)}{k\tau\xi + d(\mathcal{K}\varrho^o,\mathcal{K}\nu)}\Im \\
&= \frac{|\frac{\varrho^o}{2} - \frac{\nu}{2}|}{k\tau\xi + |\frac{\varrho^o}{2} - \frac{\nu}{2}|}\Im \\
&= \frac{\frac{1}{2}|\varrho^o - \nu|}{k\tau\xi + \frac{1}{2}|\varrho^o - \nu|}\Im \\
&= \frac{|\varrho^o - \nu|}{2k\tau\xi + |\varrho^o - \nu|}\Im \\
&\preceq \frac{|\varrho^o - \nu|}{\tau\xi + |\varrho^o - \nu|}\Im \\
&= \mathcal{G}(\varrho^o,\nu,e)
\end{aligned}$$

for any $\varrho^o,\nu \in \mathbb{V}$ and $e = (\tau,\xi) \in \mathbb{S}_0$.

$$\begin{aligned}
\mathcal{H}(\mathcal{K}\varrho^o,\mathcal{K}\nu,ke) &= \frac{d(\mathcal{K}\varrho^o,\mathcal{K}\nu)}{k\tau\xi}\Im \\
&= \frac{|\frac{\varrho^o}{2} - \frac{\nu}{2}|}{k\tau\xi}\Im
\end{aligned}$$

$$= \frac{\frac{1}{2}|\varrho^o - \nu|}{k\tau\xi}\Im$$
$$= \frac{|\varrho^o - \nu|}{2k\tau\xi}\Im$$
$$\preceq \frac{|\varrho^o - \nu|}{\tau\xi}\Im$$
$$= \mathcal{H}(\varrho^o, \nu, c)$$

for each $e = (\tau, \xi) \in \mathbb{S}_0$ *and* $\varrho^o, \nu \in \mathbb{V}$. *Consequently, all the conditions given in Theorem 1 are met. Hence in* $\mathcal{K}$, 0 *is the only fixed point.*

The following example demonstrates that Theorem 1 is not unnecessary.

Example 7. *Suppose that* $\mathbb{V}$ *is equal to* $\mathbb{M}$. *We define two binary operations,* $\star$ *and* $\triangle$, *as* $e_1 \star e_2 = (\tau_1\tau_2, \xi_1\xi_2)$ *and* $e_1 \triangle e_2 = (\tau_1 + \tau_2, \xi_1 + \xi_2) - (\tau_1\tau_2, \xi_1\xi_2)$ *for any* $e_i = (\tau_i, \xi_i) \in \mathbb{T}$ *for i=1,2. Consider the CFSs* $\mathcal{E}$, $\mathcal{G}$, *and* $\mathcal{H}$, *which are defined as follows:*

$$\mathcal{E}(\varrho^o, \nu, e) = \frac{\min\{\varrho^o, \nu\}}{\max\{\varrho^o, \nu\}}\Im,\ \mathcal{G}(\varrho^o, \nu, e) = \left(1 - \frac{\min\{\varrho^o, \nu\}}{\max\{\varrho^o, \nu\}}\right)\Im,\ \mathcal{H}(\varrho^o, \nu, e) = \left(1 - \frac{2\min\{\varrho^o, \nu\}}{\min\{\varrho^o, \nu\} + \max\{\varrho^o, \nu\}}\right)\Im$$

for each $\varrho^o, \nu \in \mathbb{V}$ *and* $e = (\tau, \xi) \in \mathbb{S}_0$. *It is simple to prove that* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *is a complete CVNMS. For any* $\varrho^o \in \mathbb{V}$, *consider a mapping* $\mathcal{K} : \mathbb{V} \to \mathbb{V}$ *represented by* ${\varrho^o}^2 + 5$. *Consider the sequence* $\{e_n\}$, *defined as* $e_n = (n, n)$ *for all* $n \in \mathbb{M}$. *The construction of* e_n *ensures that there is no ambiguity in the expression* $\lim_{n\to\infty} = \infty$. *For every element* ϱ^o *in the set* $\mathbb{V}$, *with a fixed element* $\nu \in \mathbb{V}$ *such that* ϱ^o *is not equal to* ν, *and for all* $n \in \mathbb{M}$, *we may prove that*

$$\inf_{\nu\in\mathbb{V}} \mathcal{E}(\varrho^o, \nu, e_n) = \inf \frac{\min\{\varrho^o, \nu\}}{\max\{\varrho^o, \nu\}}\Im = \varnothing$$
$$\sup_{\nu\in\mathbb{V}} \mathcal{G}(\varrho^o, \nu, e_n) = \sup\left(1 - \frac{\min\{\varrho^o, \nu\}}{\max\{\varrho^o, \nu\}}\right)\Im = \Im$$

and

$$\sup_{\nu\in\mathbb{V}} \mathcal{H}(\varrho^o, \nu, e_n) = \sup\left(1 - \frac{2\min\{\varrho^o, \nu\}}{\min\{\varrho^o, \nu\} + \max\{\varrho^o, \nu\}}\right)\Im = \Im.$$

Therefore, we get

$$\lim_{n\to\infty} \inf_{\nu\in\mathbb{V}} \mathcal{E}(\varrho^o, \nu, e_n) = \varnothing \neq \Im$$
$$\lim_{n\to\infty} \sup_{\nu\in\mathbb{V}} \mathcal{G}(\varrho^o, \nu, e_n) = \Im \neq \varnothing$$

and

$$\lim_{n\to\infty} \sup_{\nu\in\mathbb{V}} \mathcal{H}(\varrho^o, \nu, e_n) = \Im \neq \varnothing$$

for each $\varrho^o \in \mathbb{V}$. *For every* $k \in (0, 1)$, $\varrho^o, \nu \in \mathbb{V}$ *and* $e \in \mathbb{S}_0$, *note that*

$$\mathcal{E}(\mathcal{K}\varrho^o, \mathcal{K}\nu, ke) = \frac{\min\{{\varrho^o}^2 + 5, \nu^2 + 5\}}{\max\{{\varrho^o}^2 + 5, \nu^2 + 5\}}\Im$$

$$\begin{aligned}&\succeq \frac{\min\{\varrho^o,\nu\}}{\max\{\varrho^o,\nu\}}\Im\\ &=\mathcal{E}(\varrho^o,\nu,e)\end{aligned}$$

$$\begin{aligned}\mathcal{G}(\mathcal{K}\varrho^o,\mathcal{K}\nu,ke)&=\left(1-\frac{\min\{\varrho^{o2}+5,\nu^2+5\}}{\max\{\varrho^{o2}+5,\nu^2+5\}}\right)\Im\\ &\preceq\left(1-\frac{\min\{\varrho^o,\nu\}}{\max\{\varrho^o,\nu\}}\right)\Im\\ &=\mathcal{G}(\varrho^o,\nu,e)\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(\mathcal{K}\varrho^o,\mathcal{K}\nu,ke)&=\left(1-\frac{2\min\{\varrho^{o2}+5,\nu^2+5\}}{\min\{\varrho^{o2}+5,\nu^2+5\}+\max\{\varrho^{o2}+5,\nu^2+5\}}\right)\Im\\ &\preceq\left(1-\frac{2\min\{\varrho^o,\nu\}}{\min\{\varrho^o,\nu\}+\max\{\varrho^o,\nu\}}\right)\Im\\ &=\mathcal{H}(\varrho^o,\nu,e).\end{aligned}$$

Equation (1) *is satisfied by the mapping $\mathcal{K}$, however, $\mathbb{V}$ has no fixed point.*

In order to obtain the next result, Ω is defined as the set of all mappings $\varphi:\mathbb{T}\to\mathbb{T}$, where φ is continuous, $\varphi(\Im)=\Im, \varphi(c)\succ c$ for every $e\in\mathbb{T}_0$, and $\lim_{n\to}\varphi^n(e)=\Im$ for every $e\in\mathbb{T}_0$. Similarly, F is defined as the set of all mappings $\sigma:\mathbb{T}\to\mathbb{T}$ in which σ is continuous, $\sigma(\varnothing)=\varnothing, \sigma(e)\prec e$ for every $e\in\mathbb{T}$, and $\lim_{n\to}\sigma^n(e)=\varnothing$ for every $e\in\mathbb{T}$.

Theorem 2. *Assume that the CVNMS $(\mathbb{V},\mathcal{E},\mathcal{G},\mathcal{H},\star,\triangle)$ is complete. If the following conditions are satisfied by a mapping $\mathcal{K}:\mathbb{V}\to\mathbb{V}$:*

$$\mathcal{E}(\mathcal{K}\varrho^o,\mathcal{K}\nu,e)\succeq\varphi(\mathcal{E}(\varrho^o,\nu,e)),\ \mathcal{G}(\mathcal{K}\varrho^o,\mathcal{K}\nu,e)\preceq\sigma(\mathcal{G}(\varrho^o,\nu,e))\text{ and }\mathcal{H}(\mathcal{K}\varrho^o,\mathcal{K}\nu,e)\preceq\sigma(\mathcal{H}(\varrho^o,\nu,e))\tag{6}$$

for each $\varrho^o,\nu\in\mathbb{V}$ and $e\in\mathbb{S}_0$, with $\sigma\in F$ and $\varphi\in\Omega$. Then the mapping $\mathcal{K}$ has a fixed point in the set $\mathbb{V}$.

Proof. *Let ${\varrho^o}_0\in\mathbb{V}$ be Any given point. In $\mathbb{V}$, a sequence $\{\varrho^o_n\}$ is defined by $\varrho^o_n=h{\varrho^o}_{n-1}$ for all $n\in\mathbb{M}$. The existence of an element $n_0\in\mathbb{M}$ such that ${\varrho^o}_{n_0}={\varrho^o}_{n_0-1}$ makes sure that ${\varrho^o}_{n_0}$ is a fixed point in $\mathcal{K}$. Now, we assume that ϱ^o_n is not equal to ${\varrho^o}_{n-1}$ for every $n\mathbb{M}$, and we prove that the sequence $\{\varrho^o_n\}$ is Cauchy. For each $n\in\mathbb{M}$ and a given $e\in\mathbb{S}_0$, let us define*

$$\begin{aligned}\mathbb{A}_n&:=\{\mathcal{E}(\varrho^o_n,\varrho^o_m,e):m>n\}\subset\mathbb{T},\\ \mathbb{B}_n&:=\{\mathcal{G}(\varrho^o_n,\varrho^o_m,e):m>n\}\subset\mathbb{T},\\ \mathbb{C}_n&:=\{\mathcal{H}(\varrho^o_n,\varrho^o_m,e):m>n\}\subset\mathbb{T}.\end{aligned}$$

As $\varnothing \prec \mathcal{E}(\varrho_n^o, \varrho_m^o, e) \preceq \Im$ *According to Remarks* 1, *the greatest lower bound of* $\mathbb{A}_n$, *that is,* $\inf \mathbb{A}_n = \acute{x}_n$, *exists for any* $n \in \mathbb{N}$ *such that* $n < m$. *In the same way, since* $\varnothing \preceq \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec \Im$ *for each* $n \in \mathbb{M}$ *such that* $n < m$, *According to Remarks* 1, *the least upper bound* $\mathbb{B}_n$, *that is,* $\sup \mathbb{B}_n = \acute{y}_n$ *exist for any* $n \in \mathbb{N}$. *Also, since* $\varnothing \preceq \mathcal{H}(\varrho_n^o, \varrho_m^o, e) \prec \Im$ *for each* $n \in \mathbb{M}$ *such that* $n < m$, *According to Remarks* 1, *the sup*$(\mathbb{C}_n)$, *that is,* $\sup \mathbb{C}_n = \acute{c}_n$ *exist for any* $n \in \mathbb{N}$. *According to equation* (6), *for each* $n, m \in \mathbb{M}$ *such that* $m > n$, *this implies that*

$$\mathcal{E}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \mathcal{E}(\mathcal{K}\varrho_n^o, \mathcal{K}\varrho_m^o, e) \succeq \varphi(\mathcal{E}(\varrho_n^o, \varrho_m^o, e)) \succ \mathcal{E}(\varrho_n^o, \varrho_m^o, e) \tag{7}$$

$$\mathcal{G}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \mathcal{G}(\mathcal{K}\varrho_n^o, \mathcal{K}\varrho_m^o, e) \preceq \sigma(\mathcal{G}(\varrho_n^o, \varrho_m^o, e) \prec \mathcal{G}(\varrho_n^o, \varrho_m^o, e) \tag{8}$$

and

$$\mathcal{H}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) = \mathcal{H}(\mathcal{K}\varrho_n^o, \mathcal{K}\varrho_m^o, e) \preceq \sigma(\mathcal{H}(\varrho_n^o, \varrho_m^o, e) \prec \mathcal{H}(\varrho_n^o, \varrho_m^o, e). \tag{9}$$

From this, we can conclude that

$$\mathcal{E}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \succ \mathcal{E}(\varrho_n^o, \varrho_m^o, e)$$

$$\mathcal{G}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \prec \mathcal{G}(\varrho_n^o, \varrho_m^o, e)$$

and

$$\mathcal{H}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) \prec \mathcal{H}(\varrho_n^o, \varrho_m^o, e)$$

for all $n, m \in \mathbb{M}$ *such that* $m > n$ *with* $e \in \mathbb{S}_0$. *By taking the inf*$(\mathcal{E})$, *sup*$(\mathcal{G})$, *and sup*$(\mathcal{H})$, *we obtain*

$$\varnothing \preceq \acute{x}_n \preceq \acute{x}_{n+1} \preceq \Im, \quad \varnothing \preceq \acute{y}_{n+1} \preceq \acute{y}_n \preceq \Im, \quad \varnothing \preceq \acute{c}_{n+1} \preceq \acute{c}_n \preceq \Im$$

for any $n \in \mathbb{M}$. *Thus* $\{\acute{x}_n\}, \{\acute{y}_n\}$ *and* $\{\acute{c}_n\}$ *are monotonic sequences in* $\mathbb{S}$. *According to Remarks 1, there are complex numbers* $\acute{x}, \acute{y}, \acute{c} \in \mathbb{S}$ *satisfying* $\lim_{n\to\infty} \acute{x}_n = \acute{x}, \lim_{n\to\infty} \acute{y}_n = \acute{y}$ *and* $\lim_{n\to\infty} \acute{c}_n = \acute{c}$. *By using equations* (7), (8), *and* (9), *and employing equation* (6) *gradually, we have*

$$\begin{aligned}
\mathcal{E}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) &\succeq \varphi(\mathcal{E}(\varrho_n^o, \varrho_m^o, e)) \\
&\succeq \varphi^2(\mathcal{E}({\varrho^o}_{n-1}, {\varrho^o}_{m-1}, e)) \\
&. \\
&. \\
&. \\
&\succeq \varphi(\mathcal{E}({\varrho^o}_0, {\varrho^o}_{m-n}, e))
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) &\preceq \sigma(\mathcal{G}(\varrho_n^o, \varrho_m^o, e)) \\
&\preceq \sigma^2(\mathcal{G}({\varrho^o}_{n-1}, {\varrho^o}_{m-1}, e))
\end{aligned}$$

$$
\begin{aligned}
&.\\
&.\\
&.\\
&\preceq \sigma(\mathcal{G}({\varrho^o}_0, {\varrho^o}_{m-n}, e))
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) &\preceq \sigma(\mathcal{H}(\varrho^o_n, \varrho^o_m, e))\\
&\preceq \sigma^2(\mathcal{H}({\varrho^o}_{n-1}, {\varrho^o}_{m-1}, e))\\
&.\\
&.\\
&.\\
&\preceq \sigma(\mathcal{H}({\varrho^o}_0, {\varrho^o}_{m-n}, e))
\end{aligned}
$$

for every $n \in \mathbb{M}$ such that $m > n$ and $e \in \mathbb{S}_0$. This implies that

$$\acute{x}_{n+1} \succeq \inf_{m>n} \varphi^n(\mathcal{E}({\varrho^o}_0, {\varrho^o}_{m-n}, e)$$

$$\acute{y}_{n+1} \preceq \sup_{m>n} \sigma^n(\mathcal{G}({\varrho^o}_0, {\varrho^o}_{m-n}, e)$$

and

$$\acute{c}_{n+1} \preceq \sup_{m>n} \sigma^n(\mathcal{H}({\varrho^o}_0, {\varrho^o}_{m-n}, e),$$

for every $n \in \mathbb{M}$ along with $c \in \mathbb{S}_0$. On both sides of the above inequality, as n approaches infinity, we deduce that

$$
\begin{aligned}
\acute{x} &\succeq \lim_{n\to\infty} \inf_{m>n} \varphi^n(\mathcal{E}({\varrho^o}_0, {\varrho^o}_{m-n}, e))\\
&= \lim_{n\to\infty} \varphi^n(\mathcal{E}({\varrho^o}_0, {\varrho^o}_{m-n}, e))\\
&= \Im
\end{aligned}
$$

$$
\begin{aligned}
\acute{y} &\preceq \lim_{n\to\infty} \sup_{m>n} \sigma^n(\mathcal{G}({\varrho^o}_0, {\varrho^o}_{m-n}, e))\\
&= \lim_{n\to\infty} \sigma^n(\mathcal{G}({\varrho^o}_0, {\varrho^o}_{m-n}, e))\\
&= \varnothing
\end{aligned}
$$

and

$$\acute{c} \preceq \lim_{n\to\infty} \sup_{m>n} \sigma^n(\mathcal{H}({\varrho^o}_0, {\varrho^o}_{m-n}, e))$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sigma^n(\mathcal{H}({\varrho^o}_0, {\varrho^o}_{m-n}, e)) \\
&= \varnothing.
\end{aligned}
$$

Hence, $\acute{x} = \Im, \acute{y} = \varnothing$ *and* $\acute{c} = \varnothing$. *Thus,*

$$
\begin{aligned}
\lim_{n\to\infty} \inf_{m>n} \mathcal{E}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) &= \lim_{n\to\infty} \acute{x}_n = \Im, \\
\lim_{n\to\infty} \sup_{m>n} \mathcal{G}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) &= \lim_{n\to\infty} \acute{y}_n = \varnothing, \\
\lim_{n\to\infty} \sup_{m>n} \mathcal{H}({\varrho^o}_{n+1}, {\varrho^o}_{m+1}, e) &= \lim_{n\to\infty} \acute{y}_n = \varnothing,
\end{aligned}
$$

for all $e \in \mathbb{S}_0$ *which show sequence* $\{\varrho^o_n\}$ *is Cauchy.*

Given that $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *is complete,* 2 *implies the existence of* $\varrho^o \in \mathbb{V}$ *satisfying*

$$
\lim_{n\to\infty} \mathcal{E}(\varrho^o_n, \varrho^o, e) = \Im, \lim_{n\to\infty} \mathcal{G}(\varrho^o_n, \varrho^o_m, e) = \varnothing \text{ and } \lim_{n\to\infty} \mathcal{H}(\varrho^o_n, \varrho^o_m, e) = \varnothing \tag{10}
$$

for any $e \in \mathbb{S}_0$.

As a result of the conditions (5), (10) *and* (15) *of Definition* 7 *and equation* (6), *for any* $e \in \mathbb{S}_0$, *we can conclude that*

$$
\begin{aligned}
\mathcal{E}(\varrho^o, h\varrho^o, e) &\succeq \mathcal{E}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) * \mathcal{E}\left({\varrho^o}_{n+1}, \mathcal{K}\varrho^o, \frac{e}{2}\right) \\
&= \mathcal{E}a\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) * \mathcal{E}\left(\mathcal{K}\varrho^o_n, \mathcal{K}\varrho^o, \frac{e}{2}\right) \\
&\succeq \mathcal{E}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) * \varphi\left(\mathcal{E}\left(\varrho^o_n, \varrho^o, \frac{e}{2}\right)\right) \\
&\succ \mathcal{E}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) * \mathcal{E}\left(\varrho^o_n, \varrho^o, \frac{e}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}(\varrho^o, \mathcal{K}\varrho^o, e) &\preceq \mathcal{G}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{G}\left({\varrho^o}_{n+1}, \mathcal{K}\varrho^o, \frac{e}{2}\right) \\
&= \mathcal{G}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{G}\left(\mathcal{K}\varrho^o_n, \mathcal{K}\varrho^o, \frac{e}{2}\right) \\
&\preceq \mathcal{G}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \sigma\left(\mathcal{G}\left(\varrho^o_n, \varrho^o, \frac{e}{2}\right)\right) \\
&\prec \mathcal{G}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{G}\left(\varrho^o_n, \varrho^o, \frac{e}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}(\varrho^o, \mathcal{K}\varrho^o, e) &\preceq \mathcal{H}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{H}\left({\varrho^o}_{n+1}, h\varrho^o, \frac{e}{2}\right) \\
&= \mathcal{H}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{H}\left(\mathcal{K}\varrho^o_n, \mathcal{K}\varrho^o, \frac{e}{2}\right) \\
&\preceq \mathcal{H}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \sigma\left(\mathcal{H}\left(\varrho^o_n, \varrho^o, \frac{e}{2}\right)\right) \\
&\prec \mathcal{H}\left(\varrho^o, {\varrho^o}_{n+1}, \frac{e}{2}\right) \triangle \mathcal{H}\left(\varrho^o_n, \varrho^o, \frac{e}{2}\right).
\end{aligned}
$$

Considering the preceding inequalities and applying $lim_{n\to\infty}$, by using equation (10), we can conclude that

$$\mathcal{E}(\varrho^o, \mathcal{K}\varrho^o, e) = \Im,\ \mathcal{G}(\varrho^o, \mathcal{K}\varrho^o, e) = \varnothing \text{ and } \mathcal{H}(\varrho^o, \mathcal{K}\varrho^o, e) = \varnothing$$

for any $e \in \mathbb{S}_0$. By applying the conditions (3), (8) and (13) of Definition 7, it can be derived that $\varrho^o = \mathcal{K}\varrho^o$, that is, $\varrho^o$is a fixed point of $\mathcal{K}$.

To prove the uniqueness of fixed point, suppose that z and ϱ^o are two distinct fixed points of $\mathcal{K}$. According to equation (6) for every $c \in \mathbb{S}_0$, it implies

$$\mathcal{E}(\varrho^o, z, e) = \mathcal{E}(\mathcal{K}\varrho^o, \mathcal{K}z, e) \succeq \varphi(\mathcal{E}(\varrho^o, z, e)) \succ \mathcal{E}(\varrho^o, z, e)$$

$$\mathcal{G}(\varrho^o, z, e) = \mathcal{G}(\mathcal{K}\varrho^o, \mathcal{K}z, e) \preceq \sigma(\mathcal{G}(\varrho^o, z, e)) \prec \mathcal{G}(\varrho^o, z, e)$$

and

$$\mathcal{H}(\varrho^o, z, e) = \mathcal{H}(\mathcal{K}\varrho^o, \mathcal{K}z, e) \preceq \sigma(\mathcal{H}(\varrho^o, z, e)) \prec \mathcal{H}(\varrho^o, z, e)$$

It is a contradiction. Consequently, $x = z$, which shows that fixed point is unique.

5. Common Fixed-Point Results

This section investigates several standard fixed-point theorems for two mappings that satisfy the given contraction condition on CVNMSs. It extends the concept of fuzzy Banach contraction to these spaces.

Definition 10. *Let $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ be a CVNMS. A neutrosophic Banach contraction is defined as a pair of two mappings, $\mathcal{I}$ and $\mathcal{J}$, both mapping from $\mathbb{V}$ to $\mathbb{V}$, such that there exists a real number k in the interval $(0, 1)$ where*

$$\begin{aligned} \Im - \mathcal{E}(\mathcal{I}\varrho^o, \mathcal{J}\nu, e) &\preceq k(\Im - \mathcal{E}(\mathcal{I}\varrho^o, \mathcal{J}\nu, e), \\ \mathcal{G}(\mathcal{I}\varrho^o, \mathcal{J}\nu, e) &\preceq k\mathcal{G}(\mathcal{I}\varrho^o, \mathcal{J}\nu, e), \\ \mathcal{H}(\mathcal{I}\varrho^o, \mathcal{J}\nu, e) &\preceq k\mathcal{H}(\mathcal{I}\varrho^o, \mathcal{J}\nu, e) \end{aligned} \tag{11}$$

holds for any $\varrho^o, \nu \in \mathbb{V}$ and $e \in \mathbb{S}_0$.

Theorem 3. *Suppose that $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ is a complete CVNMS and a pair of two mappings $\mathcal{I}, \mathcal{J} : \mathbb{V} \to \mathbb{V}$ is a neutrosophic Banach contraction. Then both the mappings $\mathcal{I}$ and $\mathcal{J}$ have a unique shared fixed point that belongs to $\mathbb{V}$.*

Proof. *Take an arbitrarily selected point $\varrho^o \in \mathbb{V}$. For any $n \in \mathbb{M}_0$, a sequence $\{\varrho^o_n\}$ is defined in $\mathbb{V}$ as*

$$\begin{aligned} {\varrho^o}_{2n+1} &= \mathcal{I}{\varrho^o}_{2n}, \\ {\varrho^o}_{2n+2} &= \mathcal{J}{\varrho^o}_{2n+1}. \end{aligned}$$

It is guaranteed that ${\varrho^o}_{n_0}$ is a common fixed point of $\mathcal{I}$, if there is a $n_0 \in \mathbb{M}$ such that ${\varrho^o}_{n_0} = {\varrho^o}_{n_0+1}$. Using the equation (11), we also have

$$\Im - \mathcal{E}({\varrho^o}_{2n+1}, {\varrho^o}_{2n+2}, e) = \Im - \mathcal{E}(\mathcal{I}{\varrho^o}_{2n}, \mathcal{J}{\varrho^o}_{2n+1}, e)$$

$$\begin{aligned}
&\preceq k(\Im - \mathcal{E}({\varrho^o}_{2n}, {\varrho^o}_{2n+1}, e)) \\
&= k(\Im - \Im) \\
&= \varnothing
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}({\varrho^o}_{2n+1}, {\varrho^o}_{2n+2}, e) &= \mathcal{G}(\mathcal{I}{\varrho^o}_{2n}, \mathcal{J}{\varrho^o}_{2n+1}, e) \\
&\preceq k\mathcal{G}({\varrho^o}_{2n}, {\varrho^o}_{2n+1}, e) \\
&= k(\varnothing) \\
&= \varnothing
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}({\varrho^o}_{2n+1}, {\varrho^o}_{2n+2}, e) &= \mathcal{H}(\mathcal{I}{\varrho^o}_{2n}, \mathcal{J}{\varrho^o}_{2n+1}, e) \\
&\preceq k\mathcal{H}({\varrho^o}_{2n}, {\varrho^o}_{2n+1}, e) \\
&= k(\varnothing) \\
&= \varnothing
\end{aligned}$$

for each $e \in \mathbb{S}_0$. *Consequently,* $\mathcal{E}({\varrho^o}_{2n+1}, {\varrho^o}_{2n+2}, e) = \Im, \mathcal{G}({\varrho^o}_{2n+1}, {\varrho^o}_{2n+2}, e) = \varnothing$ *and* $\mathcal{H}({\varrho^o}_{2n+1}, {\varrho^o}_{2n+2}, e) = \varnothing$. *By conditions* (3), (8) *and* (13) *of Definition* 7, ${\varrho^o}_{2n+1} = {\varrho^o}_{2n+2} = \mathcal{J}{\varrho^o}_{2n+1}$, *this shows that* ${\varrho^o}_{2n+1}$ *is a fixed point of* $\mathcal{J}$. *We can infer that* ${\varrho^o}_{2n}$ *is a common fixed point of* $\mathcal{I}$ *and* $\mathcal{J}$ *because* ${\varrho^o}_{2n} = {\varrho^o}_{2n+1}$. *Similarly, if there exists an element* $n \in \mathbb{M}_0$ *such that* ${\varrho^o}_{2n+1} = {\varrho^o}_{2n+2}$, *we can demonstrate using equation* (11) *that* ${\varrho^o}_{2n+1}$ *is a shared fixed point of* $\mathcal{I}$ *and* $\mathcal{J}$.

Suppose that $\varrho^o_n \neq {\varrho^o}_{n+1}$ *for every* $n \in \mathbb{M}_0$. *There are two scenarios that we will examine. Suppose that* n *is an odd number in the first case. By substituting* $\varrho^o = {\varrho^o}_{n-1}$ *and* $\nu = \varrho^o_n$ *into equation* (11), *for every* $e \in \mathbb{S}_0$, *we obtain*

$$\begin{aligned}
\Im - \mathcal{E}(\varrho^o_n, {\varrho^o}_{n+1}, e) &= \Im - \mathcal{E}(\mathcal{I}{\varrho^o}_{n-1}, \mathcal{J}\varrho^o_n, e) \\
&\preceq k(\Im - \mathcal{E}({\varrho^o}_{n-1}, \varrho^o_n, e)) \\
&\prec \Im - \mathcal{E}({\varrho^o}_{n-1}, \varrho^o_n, e),
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}(\varrho^o_n, {\varrho^o}_{n+1}, e) &= \mathcal{G}(\mathcal{I}{\varrho^o}_{n-1}, \mathcal{J}\varrho^o_n, e) \\
&\preceq k\mathcal{G}({\varrho^o}_{n-1}, \varrho^o_n, e) \\
&\prec \mathcal{G}({\varrho^o}_{n-1}, \varrho^o_n, e)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}(\varrho^o_n, {\varrho^o}_{n+1}, e) &= \mathcal{H}(\mathcal{I}{\varrho^o}_{n-1}, \mathcal{J}\varrho^o_n, e) \\
&\preceq k\mathcal{H}({\varrho^o}_{n-1}, \varrho^o_n, e) \\
&\prec \mathcal{H}({\varrho^o}_{n-1}, \varrho^o_n, e).
\end{aligned}$$

It follows that

$$\mathcal{E}(\varrho^o_n, {\varrho^o}_{n+1}, e) \succ \mathcal{E}({\varrho^o}_{n-1}, \varrho^o_n, e)$$
$$\mathcal{G}(\varrho^o_n, {\varrho^o}_{n+1}, e) \prec \mathcal{G}({\varrho^o}_{n-1}, \varrho^o_n, e)$$

and

$$\mathcal{H}(\varrho^o_n, {\varrho^o}_{n+1}, e) \prec \mathcal{H}({\varrho^o}_{n-1}, \varrho^o_n, e)$$

for any $e \in \mathbb{S}$. For the second case, let us assume that n is an even number. By substituting $\varrho^o = \varrho^o_n$ and $\nu = {\varrho^o}_{n-1}$ into equation (11), for every $e \in \mathbb{S}_0$, we obtain

$$\begin{aligned}\Im - \mathcal{E}({\varrho^o}_{n+1}, \varrho^o_n, e) &= \Im - \mathcal{E}(\mathcal{I}\varrho^o_n, \mathcal{J}{\varrho^o}_{n-1}, e)\\ &\preceq k(\Im - \mathcal{E}(\varrho^o_n, {\varrho^o}_{n-1}, e))\\ &\prec \Im - \mathcal{E}(\varrho^o_n, {\varrho^o}_{n-1}, e),\end{aligned}$$

$$\begin{aligned}\mathcal{G}({\varrho^o}_{n+1}, \varrho^o_n, e) &= \mathcal{G}(\mathcal{I}\varrho^o_n, \mathcal{J}{\varrho^o}_{n-1}, e)\\ &\preceq k\mathcal{G}(\varrho^o_n, {\varrho^o}_{n-1}, e)\\ &\prec \mathcal{G}(\varrho^o_n, {\varrho^o}_{n-1}, e)\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}({\varrho^o}_{n+1}, \varrho^o_n, e) &= \mathcal{H}(\mathcal{I}\varrho^o_n, \mathcal{J}{\varrho^o}_{n-1}, e)\\ &\preceq k\mathcal{H}(\varrho^o_n, {\varrho^o}_{n-1}, e)\\ &\prec \mathcal{H}(\varrho^o_n, {\varrho^o}_{n-1}, e).\end{aligned}$$

It follows that

$$\mathcal{E}(\varrho^o_n, {\varrho^o}_{n+1}, e) \succ \mathcal{E}({\varrho^o}_{n-1}, \varrho^o_n, e)$$
$$\mathcal{G}(\varrho^o_n, {\varrho^o}_{n+1}, e) \prec \mathcal{G}(\varrho^o_n, {\varrho^o}_{n-1}, e)$$

and

$$\mathcal{H}(\varrho^o_n, {\varrho^o}_{n+1}, e) \prec \mathcal{H}(\varrho^o_n, {\varrho^o}_{n-1}, e)$$

for any $e \in \mathbb{S}$. Therefore, we conclude that

$$\mathcal{E}(\varrho^o_n, {\varrho^o}_{n+1}, e) \succ \mathcal{E}({\varrho^o}_{n-1}, \varrho^o_n, e),\ \ \mathcal{G}(\varrho^o_n, {\varrho^o}_{n+1}, e) \prec \mathcal{G}(\varrho^o_n, {\varrho^o}_{n-1}, e),\ \ \mathcal{H}(\varrho^o_n, {\varrho^o}_{n+1}, e) \prec \mathcal{H}(\varrho^o_n, {\varrho^o}_{n-1}, e)$$

for every $n \in \mathbb{M}_0$ and $e \in \mathbb{S}_0$. Denote $\mathcal{E}(\varrho^o_n, {\varrho^o}_{n+1}, e) = \mathbb{A}_n, \mathcal{G}(\varrho^o_n, {\varrho^o}_{n+1}, e) = \mathbb{B}_n$ and $\mathcal{H}(\varrho^o_n, {\varrho^o}_{n+1}, e) = \mathbb{C}_n$ for each $n \in \mathbb{M}_0$. Since

$$\Im \succeq \mathbb{A}_n \succ \mathbb{A}_{n-1} \succ \varnothing$$
$$\varnothing \preceq \mathbb{B}_n \prec \mathbb{B}_{n-1} \prec \Im$$
$$\varnothing \preceq \mathbb{C}_n \prec \mathbb{C}_{n-1} \prec \Im$$

for every $n \in \mathbb{M}_0$*, It concludes that sequences* $\{\mathbb{A}_n\}, \{\mathbb{B}_n\}$ *and* $\{\mathbb{C}_n\}$ *are monotonic in* $\mathbb{S}$*. By Remarks* 1*, one is possible to locate* $\acute{x}, \acute{y}, \acute{c} \in \mathbb{S}$ *satisfying*

$$\lim_{n\to\infty} \mathbb{A}_n = \acute{x},\ \lim_{n\to\infty} \mathbb{B}_n = \acute{y},\ \lim_{n\to\infty} \mathbb{C}_n = \acute{c}.$$

By using equation equation (11)*, for* $n \in \mathbb{M}_0$ *and* $c \in \mathbb{S}_0$ *we obtain*

$$\begin{gathered}\Im - \mathcal{E}(\varrho^o_n, {\varrho^o}_{n+1}, e) \preceq k(\Im - \mathcal{E}({\varrho^o}_{n-1}, \varrho^o_n, e))\\ \Im - \mathbb{A}_n \preceq k(\Im - \mathbb{A}_{n-1})\end{gathered}$$

$$\begin{gathered}\mathcal{G}(\varrho^o_n, {\varrho^o}_{n+1}, e) \preceq k\mathcal{G}({\varrho^o}_{n-1}, \varrho^o_n, e)\\ \mathbb{B}_n \preceq k(\mathbb{B}_{n-1})\end{gathered}$$

and

$$\begin{gathered}\mathcal{H}(\varrho^o_n, {\varrho^o}_{n+1}, e) \preceq k\mathcal{H}({\varrho^o}_{n-1}, \varrho^o_n, e)\\ \mathbb{C}_n \preceq k(\mathbb{C}_{n-1}).\end{gathered}$$

As the value of n *approaches infinity for inequalities, we get*

$$\Im - \acute{x} \preceq k(\Im - \acute{x})$$

$$\acute{y} \preceq k\acute{y}$$

and

$$\acute{c} \preceq k\acute{c}.$$

As $k \in (0,1)$*, if* $\acute{x} \prec \Im, \acute{y} \succ \varnothing$*, and* $\acute{c} \succ \varnothing$*, it is a contradiction. Therefore* $\acute{x} = \Im, \acute{y} = \varnothing$ *and* $\acute{c} = \varnothing$ *it indicates that*

$$\begin{gathered}\lim_{n\to\infty} \mathcal{E}(\varrho^o_n, {\varrho^o}_{n+1}, e) = \Im,\\ \lim_{n\to\infty} \mathcal{G}(\varrho^o_n, {\varrho^o}_{n+1}, e) = \varnothing,\\ \lim_{n\to\infty} \mathcal{H}(\varrho^o_n, {\varrho^o}_{n+1}, e) = \varnothing\end{gathered}$$

for every $n \in \mathbb{M}_0$ *and* $e \in \mathbb{S}_0$*.*

Now we will show that the $\{\varrho^o_n\}$ *is a Cauchy sequence. For every* $n \in \mathbb{M}_0$ *as well as fixed* $e \in \mathbb{S}_0$*, consider*

$$\begin{gathered}\mathbb{D}_n = \{\mathcal{E}(\varrho^o_n, \varrho^o_m, e) : m > n\} \subseteq \mathbb{T},\\ \mathbb{E}_n = \{\mathcal{G}(\varrho^o_n, \varrho^o_m, e) : m > n\} \subseteq \mathbb{T},\\ \mathbb{F}_n = \{\mathcal{H}(\varrho^o_n, \varrho^o_m, e) : m > n\} \subseteq \mathbb{T}.\end{gathered}$$

Since $\varnothing \prec \mathcal{E}(\varrho^o_n, \varrho^o_m, e) \preceq \Im, \varnothing \preceq \mathcal{G}(\varrho^o_m, \varrho^o_n, e) \prec \Im$ *and* $\varnothing \preceq \mathcal{H}(\varrho^o_m, \varrho^o_n, e) \prec \Im$*, by Remarks* 1*, The infimum of the CFS* $\mathcal{E}(\varrho^o_n, \varrho^o_m, e)$*, the supremum of the CFS* $\mathcal{G}(\varrho^o_n, \varrho^o_m, e)$*, and the* *supremum of the CFS $\mathcal{H}(\varrho_n^o, \varrho_m^o, e)$ all exist. By iteratively applying the condition (5) of Definition 7 for any positive integers $m > n$, we can obtain*

$$\mathcal{E}(\varrho_n^o, \varrho_m^o, e) \succeq \mathcal{E}\left(\varrho_n^o, {\varrho^o}_{n+1}, \frac{e}{m-n}\right) \star \mathcal{E}\left({\varrho^o}_{n+1}, {\varrho^o}_{n+2}, \frac{e}{m-n}\right) \star ... \star \mathcal{E}\left({\varrho^o}_{m-1}, \varrho_m^o, \frac{e}{m-n}\right).$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty} \inf_{m>n} \mathcal{E}(\varrho_n^o, \varrho_m^o, e) &\succeq \Im \star \Im \star ... \star \Im \\ &= \Im\end{aligned}$$

which leads to

$$\lim_{n\to\infty} \inf_{m>n} \mathcal{E}(\varrho_n^o, \varrho_m^o, e) = \Im$$

for every $e \in \mathbb{S}_0$. Furthermore, By iteratively applying the condition (10) of Definition 7 for any positive integers $m > n$, we can obtain

$$\mathcal{G}(\varrho_n^o, \varrho_m^o, e) \preceq \mathcal{G}\left(\varrho_n^o, {\varrho^o}_{n+1}, \frac{c}{m-n}\right) \triangle \mathcal{G}\left({\varrho^o}_{n+1}, {\varrho^o}_{n+2}, \frac{e}{m-n}\right) \triangle ... \triangle \mathcal{G}\left({\varrho^o}_{m-1}, \varrho_m^o, \frac{e}{m-n}\right).$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty} \sup_{m>n} \mathcal{G}(\varrho_n^o, \varrho_m^o, e) &\preceq \varnothing \triangle \varnothing \triangle ... \triangle \varnothing \\ &= \varnothing\end{aligned}$$

which leads to

$$\lim_{n\to\infty} \sup_{m>n} \mathcal{G}(\varrho_n^o, \varrho_m^o, e) = \varnothing$$

for every $e \in \mathbb{S}_0$. Also By iteratively applying the condition (15) of Definition 7 for any positive integers $m > n$, we can obtain

$$\mathcal{H}(\varrho_n^o, \varrho_m^o, e) \preceq \mathcal{H}\left(\varrho_n^o, {\varrho^o}_{n+1}, \frac{e}{m-n}\right) \triangle \mathcal{H}\left({\varrho^o}_{n+1}, {\varrho^o}_{n+2}, \frac{e}{m-n}\right) \triangle ... \triangle \mathcal{H}\left({\varrho^o}_{m-1}, \varrho_m^o, \frac{e}{m-n}\right).$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty} \sup_{m>n} \mathcal{H}(\varrho_n^o, \varrho_m^o, e) &\preceq \varnothing \triangle \varnothing \triangle ... \triangle \varnothing \\ &= \varnothing\end{aligned}$$

which leads to

$$\lim_{n\to\infty} \sup_{m>n} \mathcal{H}(\varrho_n^o, \varrho_m^o, e) = \varnothing$$

for every $e \in \mathbb{S}_0$. Hence, sequence $\{\varrho_n^o\}$ is Cauchy.

Given that $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *is complete, Lemma* 2 *implies the existence of* $u \in \mathbb{V}$ *satisfying*

$$\lim_{n\to\infty} \mathcal{E}(\varrho_n^o, u, e) = \Im, \ \lim_{n\to\infty} \mathcal{G}(\varrho_n^o, u, e) = \varnothing \text{ and } \lim_{n\to\infty} \mathcal{H}(\varrho_n^o, u, e) = \varnothing$$

for all $e \in \mathbb{S}_0$. *For any* $n \in \mathbb{M}_0$ *and* $e \in \mathbb{S}_0$, *by equation* (11) *we yield*

$$\begin{aligned} \Im - \mathcal{E}(Fu, G{\varrho^o}_{2n+1}, e) &\preceq k(\Im - \mathcal{E}(u, {\varrho^o}_{2n+1}, e)) \\ &\prec \Im - \mathcal{E}(u, {\varrho^o}_{2n+1}, e) \end{aligned}$$

$$\begin{aligned} \mathcal{G}(\mathcal{I}u, \mathcal{J}{\varrho^o}_{2n+1}, e) &\preceq k\mathcal{G}(u, {\varrho^o}_{2n+1}, e) \\ &\prec \mathcal{G}(u, {\varrho^o}_{2n+1}, e) \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}(\mathcal{I}u, \mathcal{J}{\varrho^o}_{2n+1}, e) &\preceq k\mathcal{H}(u, {\varrho^o}_{2n+1}, e) \\ &\prec \mathcal{H}(u, {\varrho^o}_{2n+1}, e). \end{aligned}$$

This implies that

$$\mathcal{E}(\mathcal{I}u, \mathcal{J}{\varrho^o}_{2n+1}, e) \succ \mathcal{E}(u, {\varrho^o}_{2n+1}, e) \tag{12}$$

$$\mathcal{G}(\mathcal{I}u, \mathcal{J}{\varrho^o}_{2n+1}, e) \prec \mathcal{G}(u, {\varrho^o}_{2n+1}, e) \tag{13}$$

and

$$\mathcal{H}(\mathcal{I}u, \mathcal{J}{\varrho^o}_{2n+1}, e) \prec \mathcal{H}(u, {\varrho^o}_{2n+1}, e) \tag{14}$$

for each $n \in \mathbb{M}_0$ *and* $e \in \mathbb{S}_0$. *As a result of conditions* (5), (10) *and* (15) *of Definition* 7, *equations* (12), (13) *and* (14), *for each* $n \in \mathbb{M}_0$ *and* $e \in \mathbb{S}_0$, *we may deduce that*

$$\begin{aligned} \mathcal{E}(u, \mathcal{I}u, e) &\succeq \mathcal{E}\left(u, {\varrho^o}_{2n+2}, \frac{e}{2}\right) \star \mathcal{E}\left({\varrho^o}_{2n+2}, \mathcal{I}u, \frac{e}{2}\right) \\ &= \mathcal{E}\left(u, {\varrho^o}_{2n+2}, \frac{e}{2}\right) \star \mathcal{E}\left(\mathcal{J}{\varrho^o}_{2n+1}, \mathcal{I}u, \frac{e}{2}\right) \\ &= \mathcal{E}\left(u, {\varrho^o}_{2n+2}, \frac{e}{2}\right) \star \mathcal{E}\left(\mathcal{I}u, \mathcal{J}{\varrho^o}_{2n+1}, \frac{e}{2}\right) \\ &\succeq \mathcal{E}\left(u, {\varrho^o}_{2n+2}, \frac{e}{2}\right) \star \mathcal{E}\left(u, {\varrho^o}_{2n+1}, \frac{e}{2}\right) \end{aligned}$$

$$\begin{aligned} \mathcal{G}(u, \mathcal{I}u, e) &\preceq \mathcal{G}\left(u, {\varrho^o}_{2n+2}, \frac{e}{2}\right) \triangle \mathcal{G}\left({\varrho^o}_{2n+2}, \mathcal{I}u, \frac{e}{2}\right) \\ &= \mathcal{G}\left(u, {\varrho^o}_{2n+2}, \frac{e}{2}\right) \triangle \mathcal{G}\left(\mathcal{J}{\varrho^o}_{2n+1}, \mathcal{I}u, \frac{e}{2}\right) \\ &= \mathcal{G}\left(u, {\varrho^o}_{2n+2}, \frac{e}{2}\right) \triangle \mathcal{G}\left(\mathcal{I}u, \mathcal{J}{\varrho^o}_{2n+1}, \frac{e}{2}\right) \\ &\preceq \mathcal{G}\left(u, {\varrho^o}_{2n+2}, \frac{e}{2}\right) \triangle \mathcal{G}\left(u, {\varrho^o}_{2n+1}, \frac{e}{2}\right) \end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(u,\mathcal{I}u,e) &\preceq \mathcal{H}\left(u,{\varrho^o}_{2n+2},\frac{e}{2}\right) \triangle\, \mathcal{H}\left({\varrho^o}_{2n+2},\mathcal{I}u,\frac{e}{2}\right)\\ &= \mathcal{H}\left(u,{\varrho^o}_{2n+2},\frac{e}{2}\right) \triangle\, \mathcal{H}\left(\mathcal{J}{\varrho^o}_{2n+1},\mathcal{I}u,\frac{e}{2}\right)\\ &= \mathcal{H}\left(u,{\varrho^o}_{2n+2},\frac{e}{2}\right) \triangle\, \mathcal{H}\left(\mathcal{I}u,\mathcal{J}{\varrho^o}_{2n+1},\frac{e}{2}\right)\\ &\preceq \mathcal{H}\left(u,{\varrho^o}_{2n+2},\frac{e}{2}\right) \triangle\, \mathcal{H}\left(u,{\varrho^o}_{2n+1},\frac{e}{2}\right)\end{aligned}$$

As the value of n approaches infinity for inequalities, we get

$$\mathcal{E}(u,\mathcal{I}u,e)=\Im,\ \mathcal{G}(u,\mathcal{I}u,e)=\varnothing \text{ and } \mathcal{H}(u,\mathcal{I}u,e)=\varnothing$$

for every $e\in\mathbb{S}_0$. Under conditions (3), (8) and (13) of Definition 7, it indicates that u is equal to $\mathcal{I}u$. By applying the same methods as previously, one can establish that

$$\mathcal{E}(u,\mathcal{J}u,e)=\Im,\ \mathcal{G}(u,\mathcal{J}u,e)=\varnothing \text{ and } \mathcal{H}(u,\mathcal{J}u,e)=\varnothing$$

for every $e\in\mathbb{S}_0$. By the conditions (3), (8) and (13) of Definition 7, imply that u is equal to $\mathcal{J}u$. Consequently, it follows that $u=\mathcal{I}u=\mathcal{J}u$ which shows that u is a is common fixed point of both functions $\mathcal{I}$ and $\mathcal{J}$.

In order to prove uniqueness, assume that u and v are two distinct fixed points of $\mathcal{K}$, It is possible to locate $e\in\mathbb{S}_0$ satisfying $\mathcal{E}(u,v,e)\neq\Im, \mathcal{G}(u,v,e)\neq\varnothing$ and $\mathcal{H}(u,v,e)\neq\varnothing$ By equation (11),

$$\begin{aligned}\Im-\mathcal{E}(u,v,e) &= \Im-\mathcal{E}(\mathcal{I}u,\mathcal{J}v,e)\\ &\preceq k(\Im-\mathcal{E}(u,v,e))\\ &\prec \Im-\mathcal{E}(u,v,e)\end{aligned}$$

$$\begin{aligned}\mathcal{G}(u,v,e) &= \mathcal{G}(\mathcal{I}u,\mathcal{J}v,e)\\ &\preceq k(\mathcal{G}(u,v,e))\\ &\prec \mathcal{G}(u,v,e)\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(u,v,e) &= \mathcal{H}(\mathcal{I}u,\mathcal{J}v,e)\\ &\preceq k(\mathcal{H}(u,v,e))\\ &\prec \mathcal{H}(u,v,e)\end{aligned}$$

which contradicts our assumption. Thus $\mathcal{E}(u,v,e)=\Im, \mathcal{G}(u,v,e)=\varnothing$ and $\mathcal{H}(u,v,e)=\varnothing$ for all $e\in\mathbb{S}_0$. By the conditions (3), (8) and (13) of Definition 7, we may deduce that u is equal to v which show that the $\mathcal{K}$ has a unique fixed point.

Corollary 1. *Suppose that* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *is a complete CVNMS. A mapping* $\mathcal{I} : \mathbb{V} \to \mathbb{V}$ *satisfying*

$$\begin{aligned} \Im - \mathcal{E}(\mathcal{I}\varrho^o, \mathcal{I}\nu, e) &\preceq k(\Im - \mathcal{E}(\varrho^o, \nu, e)), \\ \mathcal{G}(\mathcal{I}\varrho^o, \mathcal{I}\nu, e) &\preceq k\mathcal{G}(\varrho^o, \nu, e), \\ \mathcal{H}(\mathcal{I}\varrho^o, \mathcal{I}\nu, e) &\preceq k\mathcal{H}(\varrho^o, \nu, e) \end{aligned}$$

for each $\varrho^o, \nu \in \mathbb{V}$ *and* $e \in \mathbb{S}_0$*, where* $k \in (0, 1)$*. In that case, the mapping* $\mathcal{I}$ *has a single fixed point in* $\mathbb{V}$*.*
Proof. The desired outcome may be obtained by replacing $\mathcal{I} = \mathcal{J}$ *into Theorem* 3.

An example of the idea presented in Corollary 1 is shown below.

Example 8. *Consider* $\mathbb{V} = [0, 1]$*. Define two binary operations* $\star$ *and* $\triangle$ *by* $e_1 \star e_2 = (\tau_1\tau_2, \xi_1\xi_2)$ *and* $e_1 \triangle e_2 = (\max(\tau_1, \tau_2), \max(\xi_1, \xi_2))$ *for any* $e_i = (\tau_i, \xi_i) \in \mathbb{T}$ *where i=1,2. Let CFSs* $\mathcal{E}, \mathcal{G}$ *and* $\mathcal{H}$ *be defined as follows:*

$$\mathcal{E}(\varrho^o, \nu, e) = \left(\frac{\tau\xi + \min\{\varrho^o, \nu\}}{\tau\xi + \max(\varrho^o, \nu)}\right)\Im, \quad \mathcal{G}(\varrho^o, \nu, e) = 1 - \left(\frac{\tau\xi + \min\{\varrho^o, \nu\}}{\tau\xi + \max(\varrho^o, \nu)}\right)\Im,$$

$$\mathcal{H}(\varrho^o, \nu, e) = \left(\frac{\max\{\varrho^o, \nu\} - \min\{\varrho^o, \nu\}}{\tau\xi + \max\{\varrho^o, \nu\}}\right)\Im$$

for all $\varrho^o, \nu \in \mathbb{V}$ *and* $e = (\tau, \xi) \in \mathbb{S}_0$*. It is not difficult to show that* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *is a complete CVNMS.*

Consider a mapping $\mathcal{I} : \mathbb{V} \to \mathbb{V}$ *expressed by* $\frac{\varrho^o}{2}$ *for all* $\varrho^o \in \mathbb{V}$*. For any* $\varrho^o, \nu \in \mathbb{V}$ *satisfying* $\varrho^o \leq \nu$*, it is clear that* $\mathcal{I}\varrho^o \leq \mathcal{I}\nu$*. It follows that*

$$\begin{aligned} \mathcal{E}(\mathcal{I}\varrho^o, \mathcal{I}\nu, e) &= \left(\frac{\tau\xi + \min\{\mathcal{I}\varrho^o, \mathcal{I}\nu\}}{\tau\xi + \max(\mathcal{I}\varrho^o, \mathcal{I}\nu)}\right)\Im, \\ &= \left(\frac{\tau\xi + \mathcal{I}\varrho^o}{\tau\xi + \mathcal{I}\nu}\right)\Im, \\ &\succeq \left(\frac{\tau\xi + \varrho^o}{\tau\xi + \nu}\right)\Im, \\ &= \mathcal{E}(\varrho^o, \nu, e). \end{aligned}$$

If we choose any $k \in (\frac{1}{2}, 1)$*, we have*

$$\Im - \mathcal{E}(\mathcal{I}\varrho^o, \mathcal{I}\nu, e) \preceq k(\Im - \mathcal{E}(\varrho^o, \nu, e))$$

for every $\varrho^o, \nu \in \mathbb{V}$ *and* $e = (\tau, \xi) \in \mathbb{S}_0$*. Similarly, we can deduce that*

$$\mathcal{G}(\mathcal{I}\varrho^o, \mathcal{I}\nu, e) \preceq k\mathcal{G}(\varrho^o, \nu, e)$$

and

$$\mathcal{H}(\mathcal{I}\varrho^o, \mathcal{I}\nu, e) \preceq k\mathcal{H}(\varrho^o, \nu, e)$$

for all $\varrho^o, \nu \in \mathbb{V}$ *and* $e = (\tau, \xi) \in \mathbb{S}_0$*. Thus, all the requirements stated in Corollary* 1 *are satisfied. Specifically, the number 0 is the only fixed point of* $\mathcal{I}$*.*

Theorem 4. *Let* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *be a CVNMS. If there is a commuting pair of self-mappings,* $\mathcal{I}, \mathcal{J} : \mathbb{V} \to \mathbb{V}$ *satisfying*

$$\begin{aligned} \Im - \mathcal{E}(\mathcal{I}^n \varrho^o, \mathcal{J}^n \nu, e) &\preceq k(\Im - \mathcal{E}(\varrho^o, \nu, e)), \\ \mathcal{G}(\mathcal{I}^n \varrho^o, \mathcal{J}^n \nu, e) &\preceq k\mathcal{G}(\varrho^o, \nu, e), \\ \mathcal{H}(\mathcal{I}^n \varrho^o, \mathcal{J}^n \nu, e) &\preceq k\mathcal{H}(\varrho^o, \nu, e) \end{aligned}$$

for every $\varrho^o, \nu \in \mathbb{V}, e \in \mathbb{S}_0$ *and* $n \in \mathbb{M}$*, where* k *is any real number from* $(0, 1)$*. Then there is a single shared fixed point of mappings* $\mathcal{I}$ *and* $\mathcal{J}$ *inside* $\mathbb{V}$.

Proof. Both $\mathcal{I}^n$ *and* $\mathcal{J}^n$ *satisfy all the conditions stated in Theorem* 3*. Thus, they have a shared fixed point* μ *in* $\mathbb{V}$*, for instance,* $\mathcal{I}^n\mu = \mathcal{J}^n\mu = \mu$*. According to the provided information*

$$\mathcal{I}^n\mathcal{I}\mu = \mathcal{I}\mathcal{I}^n\mu = \mathcal{I}\mu,$$

it may be deduced that $\mathcal{I}\mu$ *is a point fixed by* $\mathcal{I}^n$*. Since both the mappings* $\mathcal{I}$ *and* $\mathcal{J}$ *commute, We may express the equation as follows:*

$$\mathcal{J}^n\mathcal{I}\mu = \mathcal{I}\mathcal{J}^n\mu = \mathcal{I}\mu$$

this demonstrates that $\mathcal{I}\mu$ *is a point fixed by* $\mathcal{J}^n$*. Thus,* $\mathcal{I}\mu$ *acts as a common fixed point of* $\mathcal{I}^n$ *and* $\mathcal{J}^n$*.*

Similarly, According to the provided information

$$\mathcal{J}^n\mathcal{J}\mu = \mathcal{J}\mathcal{J}^n\mu = \mathcal{J}\mu,$$

it may be deduced that $\mathcal{J}\mu$ *is a point fixed by* $\mathcal{J}^n$*. Since both the mappings* $\mathcal{I}$ *and* $\mathcal{J}$ *commute,We may express the equation as follows:*

$$\mathcal{I}^n\mathcal{J}\mu = \mathcal{J}\mathcal{I}^n\mu = \mathcal{J}\mu$$

this demonstrates that $\mathcal{J}\mu$ *is a point fixed by* $\mathcal{I}^n$*. Thus,* $\mathcal{J}\mu$ *acts as a common fixed point of* $\mathcal{I}^n$ *and* $\mathcal{J}n$*.*

Given that the common fixed point of $\mathcal{I}^n$ *and* $\mathcal{J}^n$ *is unique, it follows that* $\mu = \mathcal{J}\mu = \mathcal{I}\mu$*. Consequently,* μ *as the common fixed point for both* $\mathcal{I}$ *and* $\mathcal{J}$*. If* $\mathcal{I}$ *and* $\mathcal{J}$ *have any common fixed point, that point will also be a fixed point of* $\mathcal{I}^n$ *and* $\mathcal{J}^n$*. The common fixed point of* $\mathcal{I}$ *and* $\mathcal{J}$ *is uniquely defined for this purpose.*

Corollary 2. *Let* $(\mathbb{V}, \mathcal{E}, \mathcal{G}, \mathcal{H}, \star, \triangle)$ *be a CVNMS. If there is a mapping* $\mathcal{I} : \mathbb{V} \to \mathbb{V}$ *satisfying*

$$\begin{aligned} \Im - \mathcal{E}(\mathcal{I}^n \varrho^o, \mathcal{I}^n \nu, e) &\preceq k(\Im - \mathcal{E}(\varrho^o, \nu, e)), \\ \mathcal{G}(\mathcal{I}^n \varrho^o, \mathcal{I}^n \nu, e) &\preceq k\mathcal{G}(\varrho^o, \nu, e), \\ \mathcal{H}(\mathcal{I}^n \varrho^o, \mathcal{I}^n \nu, e) &\preceq k\mathcal{H}(\varrho^o, \nu, e) \end{aligned}$$

for each $\varrho^o, \nu \in \mathbb{V}, e \in \mathbb{S}_0$ *and* $n \in \mathbb{M}$*, where* $0 < k < 1$*. Then mapping* $\mathcal{I}$ *has a single common fixed point within* $\mathbb{V}$*.*

Proof. The desired outcome may be obtained by replacing $\mathcal{I} = \mathcal{J}$ *in Theorem* 4*.*

6. Application to Fredholm integral equations of the second kind

In this section, we use Theorem 1 to show that Fredholm integral equations have a unique solution. The collection of all continuous functions mapping the interval $[0,1]$ to R is represented by the set $C([0,1],\mathbb{R})$. An illustration of a second-kind nonlinear Fredholm integral equation is given below:

$$\varphi(q) = Q(q) + \acute{c}\int_0^1 \acute{w}(q,r)\varpi(r,\varphi(r))dr \tag{15}$$

where Q denotes is a continuous real-valued function on $[0,1], \acute{w}(q,r)$ denotes the kernel of the integral function,$\varpi(r,\varphi(r))$ denotes nonlinear and continuous function defined on $[0,1]\times\mathbb{R}$ and $\varphi(q)$ symbolizes the function we want to be identified.

Theorem 5. *Assume that the set $\mathbb{V} = C([0,1]\times\mathbb{R})$. Assume that the following circumstances are fulfilled:*

(1) A member $\acute{x} \in (0,1)$ can be identified in the following:

$$|\varpi(r,\varphi(r)) - \varpi(r,\sigma(r))| \le \acute{x}|\varphi(r) - \sigma(r)|$$

for any $\varphi,\sigma \in \mathbb{V}$ and $r \in [0,1]$;

(2) $\int_0^1 \acute{w}(q,r)dr \le \acute{y}$;

(3) $\acute{c}^2\acute{y}^2\acute{x}^2 \le k < 1$.

As a result, the integral equation (15) possesses a unique solution inside the set $\mathbb{V}$.
Proof. Consider a mapping $\mathcal{I} : \mathbb{V} \to \mathbb{V}$ defined as

$$\mathcal{I}\varphi(q) = Q(q) + \acute{c}\int_0^1 \acute{w}(q,r)\varpi(r,\varphi(r))dr$$

for each $\varphi(q) \in \mathbb{V}$ and $q \in [0,1]$. The complex-valued t-norm is defined as $\star_x$, whereas the complex-valued t-conorm is defined as $\triangle_x$. Moreover, $\mathcal{E}(\varrho^o,\nu,e), \mathcal{G}(\varrho^o,\nu,e)$ and $\mathcal{H}(\varrho^o,\nu,e)$ defined by

$$\begin{aligned}
\mathcal{E}(\varphi(q),\sigma(q),e) =& \frac{\tau+\xi}{\tau+\xi+|\varphi(q)-\sigma(q)|^2}\Im,\\
\mathcal{G}(\varphi(q),\sigma(q),e) =& \frac{|\varphi(q)-\sigma(q)|^2}{\tau+\xi+|\varphi(q)-\sigma(q)|^2}\Im,\\
\mathcal{H}(\varphi(q),\sigma(q),e) =& \frac{|\varphi(q)-\sigma(q)|^2}{\tau+\xi}\Im
\end{aligned}$$

for each $\varphi,\sigma \in \mathbb{V}, e = (\tau,\xi) > 0$ and $q \in [0,1]$. It is easily established that $(\mathbb{V},\mathcal{E},\mathcal{G},\mathcal{H},\star,\triangle)$ is a CVNMS.

For each $\varphi, \sigma \in \mathbb{V}$ *and* $q \in [0,1]$, *it follows that*

$$\begin{aligned}|\mathcal{I}\varphi(q) - \mathcal{I}\sigma(q)|^2 &= |Q(q) + \acute{c}\int_0^1 \acute{w}(q,r)\varpi(r,\varphi(r))dr - Q(q) - \acute{c}\int_0^1 \acute{w}(q,r)\varpi(r,\sigma(r))dr|^2 \\ &= \acute{c}^2 |\int_0^1 \acute{w}(q,r)\varpi(r,\varphi(r))dr - \int_0^1 \acute{w}(q,r)\varpi(r,\sigma(r))dr|^2 \\ &\leq \acute{c}^2 \left(\int_0^1 \acute{w}(q,r)dr\right)^2 |\varpi(r,\varphi(r)) - \varpi(r,\sigma(r))|^2 \\ &\leq \acute{c}^2 \acute{y}^2 \acute{x}^2 |\varphi(r) - \sigma(r)|^2 \\ &\leq k|\varphi(r) - \sigma(r)|^2.\end{aligned}$$

Now, for each $\varphi, \sigma \in \mathbb{V}$ *and* $e \in \mathbb{S}_0$, *it leads to*

$$\begin{aligned}\mathcal{E}(\mathcal{I}\varphi(q), \mathcal{I}\sigma(q), ke) &= \frac{k(\tau+\xi)}{k(\tau+\xi) + |\mathcal{I}\varphi(q) - \mathcal{I}\sigma(q)|^2}\Im \\ &\succeq \frac{k(\tau+\xi)}{k(\tau+\xi) + k|\varphi(q) - \sigma(q)|^2}\Im \\ &= \frac{(\tau+\xi)}{(\tau+\xi) + |\varphi(q) - \sigma(q)|^2}\Im \\ &= \mathcal{E}(\varphi(q), \sigma(q), e)\end{aligned}$$

$$\begin{aligned}\mathcal{G}(\mathcal{I}\varphi(q), \mathcal{I}\sigma(q), ke) &= \frac{|\mathcal{I}\varphi(q) - \mathcal{I}\sigma(q)|^2}{k(\tau+\xi) + |\mathcal{I}\varphi(q) - \mathcal{I}\sigma(q)|^2}\Im \\ &= \left(1 - \frac{k(\tau+\xi)}{k(\tau+\xi) + |\mathcal{I}\varphi(q) - \mathcal{I}\sigma(q)|^2}\right)\Im \\ &\preceq \left(1 - \frac{k(\tau+\xi)}{k(\tau+\xi) + k|\varphi(q) - \sigma(q)|^2}\right)\Im \\ &= \left(1 - \frac{k(\tau+\xi)}{k(\tau+\xi) + k|\varphi(q) - \sigma(q)|^2}\right)\Im \\ &= \frac{|\varphi(q) - \sigma(q)|^2}{\tau+\xi + |\varphi(q) - \sigma(q)|^2}\Im \\ &= \mathcal{G}(\varphi(q), \sigma(q), e),\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(\mathcal{I}\varphi(q), \mathcal{I}\sigma(q), ke) &= \frac{|\mathcal{I}\varphi(q) - \mathcal{I}\sigma(q)|^2}{k(\tau+\xi)}\Im \\ &\preceq \frac{|\mathcal{I}\varphi(q) - \mathcal{I}\sigma(q)|^2}{(\tau+\xi)}\Im \\ &= \mathcal{H}(\varphi(q), \sigma(q), e).\end{aligned}$$

Consequently, every condition listed in Theorem 1 *is satisfied, suggesting that there is only one solution to the equation* (15) *exists in the set* $C([0,1], \mathbb{R})$.

7. Conclusion

In this paper, we introduced the concept of CVNMSs as a generalization of CVFMSs, complex-valued IFMSs, and NMSs. Further, we proved the Banach contraction theorem and common fixed point theorems in the setting of CVNMSs. We provide several non-trivial examples to demonstrate how the new strategy outperforms literature-based methods. Furthermore, we find the existence and uniqueness of the solution of the integral equation by applying the main result. Our findings broaden the scope of previous research beyond fuzzy metric, intuitionistic fuzzy metric, and NMSs. This work is extendable in the context of complex-valued neutrosophic b-metric spaces, complex-valued neutrosophic controlled metric spaces, complex-valued neutrosophic partial metric spaces, and many other structures.

Conflict of interest

The authors declare that they have no conflicts of interest.

Authors Contribution

All authors contributed equally in this manuscript.

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