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New Existence Results on R–L Fractional Derivative under Weak Topology Attribute

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Abstract. In this article, the research examines whether Riemann–Liouville-type fractional derivatives can be used to solve an initial value problem under weak topology conditions. To prove the existence of an integrable description and a new type of iteration of a Leray–Schauder nonlinear alternative for weak topology, we will first transform the given problem into the sum of two integral operators, and then employ the modified version of Krasnoselskii's fixed point hypothesis in weak topology. Finally, an example is given to illustrate the effectiveness of our main findings.

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1. Introduction

The study of fractional differential equations has been gained importance in applied mathematics and modeling, of numerous phenomena in various sciences. Fractional calculus, define integral and derivative operators in another way, compared to regular calculus concepts. Fractional exponents are an expansion of exponents with integer values. Fractional differential equations have been researched by several authors [1–3] and gave their research in diffrent applications.

Over the last few decades, FDEs act as an indispensable instrument in the fields of creation, electrical networks [4], medicine [5] ,neurological disorder [6], HIV/AIDS transmission [7], human papilloma virus [8], optics, probability and statistics , control theory

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of dynamical systems, rheology [9], and diffusive transport that is comparable to diffusion [10]. Karthikraja et. al investigated the existence and uniqueness result in [11]. Furthermore, the application of fixed point theorems under weak topology to fractional differential equations hasn't been sufficiently generalized to our knowledge.

Many authors have studied the weak topology measures of fractional differential equations. Using fixed point hypothesis in Banach regions, the majority of look into papers focus on the subsistence of results, stability and uniqueness. To demonstrate the existence of methods to integral equations, point fixed theory by weak topology has been studied in several papers and monographs [9, 12], and its references.The boundary value problem which involving nonlinear Riemann-Liouville non-integer derivative is the subject of this paper, which examines the possibility of solutions. Here we use the De Blasi measure, which is to measure the noncompactness weak topology in banach regions. The Krasnoselskii type fixed point theorem was modified by Latrach and Taoudi in [2], and they used it to look into the following equations:

$$\mathfrak{u}(t) = \mathfrak{f}(t, \mathfrak{u}(t)) + \gamma \int_{\phi} \mathcal{M}(t,s)\mathfrak{g}(s, \mathfrak{u}(s))ds$$

In [3] El-Sayed et al., investigated Existence of a bounded variation solution of a nonlinear integral equation in $\mathcal{L}_1(R^+)$ by using measure of noncompactness

$$\mathfrak{u}(t) = \mathfrak{p}(t) + \mathfrak{g}(t)\mathfrak{h}(t, \mathfrak{u}(t)) + \int_0^{\Phi} \kappa(t,s)\mathfrak{f}(s, \mathfrak{u}(s))ds \quad t \in R^+$$

Sufficient conditions on the functions $\mathfrak{f}$ and $\mathfrak{g}$ that show the existence of at least one integrable solution for IVP are provided by the theory of fractional differential equations, the theory of fixed point under weak topology point, and the weak non-compactness of De Blasi measure . We provide some basic ideas and lemmas regarding weak topology and fractional calculus theory for this reason. Then, using some helpful definitions and lemmas of fractional integral and derivative, a new iteration of a Leray-Schauder nonlinear alternative for the weak topology, will be demonstrated and then IVP converted into a type of Volterra integral equation. In section 4, mainly devoted to fixed point theorems for random operators with Volterra type integral equation. Next, we outline our primary finding, which is predicated on a fixed point theorem variation created in [13]. Finally give numerical solution based on our main result.

2. Preliminaries

We introduce notations and definitions used throughout this paper in this section.let $\overline{\mathcal{G}}$ be the Lebesgue integrable functions mapping from $I \to R^n$ and its norm is denoted by $\mathcal{L}^1(I, \mathcal{S})$.

$$\|\overline{\mathcal{G}}\|_{\mathcal{L}^1} = \int_0^T \int_0^T \|\overline{\mathcal{G}}(\upsilon_1, \upsilon_2)\| d\upsilon_1 d\upsilon_2$$

where $\|.\|$ is an appropriate complete norm on X^n. We refer to the set of real numbers as $\mathbb{R}$ throughout the entire work. Natural numbers are represented by the symbol $\mathbb{N}$, (positive integers).We'll set aside a region with the standard Banach $\|.\|_{\mathcal{S}}$, δ null elements by the symbol $\mathcal{S}$. Typically, we substitute $\|.\|$ for $\|.\|_{\mathcal{S}}$. The closed ball with a radius of r and a symbol $\mathcal{B}_r$ denotes it when $r > 0; \mathcal{D}(\mathcal{A})$ denotes the operator $\mathcal{A}$'s domain. While $\mathcal{X}(\mathcal{S})$ means for its subfamily, which involves all relatively compact weakly sets, we will refer to $\mathcal{H}_{\mathcal{S}}$ as the collection of all Â $\mathcal{S}$Â nonempty, bounded subsets. Furthermore, symbols $\overline{\mathbb{H}_{\mathbb{W}}}$, $conv(\mathbb{H})$ support the weak closure and convex hull, respectively, an arbitrarily subset $\mathbb{H} \subset X$. Additionally, we employ the conventional notation $\mathbb{H}_1 + \mathbb{H}_2, \mu\mathbb{H}(\mu \in \mathbb{R})$ for algebraic operations on sets.

Let's discuss the idea of [13], which is represented by the function $\overline{\omega}_{\mathfrak{m}} : \mathcal{H}_{\mathcal{S}} \to \mathbb{R}^+$, defined as follows.

$$\overline{\omega}_{\mathfrak{m}}(\mathbb{H}) = \inf \{r > 0 : \exists\, \mathbb{P} \in \mathcal{Q}(\mathcal{S}) \ni : \mathbb{H} \subset \mathbb{P} + \mathcal{B}_r\}$$

Lemma 1. *Let $\mathbb{H}_1, \mathbb{H}_2$ represent two components of $\mathcal{H}_{\mathcal{S}}$. After that, the subsequent conditions are met:*

(1) $\mathbb{H}_1 \subseteq \mathbb{H}_2 \implies \omega_{\mathfrak{m}}(\mathbb{H}_1) \leq \omega_{\mathfrak{m}}(\mathbb{H}_2)$

(2) $\overline{\omega}_{\mathfrak{m}}\mathbb{H}_1 = 0 \iff \overline{\mathbb{H}_1^m} \in H(\mathcal{S})$

(3) $\overline{\omega}_{\mathfrak{m}}(\overline{\mathbb{H}_1^m}) = \overline{\omega}_{\mathfrak{m}}(\mathbb{H})$

(4) $\overline{\omega}_{\mathfrak{m}}(\mathbb{H}_1 \cup \mathbb{H}_2) = \max \{\overline{\omega}_{\mathfrak{m}}(\mathbb{H}_1), \overline{\omega}_{\mathfrak{m}}(\mathbb{H}_2)\}$

(5) $\overline{\omega}_{\mathfrak{m}}(\mu\mathbb{H}_1) = |\mu|\overline{\omega}_{\mathfrak{m}}(\mathbb{H}_1) \forall \mu \in \mathbb{R}$

(6) $\overline{\omega}_{\mathfrak{m}}(conv(\mathbb{H}_1)) = \overline{\omega}_{\mathfrak{m}}(\mathbb{H}_1)$

(7) $\overline{\omega}_{\mathfrak{m}}(\mathbb{H}_1 + \mathbb{H}_2) \leq \overline{\omega}_{\mathfrak{m}}(\mathbb{H}_1) + \overline{\omega}_{\mathfrak{m}}(\mathbb{H}_1)$

(8) *If $(\mathbb{H}_{\mathfrak{n}})_{\mathfrak{n} \geq 1}(\neq \emptyset)$ is a decreasing sequence bounded, weakly closed subsets of $\mathcal{S}$ with $\lim_{\mathfrak{n} \to \infty} \overline{\omega}_{\mathfrak{m}}(\mathbb{H}_{\mathfrak{n}}) = 0$. Consequently, $\mathbb{H}_\infty = \cap_{\mathfrak{n}=1}^{\infty} \mathbb{H}_{\mathfrak{n}}(\neq \emptyset)$ $\overline{\omega}_{\mathfrak{m}}(\mathbb{H}_\infty) = 0.\hat{A}$ that is, $\mathbb{H}_\infty$ is weakly compact.*

The measure $\overline{\omega}_{\mathfrak{m}}(.)$ has the following form in $\mathcal{L}^1$ space

Proof. [14] Let $\mathbb{R}^n$ be a compact space and $\overline{\varsigma} \subset \mathbb{R}^n$ and let $\mathbb{H} \subset \mathcal{L}^1(\overline{\varsigma}, \mathcal{S})$ bounded set where $\mathcal{S}$ is a Banach finite dimensional region. Then, $\omega_{\mathfrak{m}}(\mathbb{H})$ possesses the following form

$$\omega_{\mathfrak{m}}(\mathbb{H}) = \lim_{\epsilon \to 0} \sup \left\{ \sup_{\upsilon \in \mathbb{H}} \left\{ \iint_J \|\upsilon(\upsilon_1, \upsilon_2)\| d\upsilon_1 d\upsilon_2 : meas(J) \leq \epsilon \right\} \right\}$$

for all nonempty subset J of $\overline{\varsigma}$, where *meas* stands for the Lebesgue unit of measurement.

Definition 1. *[15] Let $\overline{\varsigma} \subset \mathbb{R}^n$ and let $\mathcal{S}, \mathbb{F}$ be two space of Banach. A function $f : \overline{\varsigma} \times \overline{\varsigma} \times \mathcal{S} \to \mathbb{F}$ is said to be Caratheodory, if*

(*i*) *for any $\overline{\mathcal{G}} \in \mathcal{S}$, the map measurable map $\upsilon_1, \upsilon_2 \longmapsto f(\upsilon_1, \upsilon_2, \overline{\mathcal{G}})$ from $\overline{\varsigma}$ to $\mathbb{F}$ and* *(ii) for as in nearly all $\upsilon_1, \upsilon_2 \in \overline{\varsigma}$, the map $u \longmapsto f(\upsilon_1, \upsilon_2, \overline{\mathcal{G}})$ is continuous from $\mathcal{S}$ to $\mathbb{F}$.*

Let the set of all measurable functions $\overline{\mathcal{G}} : \overline{\varsigma} \to \mathcal{S}$ be denoted by $m(\overline{\varsigma}, \mathcal{S})$. If f is a Caratheodory function, f defines a mapping for all $\upsilon_1, \upsilon_2 \in \overline{\varsigma}$ of the form $\mathfrak{N}_f : m(\upsilon_1, \upsilon_2, \mathcal{S})$ by $\mathfrak{N}_f \overline{\mathcal{G}}(\upsilon_1, \upsilon_2) = f(\upsilon_1, \upsilon_2, \overline{\mathcal{G}}(\upsilon_1, \upsilon_2))$. The operator Nemytskii's related with f is the name of this mapping.

Lemma 2. *[15] Let $\overline{\varsigma} \subset \mathbb{R}^n$ and let $\mathcal{S}$ be a separable Banach region and $p, q \geq 1$ and let the Caratheodory function $\mathbb{F} : \overline{\varsigma} \times \overline{\varsigma} \times \mathcal{S} \to \mathcal{S}$. The operator Nemytskii $\mathfrak{N}_{\mathbb{F}}$ associated to the continuous map $\mathbb{F}$ in the space $\mathcal{L}^1(\overline{\varsigma}, \mathcal{S})$ into itself iff*

$$\|\mathbb{F}(\upsilon_1, \upsilon_2, \overline{\mathcal{G}})\| \leq \mathsf{a}(\upsilon_1, \upsilon_2) + \mathsf{b}\|\overline{\mathcal{G}}\|, \forall \upsilon_1, \upsilon_2 \in I, \forall \overline{\mathcal{G}} \in \mathcal{S}$$

where $\mathsf{a} \in \mathcal{L}^1_+(\overline{\varsigma}, \mathcal{S})$ and $\mathsf{b} > 0$ constant. Here $\mathcal{L}^1_+(\overline{\varsigma}, \mathcal{S})$ refers for the positive cone of the space $\mathcal{L}^1(\overline{\varsigma}, \mathcal{S})$. Obviously, we have

$$\|\mathfrak{N}_{\mathbb{F}} \overline{\mathcal{G}}\|_{\mathcal{L}^1} \leq \|\mathsf{a}\|_{\mathcal{L}^1} + \mathsf{b}\|\overline{\mathcal{G}}\|_{\mathcal{L}^1}, \forall \overline{\mathcal{G}} \in \mathcal{L}^1(\overline{\varsigma}, \mathcal{S})$$

Definition 2. *[15] For $\zeta_1, \zeta_2 > 0$, the fractional integral of Riemann-Liouville order ζ_1, ζ_2 is defined as*

$$I_a^{\zeta} \overline{\mathcal{G}}(\upsilon_1, \upsilon_2) = \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^{\upsilon_1} \int_0^{\upsilon_2} (\upsilon_1 - s)^{\zeta_1 - 1} (\upsilon_2 - t)^{\zeta_2 - 1} \overline{\mathcal{G}}(s, t) ds dt, \tag{1}$$

Definition 3. *[15] For $0 < \zeta_1, \zeta_2 < 1$, the left R-L fractional derivative of order ζ_1, ζ_2 is defined by*

$$({}_a D^{\zeta} \overline{\mathcal{G}})(\upsilon_1, \upsilon_2) = \left(\frac{\partial^2}{\partial \upsilon_1 \partial \upsilon_2} \right) \left(\frac{1}{\Gamma(1 - \zeta_1)\Gamma(1 - \zeta_2)} \int_0^{\upsilon_1} \int_0^{\upsilon_2} (\upsilon_1 - s)^{1 - \zeta_1} (\upsilon_2 - t)^{1 - \zeta_2} \overline{\mathcal{G}}(s, t) ds dt \right). \tag{2}$$

Lemma 3. *[15] For $\zeta \in (\mathfrak{n} - 1, \mathfrak{n}]$ and $\upsilon_1, \upsilon_2 \in I$, we have the properties*

(i) $[\mathcal{D}]^{\zeta} I^{\zeta} \mathfrak{u}(\upsilon_1, \upsilon_2) = \mathfrak{u}(\upsilon_1, \upsilon_2)$

(ii) *if $\zeta < \beta$ for an integer β, then $[\mathcal{D}]^{\zeta} I^{\beta} \mathfrak{u}(\upsilon_1, \upsilon_2) = I^{\beta - \zeta} \mathfrak{u}(\upsilon_1, \upsilon_2)$*

(iii) *RL fractional derivative $[\mathcal{D}]^{\zeta} \mathfrak{u}(\upsilon_1, \upsilon_2)$ for $q > 0$, the laplace transform the power function are provided by $(\upsilon_1, \upsilon_2) \longmapsto t^q$*

- **(a)** $\mathcal{L}\left\{[\mathcal{D}]^{\zeta} \mathfrak{u}(\upsilon_1, \upsilon_2), \mathfrak{z}\right\} = \mathfrak{z}^{\zeta_1} w^{\zeta_2} \mathfrak{u}(\mathfrak{z}) - \sum_{i=0}^{n-1} \mathfrak{z}^i w^i [[\mathcal{D}]^{\zeta - i - 1} \mathfrak{u}(\upsilon_1, \upsilon_2)]_{(0,0)}$
- **(b)** $\mathcal{L}\left\{t^q, \mathfrak{z}, w\right\} = \Gamma(q_1 + 1)\Gamma(q_2 + 1) \mathfrak{z}^{-(q_1 + 1)} w^{-(q_2 + 1)}$,
  Where $\mathfrak{u}(\mathfrak{z}, w)$ be the Laplace transform of $\mathfrak{u}(\upsilon_1, \upsilon_2)$

In these deliberations, the following illustration of the Krasnoselskii fixed point theorem is significant.

Theorem 1. *[16] A non-empty, bounded, closed, and convex subset of a Banach space E is denoted by $\mathcal{M}$. Assume $\mathtt{A} : \mathcal{M} \to E$ and $\mathtt{B} : \mathcal{M} \to E$ Two weakly sequentially continuous mappings .*

(i) *The relatively weakly compact set is* $\mathtt{A}(\mathcal{M})$

(ii) $\mathtt{B}$ *is a contraction, and*

(iii) $(\mathtt{x} = \mathtt{Bx} + \mathtt{Ay}, \mathtt{y} \in \mathcal{M}) \implies \mathtt{x} \in \mathcal{M}$
Then, $\mathtt{A} + \mathtt{B}$ *has more than a fixed point in* $\mathcal{M}$.

Lemma 4. *[15] The linear fractional differential equation's singular solution*

$$D^{\zeta}\overline{\mathcal{G}}(\upsilon_1, \upsilon_2) = 0$$

is defined by

$$\forall \ \mathbb{C}_i \in \mathbb{R}, \quad \overline{\mathcal{G}}(\upsilon_1, \upsilon_2) = \mathbb{C}_1 \upsilon_1^{\zeta_1 - 1} \upsilon_2^{\zeta_2 - 1} + \mathbb{C}_2 \upsilon_1^{\zeta_1 - 2} \upsilon_2^{\zeta_2 - 2} + \cdots + \mathbb{C}_n \upsilon_1^{\zeta_1 - n} \upsilon_2^{\zeta_2 - n}, \quad i = 1, 2, \ldots, n$$

3. Existence Result

FDEs have gained importance in both theoretical and practical aspects of several scientific and engineering domains. Karthikraja et. al investigated the existence and uniqueness result in [11]. This article examines whether there are solutions for Riemann-Liouville type fractional derivatives with initial conditions in Banach space, as inspired by the aforementioned works.

$$\begin{cases} [\mathcal{D}]^{\zeta}\overline{\mathcal{G}}(\upsilon_1, \upsilon_2) = p(\upsilon_1, \upsilon_2) f(\upsilon_1, \upsilon_2, \overline{\mathcal{G}}(\upsilon_1, \upsilon_2)) + g(\upsilon_1, \upsilon_2, \mathcal{M}\overline{\mathcal{G}}(\upsilon_1, \upsilon_2)), \\ \qquad\qquad \upsilon_1, \upsilon_2 \in I = [0, T], T > 0 \\ \lim\limits_{\upsilon_1 \to 0+, \upsilon_2 \to 0+} \upsilon_1^{2-\zeta} \upsilon_2^{2-\zeta} \overline{\mathcal{G}}(\upsilon_1, \upsilon_2) = \lim\limits_{\upsilon_1 \to 0+, \upsilon_2 \to 0+} \upsilon_1^{2-\zeta} \upsilon_2^{2-\zeta} \overline{\mathcal{G}}'(\upsilon_1, \upsilon_2) = 0, \end{cases} \quad (3)$$

Here D^{ζ} is the order, $1 \leq \zeta \leq 2$ left Riemann Liouville derivative. In this case, $p(\upsilon_1, \upsilon_2)$ is a measurable function, $f(\upsilon_1, \upsilon_2, \overline{\mathcal{G}}(\upsilon_1, \upsilon_2))$ and $g(\upsilon_1, \upsilon_2, \mathcal{M}\overline{\mathcal{G}}(\upsilon_1, \upsilon_2))$ are nonlinear functions, and $\mathcal{M}$ is a linear operator with bounds that goes from $\mathcal{L}^1(I, R^n)$ to itself.

Let us consider the Banach region $\mathcal{S}$ and the operator $\mathfrak{U} : \mathcal{D}(\mathfrak{U}) \subseteq \mathcal{S} \to \mathcal{S}$. By the subsequent circumstances

$(\mathcal{C}_1)$ $\begin{cases} \textit{Accordingly} \quad (\breve{u}_n)_{n \in N} \subset \mathcal{D}(\mathfrak{U}) \text{ a sequence that weakly converges in } \mathcal{S}, \text{ likewise} \\ (\mathfrak{U}\breve{u}_n)_{n \in \mathbb{N}} \text{ contains a subsequence that strongly converges in } \mathcal{S}. \end{cases}$

$(\mathcal{C}_2)$ $\begin{cases} \textit{Accordingly} (\breve{u}_n)_{n \in N} \subset \mathcal{D}(\mathfrak{U}) \text{ a sequence that weakly converges in } \mathcal{S}, \text{ likewise} \\ (\mathfrak{U}\breve{u}_n)_{n \in \mathbb{N}} \text{ contains a subsequence that is weakly convergent } \mathcal{S}. \end{cases}$

The reader is directed to the monograph [16] for some applications on maps that satisfy the conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ that were considered in [10, 17].

Note 2. **(i)** *It is not always the case which operators satisfy* $(\mathcal{C}_1)$ *or* $(\mathcal{C}_2)$ *are weakly continuous.*

(ii) *(A2) is satisfied by all* $\overline{\omega}$*-contractive maps.*

(iii) *According to the Eberlein-Mulian Theorem, a map* $\mathbb{P}$ *satisfies* $(\mathcal{C}_2) \iff$ *Sets that are relatively weakly compact are transformed into sets that are likewise weakly compact.*

(iv) *If and only if a map* $\mathbb{P}$ *transforms relatively minutely compact sets into relatively compact ones, then it satisfied* $(\mathcal{C}_1)$.

(v) *Every bounded linear operator satisfies condition* $(\mathcal{C}_2)$.

Lemma 5. *Suppose that* X *is a Banach . A mapping* $B : X \times X \to X$ *is* $\overline{\omega}$*-contractive if it is assumed to be a contraction and to satisfy* $(\mathcal{C}_2)$.

Proof. *It maps bounded sets into bounded sets with a positive constant* $\zeta \in (0, 1)$ *if* B *is a Lipschitzian map. The symbol for a bounded subset of* X *is* $\mathcal{A}$. *Assume that* $\mathbb{P} \in \overline{\omega}(X)$ *and that* $\upsilon_1 > 0, \upsilon_2 > 0$ *so that* $\mathcal{A} \subset \mathbb{P} + B\upsilon_1\upsilon_2$. *It's obvious that*

$$B(\mathcal{A}) \subseteq B(\mathbb{P}) + B_{\zeta_1\upsilon_1}B_{\zeta_2\upsilon_2} \subseteq \overline{B(\mathbb{P})}^{\omega} + B_{\zeta_1\upsilon_1}B_{\zeta_2\upsilon_2}$$

Since B *also satisfies* $(\mathcal{C}_2), B_{\mathbb{P}}$, *is comparatively weakly compact, and for all* $\upsilon_1, \upsilon_2 > 0$, $\overline{\omega}(B(\mathcal{A})) \leq \zeta\upsilon_1\upsilon_2$, *so that* $B \subseteq \mathbb{P} + B_{\upsilon_1}B_{\upsilon_2}$ *with some* $\mathbb{P} \subseteq \mathcal{W}(X)$. *Consequently, the lemma is proved by* $\overline{\omega}B(\mathcal{A}) \leq \zeta\overline{\omega}(\mathcal{A})$

3.1. Case C has no bounds

The following variant of the Schauder fixed point theorem for weak topology will be used to illustrate our first existence result, which is a new version of a Leray-Schauder nonlinear alternative for the weak topology.

Theorem 3. *Let* $C \in X$ *which is be a closed convex and that is not empty. Presume that the continuous map* $\mathbb{F} : C \times C \to C$ *satisfies* $(\mathcal{C}_1)$. *Should* $\mathbb{F}(C \times C)$ *be relatively weakly compact, then* $x \in C$ *such that* $\mathbb{F}x = x$.

Note 4. *Given its weak relative compactness with* $(\mathcal{C}_1)$, *it is clear that* $\mathbb{F}(C \times C)$ *is encapsulating for several* ζ *(strong) measure of noncompactness. Remember that when* $\zeta(\mathbb{F}(\mathbb{H})) \geq \zeta(\mathbb{H})$ *for some bounded subset* $\mathbb{H}$ *implies* $\zeta(\mathbb{H}) = 0$, *so that* $\mathbb{H}$ *is relatively compact, then* $\mathbb{F}$ *is said to be condensing relative to a measure of noncompactness* ζ. *Therefore, theorem 1 is a direct consequence of the fixed point theorem of Darbo-Sadovskij [18]. But in [19], the authors first showed that* $\mathbb{F}_{|C}$ *is even compact where* $C = \overline{co}(\mathbb{F}C)$. *Then, using the Schauder fixed point theorem, they used the Krein-Mulian theorem to show that the latter set is weakly compact. Additionally, we point out that the poor measure of noncompactness is neither*

Theorem 5. *In a Banach space* X, *let* C *be a nonempty closed convex set, and let* $\mathfrak{S} \subset C$ *be an open subset where some* $\mathfrak{r} \in \mathfrak{S}$. *Presume that the continuous map* $\mathbb{F} : \mathfrak{S} \times \mathfrak{S} \to C$ *that meets condition* $(\mathcal{C}_1)$. *If* $\mathbb{F}(\mathfrak{S} \times \mathfrak{S})$ *is only marginally compact, then*

(a) *Either $\mathfrak{S}$ contains the solution to the equation $\mathbb{F}u = u$,*

(b) *Alternatively, for some $\mu \in (0,1)$, $\exists$ an element $u \in \partial\mathfrak{S}$ such that $u = \mu\mathbb{F}u + (1-\mu)\mathfrak{r}$*

Proof. *We will use an Urysohn function that goes back to Cech in accordance with a standard procedure to prove this theorem. This auxiliary function is used in the majority of nonlinear alternative proofs [20–22]. Assuming that (ii) is false and that $\mathbb{F}$ has no fixed point on $\partial\mathfrak{S}$, we are done. Next,*

$$u \neq \mu\mathbb{F}u + (1-\mu)\mathfrak{r} \quad u \in \partial\mathfrak{S} \text{ and } \mu \in [0,1]$$

Thus, $\mathfrak{r} \in \mathfrak{S}$,

$$\mathcal{L} = \{u \in \overline{\mathfrak{S}} : u = \upsilon_1\upsilon_2\mathbb{F}u + (1-\upsilon_1)(1-\upsilon_2)\mathfrak{r}, \upsilon_1, \upsilon_2 \in [0,1]\}$$

isn't empty. Furthermore, $\mathcal{L}$ is closed since $\mathcal{L} \cap \partial\mathfrak{S} = \phi$ and $\mathbb{F}$ is continuous. Therefore, according to Urysohn's lemma [23], $\lambda\mathcal{L} = 1$ and $\lambda(\partial\mathfrak{S}) = 0(\lambda(u) = \frac{d(x,\partial\mathfrak{S})}{d(x,\partial\mathfrak{S})-d(x,\mathcal{L})})$ exist for a continuous function $\lambda : \overline{\mathfrak{S}} \to [0,1]$. In [20], let $\mathbb{P}$ be the function is defined by

$$\lambda u = \begin{cases} \lambda(u)\mathbb{F}u + (1-\lambda(u))\mathfrak{r}, & if u \in \partial\overline{\mathfrak{S}} \\ \mathfrak{r} & if u \in C \setminus \overline{\mathfrak{S}} \end{cases} \tag{4}$$

The continuous nature of $\mathbb{P} : C \times C \to C$ is readily apparent. It suffices to show that $(\mathcal{C}_1)$ is verified by operator $\mathbb{P}$ and that $\mathbb{P}(C \times C)$ is relatively weakly compact in accordance with Theorem 1. To do this, let $(x_n)_n \in \mathbb{N}$ be a weakly concurrent sequence in C. Based on whether or not $(x_n)_n \in \mathbb{N}$ lies in $\mathfrak{S}$ for n large enough, we distinguish between two scenarios:

(a) *For every $n \in \mathbb{P}$, there is some $n_0 \in \mathbb{P}$ such that $(n \geq n_0 \implies x_n \in \mathfrak{S})$. The sequence $(x_n)_n \in \mathbb{N}$ in this instance lies in $\mathfrak{S}$ and converges weakly there. The sequence $(\mathbb{F}x_n)_{n \geq n_0}$ has a strongly convergent subsequence, such as $(\mathbb{F}x_{\kappa_n})_n, \mathbb{F}x_{\kappa_n} \to y$ in C, since F satisfies $(\mathcal{C}_1)$. We can extract a convergent subsequence, such as $(\lambda(x_{|n}))_n$, from $(\lambda(x_{|n}))_n$ by using the compactness of $[0,1]$. Consequently, the limit $\upsilon_1\upsilon_2 y + (1-\upsilon_1)(1-\upsilon_2)\mathfrak{r}$ lies in C since the sequence $(\lambda(x_{|n}))_n$ confirms that $\mathbb{P}x_{|n} = (\lambda(x_{|n}))\mathbb{F}x_{|n} + (1-\lambda(x_{|n}))\mathfrak{r}$.*

(b) *We can examine a subsequence $(x_{m_n})_n \subset C \setminus \mathfrak{S}$ such that $\mathbb{P}x_{m_n} = \mathfrak{r} \times \mathfrak{r} \to \mathfrak{r}$ in C if $(x_n)_n$ is such that for every $n \in \mathbb{P}$, there exists $m \in \mathbb{P}$ such that $x_{m_n} \in \overline{\mathfrak{S}}$. $\mathbb{P}$ confirms $(\mathcal{C}_1)$ based on (a) and (b). Using an argument akin to the one used to determine that $\mathbb{P}$ verifies $(\mathcal{C}_1)$, we utilize the fact that $\mathbb{F}(\mathfrak{S})$ is relatively weakly compact to demonstrate the set $\mathbb{P}(C)$'s weak compactness. The fact that some $u \in C$ with $u = \mathbb{P}u$ is then guaranteed by Theorem 1. $u = \lambda(u)\mathbb{F}u + (1-\lambda(u))\mathfrak{r}$ since $u \in \mathfrak{S}$ since $\mathfrak{r} \in \mathfrak{S}$. Thus, $u \in \mathcal{L}$ and $\lambda(u) = 1$, which suggests that $u = \mathbb{F}u$ and concludes the theorem's proof.*

Theorem 6. *Given a Banach area $\mathcal{S}$, let $\mathcal{M}$ be a convex, closed, enclosed, and nonempty subset of it. Consider the following pair of operators: $\mathcal{A} : \mathcal{M} \to \mathcal{S}$ and $\mathcal{B} : \mathcal{M} \to \mathcal{S}$.*

(i) The condition $\mathcal{A}$ is satisfied, $(\mathcal{C}_1)$

(ii) $\exists\ \beta \in [0,1] \ni: \omega(\mathcal{A}S + \mathcal{B}S) \le \beta\omega(S)\ \forall\ S \subseteq \mathcal{M}$,

(iii) The contraction function $\mathcal{B}$ and satisfies $(\mathcal{C}_2)$, and

(iv) $\mathcal{A}\mathcal{M} + \mathcal{B}\mathcal{M} \subseteq \mathcal{M}$

Then, there is $\overline{\mathcal{G}} \in \mathcal{M} \ni: \mathcal{A}\overline{\mathcal{G}} + \mathcal{B}\overline{\mathcal{G}} = \overline{\mathcal{G}}$.

Theorem 6 is the foundation for our existence conclusion. We first demonstrate that the problem IVP (**18**) has solutions before converting it into an equivalent integral equation.

Lemma 6. *The following integral equation of the Volterra type equals IVP* (**18**).

$$\begin{aligned}\overline{\mathcal{G}}(\upsilon_1,\upsilon_2) &= \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}p(s,t)f(s,t,\overline{\mathcal{G}}(s,t))dtds \\ &+\frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}g(s,t,M\overline{\mathcal{G}}(s,t))dtds \qquad (5)\end{aligned}$$

Proof. The equation (**18**) can be expressed as follows using **Lemma 4**

$$\begin{aligned}\overline{\mathcal{G}}(\upsilon_1,\upsilon_2) &= \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}p(s,t)f(s,t,\overline{\mathcal{G}}(s,t))dtds \\ &+\frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}g(s,t,M\overline{\mathcal{G}}(s,t))dtds \\ &+c_1\upsilon_1^{\zeta_1-1}\upsilon_2^{\zeta_2-1}+c_2\upsilon_1^{\zeta_1-2}\upsilon_2^{\zeta_2-2} \qquad (6)\end{aligned}$$

We obtain $c_2 = 0$ by using the condition $\lim_{\upsilon_1\to0+,\upsilon_2\to0+}\upsilon_1^{2-\zeta}\upsilon_2^{2-\zeta}\overline{\mathcal{G}}(\upsilon_1,\upsilon_2) = 0$, and we obtain $c_1 = 0$ by using the condition $\lim_{\upsilon_1\to0+,\upsilon_2\to0+}\upsilon_1^{2-\zeta}\upsilon_2^{2-\zeta}u'(\upsilon_1,\upsilon_2) = 0$ When we replace in (**6**), we get the integral equation.

$$\begin{aligned}\overline{\mathcal{G}}(\upsilon_1,\upsilon_2) &= \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}p(s,t)f(s,t,\overline{\mathcal{G}}(s,t))dtds \\ &+\frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_1-t)^{\zeta_2-1}g(s,t,M\overline{\mathcal{G}}(s,t))dtds\end{aligned}$$

Definition 4. *The condition $\mathcal{A}$ is satisfied $(\mathcal{C}_1)$. Due to* **Lemma 6**, *the integral equation* (**25**) *can be written in the following way.*

$$\overline{\mathcal{G}} = \mathcal{A}\overline{\mathcal{G}} + \mathcal{B}\overline{\mathcal{G}}$$

where $\mathcal{L}^1(I,\mathcal{S})$ into itself defines two operators, $\mathcal{A}$, and $\mathcal{B}$.

$$\mathcal{A} = \overline{\mathbb{I}}\mathfrak{N}_f \text{ and } \mathcal{B} = \overline{\mathbb{J}}\mathfrak{N}_g M \tag{7}$$

where $\mathfrak{N}_f$ and $\mathfrak{N}_g$ and $f(\upsilon_1,\upsilon_2,\overline{\mathcal{G}}(\upsilon_1,\upsilon_2))$ and $g(\upsilon_1,\upsilon_2,M\overline{\mathcal{G}}(\upsilon_1,\upsilon_2))$ are the Nemytskii operators, respectively. From $\mathcal{L}^1(I,\mathcal{S})$, the condition $\mathcal{A}$ is satisfied., $(\mathcal{C}_1)$, $\mathcal{L}^1(I,\mathcal{S})$ is the definition of the linear operators $\overline{\mathbb{I}}$ and $\overline{\mathbb{J}}$.

$$\overline{\mathbb{I}}\mathcal{V}(\upsilon_1,\upsilon_2) = \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}p(s,t)\mathcal{V}(s,t)\ dt\ \ ds \tag{8}$$

and

$$\overline{\mathbb{J}}\mathcal{V}(\upsilon_1,\upsilon_2) = \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}\mathcal{V}(s,t)\ dt\ \ ds \tag{9}$$

We will assume the following in what follows:

$(\mathbf{A}_1)$ *The function $p : I\times I\to\mathcal{S}$ belongs to $\mathcal{L}^\infty(I,\mathcal{S})$*

Lemma 7. *Let's assume that $(\mathcal{C}_1)$ is true. The following estimates are obtained from the linear operators $\overline{\mathbb{I}}$ and $\overline{\mathbb{J}}$ being constrained on $\mathcal{L}^1(I,\mathcal{S})$:*

$$\|\overline{\mathbb{I}}\mathcal{V}\|_{\mathcal{L}^1} \le \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|p\|_{\mathcal{L}^\infty}\|\mathcal{V}\|_{\mathcal{L}^1}$$

and

$$\|\overline{\mathbb{J}}\mathcal{V}\|_{\mathcal{L}^1} \le \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{V}\|_{\mathcal{L}^1} \qquad \forall\mathcal{V}\in\mathcal{L}^1(I,\mathcal{S}).$$

We now show that the integral equations of the Volterra type (**25**) have an integrable solution. Every response to (**25**) is, of course, also a response to (**18**). So, we take into account the following suppositions.

$(\mathbf{A}_2)$ The condition $\mathcal{A}$ is satisfied., $(\mathcal{C}_1)$, For the function $g : I\times I\times\mathcal{S}\to\mathcal{S}$, $g(\upsilon_1,\upsilon_2,0)\in\mathcal{L}^1(I,\mathcal{S})$, and g are Lipschitzian with respect to the second variable, which means that there exists a $\upsilon\in\mathbb{R}^+$

$$\|g(\upsilon_1,\upsilon_2,\overline{\mathcal{G}}_1(\upsilon_1,\upsilon_2))-g(\upsilon_1,\upsilon_2,\overline{\mathcal{G}}_2(\upsilon_1,\upsilon_2))\| \le \Upsilon\|\overline{\mathcal{G}}_1-\overline{\mathcal{G}}_2\| \quad \forall\upsilon_1,\upsilon_2\in\Upsilon \text{ and } \overline{\mathcal{G}}_1,\overline{\mathcal{G}}_2\in\mathcal{S} \tag{10}$$

$(\mathbf{A}_3)$ There is a function $\mathsf{a},\mathsf{b}\in\mathcal{L}^1_+(I)$ and a non negative constant c ∋: the function $f(\upsilon_1,\upsilon_2,\overline{\mathcal{G}}(\upsilon_1,\upsilon_1))$ is a CarathÃ©odory function.

$$\|f(\upsilon_1,\upsilon_2,\overline{\mathcal{G}}(\upsilon_1,\upsilon_2))\| \le \mathsf{a}(\upsilon_1)+\mathsf{b}(\upsilon_2)+\mathsf{c}\|\overline{\mathcal{G}}\|,\forall \quad (\upsilon_1,\upsilon_2,\overline{\mathcal{G}}(\upsilon_1,\upsilon_2))\in I\times I\times\mathcal{S} \tag{11}$$

Lemma 8. *[24] Assume that X is a Banach region with finite boundaries. Consider that $(\mathbf{A}_3)$ satisfied. Then, the operator $\mathfrak{N}_f$ Nemytskii satisfies hypothesis $\mathcal{C}_2$.*

Theorem 7. *Let us consider the compact subset $I = [0, T]$, $T > 0$ of R and let X be a finite dimensional Banach region. Assume that the circumstances* $(\mathbf{A}_1)$ *through* $(\mathbf{A}_3)$ *are met. In* $\mathcal{L}^1(I, \mathcal{S}^n)$*, IVP* (**18**) *has at least one solution, if*

$$\frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}(\mathsf{c}\|p\|_{\mathcal{L}^\infty} + \Upsilon\|M\|_{\mathcal{L}}) < 1 \tag{12}$$

In this case, $\|.\|_{\mathcal{L}}$ *stands for the typical norm of linear operator spaces.*

Proof. It suffices to demonstrate that the operators $\mathcal{A}$ and $\mathcal{B}$ provided by (**22**) satisfy each of the hypotheses of **Theorem 6** in order to prove the Theorem . Four steps are required to do this. Before, let's

$$r \geq \frac{\frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|p\|_{\mathcal{L}^\infty}(\|\mathsf{a}\|_{\mathcal{L}^1} + \|\mathsf{b}_{\mathcal{L}^1}) + \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|g(\upsilon_1, \upsilon_2, 0)\|_{\mathcal{L}^1}}{1 - (\mathsf{c}\frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|p\|_{\mathcal{L}^\infty} + \Upsilon\frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|M\|_{\mathcal{L}})} \tag{13}$$

$\mathbb{R}$ is clearly positive from (**12**). Consider the bounded, convex and closed of $\mathcal{L}^1(I, \mathcal{S}^n)$, defined by

$$B_r = \left\{\overline{\mathcal{G}} \in \mathcal{L}^1(I, \mathcal{S}) / \|\overline{\mathcal{G}}\|_{\mathcal{L}^1} \leq r\right\}$$

Step 1 : Now $\mathcal{A}$ is continuous from $\mathcal{L}^1(I, \mathcal{S})$ with itself based on Lemma and utilising $(\mathbf{A}_3)$. We now demonstrate that $\mathcal{A}$ satisfies $(\mathcal{C}_1)$. **Lemma 8** states that $(\mathfrak{N}_f\overline{\mathcal{G}}_n)_{n\in\mathbb{N}}$ having a weakly convergent subsequence, let $(\mathfrak{N}_f\overline{\mathcal{G}}_{n_k})_{k\in\mathbb{N}}$, if $(\overline{\mathcal{G}}_n)_{n\in\mathbb{N}}$ consists of a weakly converging series of $\mathcal{L}^1(I, \mathcal{S})$ for this purpose. Consequently, the sequence $(\mathbb{I}\mathfrak{N}_f\overline{\mathcal{G}}_{n_k})_{k\in\mathbb{N}}$ converges point-wise for nearly every $x, y \in I$ given the boundedness of the operator I. Using the Vitali convergence theorem , We may now say that the series $(\mathcal{A}\overline{\mathcal{G}}_{n_k})_{k\in\mathbb{N}}$ strongly converges in $\mathcal{L}^1(I, \mathcal{S})$. As a result, $\mathcal{A}$ is satisfying $(\mathcal{C}_1)$.

Step 2 : We assert that **Theorem 6**'s condition (**ii**) is satisfied. If $\mathcal{L}^1(I, \mathcal{S})$ has a bounded subset S, then we have for any $\overline{\mathcal{G}} \in S, \forall\ \epsilon > 0$ and for any subset which is nonempty J of I

$$\begin{aligned}
\iint\limits_J \|\mathfrak{N}_f\overline{\mathcal{G}}(\upsilon_1, \upsilon_2)\|d\upsilon_1 d\upsilon_2 &\leq \iint\limits_J \|f(\upsilon_1, \upsilon_2, \overline{\mathcal{G}}(\upsilon_1, \upsilon_2))\|d\upsilon_1 d\upsilon_2 \\
&\leq \iint\limits_J (\|\mathsf{a}(\upsilon_1)\| + \|\mathsf{b}(\upsilon_2)\| + \mathsf{c}(\|\overline{\mathcal{G}}(\upsilon_1, \upsilon_2)\|))d\upsilon_1 d\upsilon_2 \\
&\leq \|\mathsf{a}\|_{\mathcal{L}^1(J)} + \|\mathsf{b}\|_{\mathcal{L}^1(J)} + \mathsf{c}\iint\limits_J \|\overline{\mathcal{G}}(\upsilon_1, \upsilon_2)\|d\upsilon_1 d\upsilon_2
\end{aligned}$$

The set with a single member being weakly compact, we obtain by applying **Proposition 2.1**

$$\lim_{\epsilon\to 0}\sup\left\{\iint\limits_J (\|\mathsf{a}(\upsilon_1)\| + \mathsf{b}(\upsilon_2)\|)d\upsilon_1 d\upsilon_2 \Big/ m(J) \leq \epsilon\right\} = 0$$

then,

$$\omega(\mathfrak{N}_f(S)) \leq \mathsf{b}\omega(S) \tag{14}$$

From (**14**) and **Lemma 7**, it follows that

$$\omega_{\mathfrak{m}}(\mathcal{A}S) \leq \frac{\mathsf{b}T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2))}\|p\|_{\mathcal{L}^\infty}\omega_{\mathfrak{m}}(S) \tag{15}$$

Furthermore, using the operator $\mathfrak{N}_f M$ in the same manner as before while accounting for results in

$$\begin{aligned} \iint\limits_J \|\mathfrak{N}_f M\overline{\mathcal{G}}(\upsilon_1,\upsilon_2)\|d\upsilon_1 d\upsilon_2 &\leq \iint\limits_J \|g(\upsilon_1,\upsilon_2,M\overline{\mathcal{G}}(\upsilon_1,\upsilon_2))\|d\upsilon_1 d\upsilon_2 \\ &\leq \iint_J \|g(\upsilon_1,\upsilon_2,0)d\upsilon_1 d\upsilon_2 + \delta\iint_J \|M\overline{\mathcal{G}}(\upsilon_1,\upsilon_2)\|d\upsilon_1 d\upsilon_2 \\ &\leq \|g(\upsilon_1,\upsilon_2,0)\|_{\mathcal{L}^1} + \delta\|M\|_{\mathcal{L}}\iint_J \|\overline{\mathcal{G}}(\upsilon_1,\upsilon_2)d\upsilon_1 d\upsilon_2 \end{aligned}$$

Thus, we have

$$\omega_{\mathfrak{m}}(\mathfrak{N}_g MS) \leq \delta\|M\|_{\mathcal{L}}\omega_{\mathfrak{m}}(S)$$

Using **Lemma 7**, we obtain

$$\omega_{\mathfrak{m}}(\mathcal{B}S) \leq \frac{\delta T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|M\|_{\mathcal{L}}\omega_{\mathfrak{m}}(S) \tag{16}$$

Adding (**15**), (**16**), and **Lemma 1** results in

$$\omega_{\mathfrak{m}}(\mathcal{A}S+\mathcal{B}S) \leq \Big(\frac{\mathsf{b}T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|p\|_{\mathcal{L}^\infty} + \frac{\delta T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|M\|_{\mathcal{L}}\Big)\omega_{\mathfrak{m}}(S) \tag{17}$$

Theorem 6's condition (**ii**) is satisfied, as shown by estimation (**17**) and hypothesis (**12**).
Step 3 : We will illustrate that $\mathcal{B}$ is a mapping of contractions. Let $\overline{\mathcal{G}}_1, \overline{\mathcal{G}}_2 \in \mathcal{L}^1(I,\mathcal{S})$ for this purpose. Then, by applying **Lemma 7** and the supposition ($\mathbf{A}_2$), it follows for all $\upsilon_1,\upsilon_2 \in I$ that

$$\begin{aligned} \|\mathcal{B}\overline{\mathcal{G}}_1 - \mathcal{B}\overline{\mathcal{G}}_2\| &\leq \|\overline{\mathbb{J}}\mathfrak{N}_g M\overline{\mathcal{G}}_1 - \overline{\mathbb{J}}\mathfrak{N}_g M\overline{\mathcal{G}}_2\|_{\mathcal{L}^1} \\ &\leq \|\overline{\mathbb{J}}\|_L\|\mathfrak{N}_g M\overline{\mathcal{G}}_1 - \mathfrak{N}_g M\overline{\mathcal{G}}_2\|_{\mathcal{L}^1} \\ &\leq \frac{\delta T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|M\|_{\mathcal{L}}\|\overline{\mathcal{G}}_1 - \overline{\mathcal{G}}_2\|_{\mathcal{L}^1} \end{aligned}$$

Then, $\mathcal{B}$ is a contraction mapping with $\frac{\delta T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}$ on $\mathcal{L}^1(I,\mathcal{S})$
Step 4 : Proof that $\mathcal{A}\overline{\mathcal{G}}_1 + \mathcal{B}\overline{\mathcal{G}}_2 = B_r$ is still required. As a result, we conclude that for all $\overline{\mathcal{G}}_1, \overline{\mathcal{G}}_2 \in B_r$, given the assumptions ($\mathbf{A}_1$) through ($\mathbf{A}_3$), and the **lemma 2** and **lemma 7** , we have

$$\|\mathcal{A}\overline{\mathcal{G}}_1 + \mathcal{B}\overline{\mathcal{G}}_2\|_{\mathcal{L}^1} = \|\overline{\mathbb{I}}\mathfrak{N}_f\overline{\mathcal{G}}_1 + \overline{\mathbb{J}}\mathfrak{N}_g M\overline{\mathcal{G}}_2\|_{\mathcal{L}^1}$$

$$
\begin{aligned}
&\leq \|\bar{\mathbb{I}}\|_{\mathcal{L}}\|\mathfrak{N}_f\overline{\mathcal{G}}_1\|_{\mathcal{L}^1} + \|\bar{\mathbb{J}}\|_{\mathcal{L}}\|\mathfrak{N}_g M\overline{\mathcal{G}}_2\|_{\mathcal{L}^1} \\
&\leq \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|p\|_{\mathcal{L}^\infty}(\mathtt{a}(\upsilon_1,\upsilon_2) + \mathtt{b}\|\overline{\mathcal{G}}_1\|_{\mathcal{L}^1}) \\
&\quad + \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}(\|g(\upsilon_1,\upsilon_2,0)\|_{\mathcal{L}^1} + \delta\|M\|_{\mathcal{L}}\|\overline{\mathcal{G}}_2\|_{\mathcal{L}^1}) \\
&\leq \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|p\|_{\mathcal{L}^\infty}(\mathtt{a}(\upsilon_1,\upsilon_2) + \mathtt{b}r) + \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}(\|g(\upsilon_1,\upsilon_2,0)\|_{\mathcal{L}^1} + \delta r\|M\|_{\mathcal{L}}) \\
&\leq \Big(\frac{\mathtt{b}T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|p\|_{\mathcal{L}^\infty} + \frac{\delta T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|M\|_{\mathcal{L}}\Big)r \\
&\quad + \Big(\frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|p\|_{\mathcal{L}^\infty} + \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|g(\upsilon_1,\upsilon_2,0)\|_{\mathcal{L}^1}\Big)
\end{aligned}
$$

From equation (13) we get

$$\|\mathcal{A}\overline{\mathcal{G}}_1 + \mathcal{B}\overline{\mathcal{G}}_2\|_{\mathcal{L}^1} \leq r$$

Thus, by using **Theorem 6**, As the solution of IVP (**18**), we deduce that the operator $\mathcal{A}+\mathcal{B}$ has at least each distinctive fixed point in B_r.

4. Profound Solution

Using the fixed point theorem, which involves the sum of two operators under weak topology, the authors examined into the possibility of solutions for the above last two equations.The author hopes to investigate the existence of an integrable solution under weak topology as a result of this work. We study that the fractional derivative with the initial conditions to find weak solutions in the current study, which was inspired by the work [25].

$$
\begin{cases}
[\mathcal{D}]^{\zeta}\mathfrak{u}(\upsilon_1,\upsilon_2) = f(\upsilon_1,\upsilon_2,\mathcal{H}_1\mathfrak{u}(\upsilon_1,\upsilon_2)) + g(\upsilon_1,\upsilon_2,\mathcal{H}_2\mathfrak{u}(\upsilon_1,\upsilon_2)), \quad \upsilon_1,\upsilon_2 \in I = [0,T], T > 0 \\
[\mathcal{D}]^{\zeta-i}u|_{(0,0)} = 0 \quad i = 1,2
\end{cases}
\tag{18}
$$

Here $[\mathcal{D}]^{\zeta}$ is the order, $1 \leq \zeta \leq 2$ left Riemann Liouville derivative. In this case, $f(\upsilon_1,\upsilon_2,\mathcal{H}_1\mathfrak{u}(\upsilon_1,\upsilon_2))$ and $g(\upsilon_1,\upsilon_2,\mathcal{H}_2\mathfrak{u}(\upsilon_1,\upsilon_2))$ are nonlinear functions, and $\mathcal{H}$ represents a linear operator and it is bounded from $\mathcal{L}^1(I,R^n)$ in and of itself.

By using the theorem (1) we demonstrate in this section that there are integrable solutions to issue (18). Prior to it, the following lemma is presented.

Lemma 9. *Let $1 < \zeta \leq 2$ the IVP unique solution is*

$$
\begin{cases}
[\mathcal{D}]^{\zeta}\mathfrak{u}(\upsilon_1,\upsilon_2) \; = \varrho(\upsilon_1,\upsilon_2), (\upsilon_1,\upsilon_2) \in I = [0,T] \\
[\mathcal{D}]^{\zeta-i}u|_{(0,0)} = 0, i = 1,2
\end{cases}
\tag{19}
$$

which gives

$$\mathfrak{u}(\upsilon_1,\upsilon_2) = I^{\zeta}\varrho(\upsilon_1,\upsilon_2), (\upsilon_1,\upsilon_2) \in I \tag{20}$$

Proof. *We take* $[[\mathcal{D}]^{\zeta-i}\mathfrak{u}(\upsilon_1,\upsilon_2)]_{(0,0)} = \mathtt{m}_i, i = 1,2.$ *As in relate the Laplace transform on either side of (19) and using Lemma 3's property we get*

$$\mathfrak{z}^{\zeta}w^{\zeta}\mathfrak{u}(\mathfrak{z},w) - \sum_{i=0}^{I}\mathfrak{z}^{i}w^{i}[[\mathcal{D}]^{\zeta-i-1}\mathfrak{u}(\upsilon_1,\upsilon_2)]|_{(0,0)} = Y(\mathfrak{z},w)$$

Thus, $\mathfrak{u}(\mathfrak{z},w) and Y(\mathfrak{z},w)$ *represent the Laplace transforms of* $\mathfrak{u}(\upsilon_1,\upsilon_2)$ *and* $\varrho(\upsilon_1,\upsilon_2)$ *as in separately. That is, we can write*

$$\mathfrak{u}(\mathfrak{z},w) = \mathfrak{z}^{-\zeta_1}w^{-\zeta_2}Y(\mathfrak{z},w) + \sum_{i=0}^{I}\mathtt{m}_{i+1}\mathfrak{z}^{i-\zeta_1}w^{i-\zeta_2}$$

Applying the inverse Laplace transform while considering the convolution product, we observe

$$\begin{aligned}\mathfrak{u}(\upsilon_1,\upsilon_2) &= \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}\varrho(s,t)dsdt \\ &\quad + \sum_{i=0}^{I}\frac{\mathtt{m}_{i+1}}{\Gamma(\zeta_1-i)\Gamma(\zeta_2-i)}\mathfrak{z}^{\zeta_1-i-1}w^{\zeta_1-i-1} \\ &= \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}\varrho(s,t)dsdt \\ &\quad + \sum_{i=1}^{2}\frac{\mathtt{m}_{i+1}}{\Gamma(\zeta_1-i+1)\Gamma(\zeta_2-i+1)}\mathfrak{z}^{\zeta_1-i}w^{\zeta_1-i}\end{aligned}$$

Given that $\mathtt{m}_i = 0, i = 1,2,$ *then*

$$\mathfrak{u}(\upsilon_1,\upsilon_2) = \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}\varrho(s,t)dsdt = I^{\zeta}\varrho(\upsilon_1,\upsilon_2)$$

On the other hand, using the properties one and two of lemma 3, $\mathfrak{u}(t)$ *given by (19) satisfies the these equations (20). This concludes with the proof.*

IVP (18) is obviously equal to the operator equation shown in lemma 9:

$$u = \mathcal{A}u + \mathcal{B}u \tag{21}$$

where $\mathcal{L}^1(I,\mathcal{S})$ into itself defines two operators, $\mathcal{A}$, and $\mathcal{B}$.

$$\mathcal{A} = \bar{\mathbb{I}}\mathfrak{N}_f\mathcal{L}_1 \text{ and } \mathcal{B} = \bar{\mathbb{J}}\mathfrak{N}_g\mathcal{L}_2 \tag{22}$$

where $\mathfrak{N}_f$, $\mathfrak{N}_g$, $f(.,.,.)$ and $g(.,.,.)$ are the Nemytskii operators, respectively. From $\mathcal{L}^1(I,\mathcal{S})$, the operators $\bar{\mathbb{I}}$ and $\bar{\mathbb{J}}$ are defined into $\mathcal{L}^1(I,\mathcal{S})$ and it's linear.

$$\bar{\mathbb{I}}\mathcal{V}(\upsilon_1,\upsilon_2) = \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}\mathcal{V}(s,t)\ \mathrm{dt}\ \ \mathrm{ds} \tag{23}$$

and

$$\bar{\mathbb{J}}\mathcal{V}(\upsilon_1,\upsilon_2)=\frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}\mathcal{V}(s,t)\ \mathrm{dt}\ \ \mathrm{ds} \tag{24}$$

Consider the following possibilities as we present the paper's significant results:

($\mathbf{A}_4$) The measurable function and Lipschitzian is g , $g(.,.,0)\in\mathcal{L}^1(I,E)$, with regard to the second variable, meaning that $\exists\quad\mathfrak{F}>0$ such that

$$\|g(\upsilon_1,\upsilon_2,\mathfrak{u})-g(\upsilon_1,\upsilon_2,\mathfrak{v})\|\le\mathfrak{F}\|\mathfrak{u}-\mathfrak{v}\|\forall\upsilon_1,\upsilon_2\in I \text{ and } \mathfrak{u},\mathfrak{v}\in E$$

($\mathbf{A}_5$) There are functions $\mathcal{M}_i\in\mathcal{L}^1(I,\mathbb{R}+)$ and $\mathfrak{H}_i$ be the nondecreasing functions which is in $\mathcal{L}^\infty_{loc}(\mathbb{R}_+)$, where $i=1$ and 2 and the functions f and g complete the weak Caratheodory specifications.

$$\|f(\upsilon_1,\upsilon_2,\mathfrak{u})\|\le\mathcal{M}_1(\upsilon_1,\upsilon_2)\mathfrak{H}_1(\|\mathfrak{u}\|) \text{ and } \|g(\upsilon_1,\upsilon_2,\mathfrak{u})\|\le\mathcal{M}_2(\upsilon_1,\upsilon_2)\mathfrak{H}_2(\|u\|)$$

Theorem 8. *Assume that* $(\mathbf{A}_4)-(\mathbf{A}_5)$ *is true. Prove that the IVP (18) has at all events one integrable solution on* I.

$$\frac{\mathfrak{F}T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{L}_2\|_{\mathcal{L}}<1 \tag{25}$$

Proof. Take the operator equation in (21) as an example. Choose $R\ge R_0$ *where*

$$R_0=min\left\{\frac{\|\mathcal{M}_1\|_1\|\mathfrak{H}_1\|_\infty+\|g(.,.,0)\|_1}{\frac{\Gamma(\zeta_1)\Gamma(\zeta_2)}{T^{\zeta_1+\zeta_2}}-\mathfrak{F}\|\mathcal{L}_2\|_{\mathcal{L}}},\frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}(\|\mathcal{M}_1\|_1\|\mathfrak{H}_1\|_\infty+\|\mathcal{M}_2\|_1\|\mathfrak{H}_2\|_\infty)\right\}$$

According to the relation (25), it is obvious that $R_0>0$. *We define the convex, closed and bounded set* $\mathtt{B}_R=\left\{\mathfrak{u}\in\mathcal{L}^1(I,E):\|\mathfrak{u}\|_1\le R\right\}$, *and demonstrate* $\mathcal{A}$ *and* $\mathcal{B}$ *operator expressed by (22) by all of Theorem 1's hypotheses.*

Step1 : *We demonstrate the continuous weak sequential nature of* $\mathcal{A}=\bar{\mathbb{I}}\mathfrak{N}_f\mathcal{L}_1$ *and* $\mathcal{B}=\bar{\mathbb{J}}\mathfrak{N}_g\mathcal{L}_2$ *on* $\mathcal{L}^1(I,E)$. *To achieve this, we demonstrate that* $\mathfrak{N}_f\mathcal{L}_1$ *and* $\mathfrak{N}_g\mathfrak{L}_2$ *are weakly sequentially continuous on* $\mathcal{L}^1(I,E)$, *taking into account Lemma 1 and assumptions* $(\mathbf{A}_2)$. *In addition,* $\bar{\mathbb{I}}$ *and* $\bar{\mathbb{J}}$ *are linear continuous operators from* $\mathcal{L}^1(I,E)$ *into itself, and* $\mathcal{A}$ *and* $\mathcal{B}$ *are continuously weakly sequential on* $\mathcal{L}^1(I,E)$.

Step2 : *Now, illustrate that* $\mathcal{B}$ *is a functions of contractions generated by assuming that* $\mathfrak{u},\mathfrak{v}\in\mathcal{L}^1(I,E)$. *By the assumptions* $(\mathbf{A}_2)$ *and the Holder inequality then* $\forall(\upsilon_1,\upsilon_2)\in I$, *we then have*

$$\begin{aligned}\|\mathcal{B}\mathfrak{u}(\upsilon_1,\upsilon_2)-\mathcal{B}\mathfrak{v}((\upsilon_1,\upsilon_2))\|&\le\int_0^{\upsilon_1}\int_0^{\upsilon_2}(\upsilon_1-s)^{\zeta_1-1}(\upsilon_2-t)^{\zeta_2-1}\|g(s,t,\mathfrak{u}(s,t))-g(s,t,\mathfrak{v}(s,t))\|dsdt\\&\le\frac{\mathfrak{F}T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{L}_2\|_L\|\mathfrak{u}-\mathfrak{v}\|_1\end{aligned}$$

So, $\mathcal{B}$*is a mapping of contractions generated on* $\mathcal{L}^1(I,E)$ *with constant* $\frac{\mathfrak{F}T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{L}_2\|_L$

Step3 : *Let* $\mathcal{V}=\mathcal{A}\mathfrak{u}+\mathcal{B}\mathfrak{v}$ *Now, we prove that* $\mathcal{V}\in\mathtt{B}_R,\forall\mathfrak{u},\mathfrak{v}\in\mathtt{B}_R$, *indeed*

$$\|\mathcal{V}(\upsilon_1,\upsilon_2)\|=\|\mathcal{A}\mathfrak{u}(\upsilon_1,\upsilon_2)+\mathcal{B}v(\upsilon_1,\upsilon_2)\|$$

$$\leq \|\bar{\mathbb{I}}\mathfrak{N}_f\mathcal{L}_1\mathfrak{u}(\upsilon_1,\upsilon_2)\| + \|\bar{\mathbb{J}}\mathfrak{N}_g\mathcal{L}_2\mathfrak{v}(\upsilon_1,\upsilon_2)\|$$

by using the Assumptions $(\mathbf{A}_4),(\mathbf{A}_5)$*, we find*

$$\|\mathcal{V}(\upsilon_1,\upsilon_2)\| \leq \frac{T^{\zeta_1-1}T^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{M}_1\|_1\|\mathfrak{H}_1\|_\infty + \frac{T^{\zeta_1-1}T^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}(\mathfrak{F}\|\mathcal{L}_2\|_{\mathcal{L}}\|\mathfrak{v}\|_1 + \|g(.,.,0)\|_1), \quad (26)$$

$$\|\mathcal{V}(\upsilon_1,\upsilon_2)\| \leq \frac{T^{\zeta_1-1}T^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{M}_1\|_1\|\mathfrak{H}_1\|_\infty + \frac{T^{\zeta_1-1}T^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{M}_2\|_1\|\mathfrak{H}_2\|_\infty \quad (27)$$

Here, $\|\mathcal{L}_2\|_{\mathcal{L}}$ *represents the standard norm of linear operator spaces. In equation (26), (27) applying* $\mathcal{L}^1(I\times I,E)$*-norm on both sides*

$$\|\mathcal{V}\|_1 \leq \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{M}_1\|_1\|\mathfrak{H}_1\|_\infty + \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}(\mathfrak{F}\|\mathcal{L}_2\|_{\mathcal{L}}\|v\|_1 + \|g(.,.,0)\|_1)$$

and

$$\|\mathcal{V}\|_1 \leq \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{M}_1\|_1\|\mathfrak{H}_1\|_\infty + \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{M}_2\|_1\|\mathfrak{H}_2\|_\infty$$

Accordingly, $\mathcal{V} = \mathcal{A}u + \mathcal{B}v \in \mathtt{B}_R$ *for every* $u \in \mathtt{B}_R$ *for* $R \geq R_0$*.*
Step4 : *For each* $R \geq R_0$*, we use the De Blasi metric of weak noncompactness in* $\mathcal{L}^1(I,E)$ *to demonstrate that* $\mathcal{A}\mathtt{B}_R$ *is only moderately weakly compact. Let* S *be a bounded subset of* $\mathtt{B}_R$*, be a positive real integer, and let epsilon be a positive real number. We derive a result for every* $\mathfrak{u} \in S$ *and every nonempty subset* $\bar{\mathbb{J}}$ *of* $\bar{\mathbb{I}}$*.*

$$\begin{aligned}\iint_J \|N_f\mathcal{L}_1\mathfrak{u}(\upsilon_1,\upsilon_2)\|d\upsilon_1 d\upsilon_2 &\leq \iint_J \|\mathcal{M}_1(\upsilon_1,\upsilon_2)\mathfrak{H}_1(\|\mathcal{L}_1\mathfrak{u}(\upsilon_1,\upsilon_2)\|)\|d\upsilon_1 d\upsilon_2 \\ &\leq \|\mathfrak{H}_1\|_\infty \iint_J \|\mathcal{M}_1(\upsilon_1,\upsilon_2)\|d\upsilon_1 d\upsilon_2 \\ &\leq \|\mathfrak{H}_1\|_\infty\|\mathcal{M}_1\|_1\end{aligned}$$

By the Lemma 2 we get

$$\lim_{\epsilon\to 0}\sup\left\{\|\mathfrak{H}_1\|_\infty \iint_J \|\mathcal{M}_1(\upsilon_1,\upsilon_2)\|dt : \mathtt{meas}(J) \leq \epsilon\right\} = 0$$

The fact that $\omega(\mathfrak{N}_f\mathcal{L}_1S) = 0$ *suggests that* $\mathfrak{N}_f\mathcal{L}_1S$ *is a single somewhat compact. The fact that* I *is confined by the boundness of the Riemann-Liouville integral operator on* $\mathcal{L}^1(I)$ *further leads us to the conclusion that* $\omega(AS) = 0$ *and that* AS *is only moderately weakly compact. The theorem 1, which states that the operator* $\mathcal{A}+\mathcal{B}$ *has at least one fixed point on* I*, provides the solution to IVP (18).*

5. Example

Example 1. *We examine the following example to demonstrate how IVP (**18**) applies the obtained result.*

$$\begin{cases} [\mathcal{D}]^{\frac{3}{2}}\mathfrak{u}(\upsilon_1,\upsilon_2) = (\upsilon_1+\upsilon_2)\sqrt{2}\sin\mathfrak{u}(\upsilon_1,\upsilon_2) + \frac{\sqrt{3}\upsilon_1\upsilon_2}{43}\mathcal{M}e^{-\mathfrak{u}(\upsilon_1,\upsilon_2)}(\upsilon_1,\upsilon_2) \in I\times I, I=[0,\frac{\pi}{2}] \\ \lim\limits_{\upsilon_1\to 0+}\lim\limits_{\upsilon_2\to 0+}\upsilon_1^{2-\zeta}\upsilon_2^{2-\zeta}\mathfrak{u}(\upsilon_1,\upsilon_2) = \lim\limits_{\upsilon_1\to 0+}\lim\limits_{\upsilon_2\to 0+}\upsilon_1^{2-\zeta}\upsilon_2^{2-\zeta}u'(\upsilon_1,\upsilon_2) = 0 \end{cases} \tag{28}$$

Consider $T=\frac{\pi}{2}, (E,\|.\|) = (\mathbb{R},|.|), \zeta=\zeta_1=\zeta_2=\frac{3}{2}$,

$$p(\upsilon_1,\upsilon_2) = \upsilon_1+\upsilon_2 \quad f(\upsilon_1,\upsilon_2,\mathfrak{u}(\upsilon_1,\upsilon_2)) = \sqrt{2}\sin\mathfrak{u}(\upsilon_1,\upsilon_2) \quad g(\upsilon_1,\upsilon_2,\mathcal{M}\mathfrak{u}(\upsilon_1,\upsilon_2)) = \frac{\sqrt{3}\upsilon_1\upsilon_2}{43}e^{-\mathfrak{u}(\upsilon_1,\upsilon_2)}$$

where $\mathcal{M}:\mathcal{L}^1(I\times I,\mathbb{R})\to\mathcal{L}^1(I\times I,\mathbb{R}), \mathcal{M}\mathfrak{u}(\upsilon_1,\upsilon_2) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\mathfrak{u}(\upsilon_1,\upsilon_2)d\upsilon_1 d\upsilon_2$, *It is clear that*

$$|f(\upsilon_1,\upsilon_2,\mathfrak{u}(\upsilon_1,\upsilon_2))| \le \mathfrak{M}|\mathfrak{u}(\upsilon_1,\upsilon_2)|, \quad |g(\upsilon_1,\upsilon_2,\mathcal{M}\mathfrak{u}(\upsilon_1,\upsilon_2))| \le \mathcal{M}(\upsilon_1,\upsilon_2)|\mathfrak{u}(\upsilon_1,\upsilon_2)|$$

where, $\mathcal{M}(\upsilon_1,\upsilon_2) = \frac{\sqrt{3}\upsilon_1\upsilon_2}{43}$ *and* $\|\mathcal{M}\|_1 = \frac{\sqrt{3}\pi^4}{2752}, |\mathfrak{u}(\upsilon_1,\upsilon_2)| \le 1 \ \gamma = \frac{\sqrt{3}\upsilon_1\upsilon_2}{43}$ *and* $c = \frac{\sqrt{2}}{65}, \quad \|p\|_1 = \frac{\pi^3}{8} \ \frac{T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}(\mathsf{c}\|p\|_{\mathcal{L}^\infty} + \Upsilon\|M\|_{\mathcal{L}}) = 0.30745 < 1$ *which we concluded that the circumstances* $(\mathbf{A}_1)$ *through* $(\mathbf{A}_3)$ *are met. In* $\mathcal{L}^1(I,\mathcal{S}^n)$*, IVP (**18**) has at least one solution.*

Example 2. *We look at the following example to show how the obtained result can be applied.*

$$\begin{cases} [\mathcal{D}]^{\zeta}\mathfrak{u}(\upsilon_1,\upsilon_2) = \sqrt{2}\Big(\frac{(\upsilon_1+\upsilon_2)}{15}e^{-\mathcal{H}_1\mathfrak{u}(\upsilon_1,\upsilon_2)} + \frac{\sqrt{2}\upsilon_1^2\upsilon_2}{90}(\mathcal{H}_2\sin(\mathfrak{u}(\upsilon_1,\upsilon_2))\Big), (\upsilon_1,\upsilon_2)\in I\times I, I=[0,\frac{\pi}{2}] \\ \lim\limits_{\upsilon_1\to 0+}\lim\limits_{\upsilon_2\to 0+}\upsilon_1^{2-\zeta}\upsilon_2^{2-\zeta}\mathfrak{u}(\upsilon_1,\upsilon_2) = \lim\limits_{\upsilon_1\to 0+}\lim\limits_{\upsilon_2\to 0+}\upsilon_1^{2-\zeta}\upsilon_2^{2-\zeta}u'(\upsilon_1,\upsilon_2) = 0 \end{cases} \tag{29}$$

Consider $T=\frac{\pi}{2}, (E,\|.\|) = (\mathbb{R},|.|), \zeta=\zeta_1=\zeta_2=\frac{3}{2}$,

$$f(\upsilon_1,\upsilon_2,\mathfrak{u}(\upsilon_1,\upsilon_2)) = \sqrt{2}\frac{(\upsilon_1+\upsilon_2)}{15}e^{-\mathfrak{u}(\upsilon_1,\upsilon_2)}, \quad g(\upsilon_1,\upsilon_2,\mathfrak{u}(\upsilon_1,\upsilon_2)) = \sqrt{2}\frac{\upsilon_1^2\upsilon_2}{90}\sin(\mathfrak{u}(\upsilon_1,\upsilon_2))$$

where $\mathcal{H}_1,\mathcal{H}_2:\mathcal{L}^1(I\times I,\mathbb{R})\to\mathcal{L}^1(I\times I,\mathbb{R}), \mathcal{H}_1\mathfrak{u}(\upsilon_1,\upsilon_2) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\mathfrak{u}(\upsilon_1,\upsilon_2)d\upsilon_1 d\upsilon_2$,
$\mathcal{H}_2\mathfrak{u}(\upsilon_1,\upsilon_2) = k(\upsilon_1,\upsilon_2)\mathfrak{u}(\upsilon_1,\upsilon_2)$ *with* $k: C(I\times I,\mathbb{R})\to C(I\times I,\mathbb{R}), \ \overline{\kappa} = \max_{(\upsilon_1,\upsilon_2)\in I\times I}k(\upsilon_1,\upsilon_2)$
It is clear that

$$|f(\upsilon_1,\upsilon_2,\mathfrak{u}(\upsilon_1,\upsilon_2))| \le \mathcal{M}_1(\upsilon_1,\upsilon_2)\mathfrak{H}_1(|\mathfrak{u}(\upsilon_1,\upsilon_2)|), |g(\upsilon_1,\upsilon_2,\mathfrak{u}(\upsilon_1,\upsilon_2))| \le \mathcal{M}_2(\upsilon_1,\upsilon_2)\mathfrak{H}_2(|\mathfrak{u}(\upsilon_1,\upsilon_2)|)$$

where $\mathcal{M}_1(\upsilon_1,\upsilon_2) = \sqrt{2}\frac{\upsilon_1+\upsilon_2}{15}, \mathcal{M}_2(\upsilon_1,\upsilon_2) = \sqrt{2}\frac{\upsilon_1^2\upsilon_2}{90}\|\mathcal{M}_1\|_1 = \frac{\sqrt{2}\pi^3}{120}, \|\mathcal{M}_2\|_1 = \frac{\sqrt{2}}{90}\left(\frac{\pi^5}{192}\right)$
and $\mathfrak{H}_1,\mathfrak{H}_2$ *are positive non-decreasing functions defined on* $\mathbb{R}_+$ *by* $\mathfrak{H}_1(u) = e^{-\mathcal{H}_1\mathfrak{u}(\upsilon_1,\upsilon_2)}, \mathfrak{H}_2(u) = \sin(\upsilon_1+\mathcal{H}_2\mathfrak{u}(\upsilon_1,\upsilon_2))$*, and*

$$\|\mathfrak{H}_1\|_\infty = \|\mathfrak{H}_2\|_\infty = 1, \|\mathcal{H}_1\|_{\mathcal{L}} = \frac{\pi}{2}, \|\mathcal{H}_2\|_{\mathcal{L}} = \overline{\kappa}$$

$$|g(\upsilon_1,\upsilon_2,u)-g(\upsilon_1,\upsilon_2,v)| \leq \Big(\frac{\sqrt{2}\pi^3}{720}\Big)|u-v|, \quad \mathfrak{F}=\frac{\sqrt{2}\pi^3}{720}$$

$$\begin{aligned} g(\upsilon_1,\upsilon_2,0) &= \frac{\sqrt{2}\upsilon_1^2\upsilon_2}{90} \\ \|g(\upsilon_1,\upsilon_2,0)\|_1 &= \frac{\sqrt{2}}{90}\Big(\frac{\pi^5}{192}\Big) \end{aligned}$$

On the other hand, for $k(\upsilon_1,\upsilon_2)=\frac{1}{2}\cos(\upsilon_1,\upsilon_2)$*, we have* $\frac{\mathfrak{F}T^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\mathcal{H}_2\|_{\mathcal{L}} < 0.30053971 < 1$ *and* $R_0 = min\{2.267578, 1.890631\} = 1.890631$*. As a result, the issue has at least one integrable solution.*

6. Conclusion

The exploration commitment for this study was to explore the hypothesis of fixed point under weak topology point, the De Blasi proportion of weak non-compactness and the hypothesis of fractional differential condition. We reviewed every one of the required definitions and sayings connected with the weak topology and fractional analytic. Besides, we changed over the IVP into Volterra type essential condition and we demonstrated the alteration of the fixed point hypothesis. we acknowledge the methods and significance are intend to investigate various fractional derivative type, measures and fixed point techniques in future extensions. Lastly, we note that this type of fixed point theorem has not yet been applied to the field of differential equations of fractional orders.

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