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Connected Disjunctive Domination in Graphs

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Abstract. A set S of vertices of a graph G is a disjunctive dominating set if for every $v \in V(G) \setminus S$, v is adjacent to a vertex in S or S contains two vertices each of distance two from v. A disjunctive dominating set S is a connected disjunctive dominating set if $\langle S \rangle$ is connected. In this paper, we study the concept of connected disjunctive dominating set.

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Key Words and Phrases: Connected disjunctive dominating set, connected disjunctive domination number, join, corona, lexicographic product

1. Introduction

In 2014, Goddard et al.[1] introduced the concept of disjunctive, specifically b-disjunctive, domination in graphs. While most of the variations on dominating sets tend to increase the domination number, which in effect raise implementation costs, disjunctive domination is a relaxation of the domination number [2]. In [1], sharp bounds for the disjunctive domination number were established for general graphs, and exact values were determined for specific graphs.

In 2016, Henning and Naicker [2] introduced the disjunctive total domination. Accordingly, it allows for greater flexibility by modeling networks where one trades off redundancy and backup capability with resource optimization. The above-mentioned authors established in [2] tight upper bound on the disjunctive total domination number of a graph in terms of its order and characterized the extremal graphs, and then proved that this bound can be significantly improved if claw-freeness of a graph is imposed. The same authors also investigated the variant on the class of trees in [3]. In [4, 5], Malalay and Jamil explored

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both the disjunctive domination and disjunctive total domination, and initiated the study of restrained disjunctive domination in graphs under some binary operations.

In this present paper, we introduce and initiate the study of connected disjunctive domination. First, we investigate the concept for some special graphs, and characterize the graphs which give small values for the corresponding parameter. Then we study connected disjunctive domination for families of graphs involving some binary operations.

2. Terminology and Notations

Throughout this paper, we only consider graphs which are finite, simple and undirected. All basic terminologies which are not defined but are being used here are adapted from [6]. For a graph G, the symbols $V(G)$ and $E(G)$ refer to the *vertex-set* and *edge-set*, respectively, of G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ is called the *order* of G. A graph G is *connected* if for every pair of distinct vertices u and v of G, G contains a path from u to v. If G is connected and $u, v \in V(G)$, then $d_G(u, v)$, the *distance from* u *to* v, is the length of the shortest path connecting u and v. A vertex u is a *cut-vertex* if the removal of u from G increases the number of components of G.

Given graphs G and H, the *join* of G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. In particular, we call $G \circ K_1$ the corona of G, and write $cor(G) = G \circ K_1$. The *composition* (or *lexicographic product*) of G and H is the graph $G[H]$ with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. In any of these graphs, G and H are referred to as their basic component graphs.

Vertices u and v of a graph G are *neighbors* if $uv \in E(G)$. The *open neighborhood* of v refers to the set $N_G(v)$ consisting of all neighbors of v. The *degree* of v, denoted $deg_G(v)$, refers to the cardinality $|N_G(v)|$ of the open neighborhood of v. Vertex v is an *end-vertex* if $deg_G(v) = 1$. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = \cup_{v \in S} N_G[v]$. A subset $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. If $N_G(S) = V(G)$, then S is a *total dominating set* of G. The minimum cardinality of a dominating set of G is the *domination number* of G, and the minimum cardinality of a total dominating set is the *total domination number of* G. We write $\gamma(G)$ and $\gamma_t(G)$ to denote the domination number and total domination number, respectively, of G. A dominating set of cardinality $\gamma(G)$ is called a γ*-set* of G. Similarly, a γ_t*-set* is a total dominating set of cardinality $\gamma_t(G)$. The reader is referred to [7–10] for the history, fundamental concepts and recent developments of domination in graphs as well as its various applications, and to [11–13] for studies whose primary emphasis is on total domination in graphs.

A dominating set S is a *connected dominating set* of G provided the subgraph $\langle S \rangle$ induced by S is connected. The minimum cardinality of a connected dominating set, which is denoted by $\gamma_c(G)$, is the *connected domination number* of G. The problem of connected domination would arise in real life in the following scenario. An existing computer network with direct connections described by a graph G must have the property that any computer turned on must always be able to send a message to any other computer turned on. One can make sure a computer is always on by connecting it to an (expensive) unlimited power supply (UPS) source. The requirement is met by connecting only the computers in a connected dominating set to such power sources.

A set $S \subseteq V(G)$ is a 2-*dominating set* of G if for each $v \in V(G) \setminus S$, $|N_G(v) \cap S| \geq 2$. It is a *connected* 2-*dominating set* of G if it is a 2-dominating set and $\langle S \rangle$ is connected. The minimum cardinality of a 2-dominating set (resp. connected 2-dominating set) is the 2-*domination number* (resp. *connected* 2-*domination number*) of G, denoted by $\gamma_{\times 2}(G)$ (resp. $\gamma_{\times 2,c}(G)$). Any 2-dominating (resp. connected 2-dominating) set with cardinality $\gamma_{\times 2}(G)$ (resp. $\gamma_{\times 2,c}(G)$) is called a $\gamma_{\times 2}$-*set* (resp. $\gamma_{\times 2,c}$-set) of G. Excellent references for studies of 2-domination include [14–19].

For a vertex v of G, $N_G(v, 2) = \{u \in V(G) \setminus \{v\} : d_G(u, v) \leq 2\}$. For $S \subseteq V(G)$, $N_G(S, 2) = \cup_{v \in S} N_G(v, 2)$. A set $S \subseteq V(G)$ is a *distance-two dominating set* of G provided $V(G) \setminus S \subseteq N_G(S, 2)$, i.e., if for every $v \in V(G) \setminus S$ there exists $u \in S$ such that $d_G(u, v) \leq 2$. A distance-two dominating set S is a *connected distance-two dominating set* if $\langle S \rangle$ is connected. The minimum cardinality of a distance-two dominating set (resp. connected distance-two dominating set) is the *distance-two domination number* (respectively *connected distance-two domination number*) of G. We use the symbols $\gamma_2(G)$ and $\gamma_{2,c}(G)$ for the distance-two domination number and connected distance-two domination number, respectively, of G. A distance-two dominating set (resp. connected distance-two dominating set) of cardinality $\gamma_2(G)$ (resp. $\gamma_{2,c}(G)$) is called a γ_2-*set* (resp. $\gamma_{2,c}$-*set*) . The articles in [20, 21] are good references for studies in distance-two domination. In [21] the connected distance-two domination is called connected 2-distance domination.

A subset $S \subseteq V(G)$ is a *connected distance-two dominating set* of G if S is a distance-two dominating set of G for which $\langle S \rangle$ is connected. In [21], the same is called *connected* 2-*distance dominating set*. We used $\gamma_{2,c}(G)$ to denote the minimum cardinality of a connected distance-two dominating set of G.

A set $S \subseteq V(G)$ is a *disjunctive dominating set* of G if for every $v \in V(G) \setminus S$, v is a neighbor of a vertex in S or S has at least two vertices each at distance 2 from v. Provided G has no isolated vertex, $S \subseteq V(G)$ is a *disjunctive total dominating set* if for every $v \in V(G)$, v is adjacent to a vertex of S or S has at least two vertices each at distance 2 from v. The minimum cardinality of a disjunctive dominating set (resp. disjunctive total dominating set) is the *disjunctive domination number* (resp. *disjunctive total domination number*) of G. We write $\gamma^d(G)$ and $\gamma_t^d(G)$ to denote the disjunctive domination number and disjunctive total domination number, respectively, of G. A disjunctive dominating set of cardinality $\gamma^d(G)$ is called a γ^d-*set*. Any disjunctive total dominating set of cardinality $\gamma_t^d(G)$ is called γ_t^d-*set*.

For convenience, the symbol $N_G^d(S)$ denotes the set of all $x \in V(G)$ such that $xy \in$

$E(G)$ for some $y \in S$ or there exist distinct $u, v \in S$ with $d_G(x,u) = 2 = d_G(x,v)$. Precisely, S is a disjunctive dominating set (resp. disjunctive total dominating set) of G if and only if $V(G) \setminus S \subseteq N^d_G(S)$ (resp. $V(G) = N^d_G(S)$. Since $N_G(S) \subseteq N^d_G(S)$, dominating sets are disjunctive dominating sets. In particular, $\gamma^d(G) = 1$ if and only if $\gamma(G) = 1$; and if $\gamma(G) = 2$, then $\gamma^d(G) = 2$, but not conversely. Note, for example, that for path P_3 on 3 vertices and any graph G, $\gamma^d(P_3 \circ G) = 2$ while $\gamma(P_3 \circ G) = 3$. Also, since total dominating sets are disjunctive total dominating sets, $\gamma^d_t(G) \leq \gamma_t(G)$ for all graphs G without isolated vertices. In particular, if $\gamma_t(G) = 2$, then $\gamma^d_t(G) = 2$. The converse, however, need not be true. Note that for cycle C_5, $\gamma_t(C_5) = 3$ but $\gamma^d_t(C_5) = 2$.

3. Results

A disjunctive dominating set S is a *connected disjunctive dominating set* provided $\langle S \rangle$ is connected. The minimum cardinality of a connected disjunctive dominating set of G, denoted by $\gamma^d_c(G)$, is the connected disjunctive domination number of G. Any connected disjunctive dominating set of cardinality $\gamma^d_c(G)$ is called a γ^d_c*-set* of G.

We denote by P_n, C_n, K_n and $\overline{K_n}$ the path, cycle, complete graph and empty graph on n vertices. For positive integers $n_1, n_2, \ldots, n_k$, we denote by $K_{n_1,n_2,\ldots,n_k}$ the complete multipartite graph with partite sets $U_{n_1}, U_{n_2}, \ldots, U_{n_k}$ with $|U_{n_j}| = n_j$ for each $j \in \{1, 2, \ldots, k\}$. In particular, a star on $n+1$ vertices is denoted by $K_{1,n}$.

A *subdivision* of an edge $uv \in E(G)$ is obtained by inserting in G a new vertex w and replacing the edge uv by the edges uw and wv. A *spider* is the graph obtained from a star by subdividing all of the edges. A *wounded spider* is any graph obtained from a spider by removing at least one endvertex.

For convenience, let $S_{k,0}$ denote any spider with k endvertices, and let $S_{k,j}$ $(1 \leq j \leq k)$ denote the wounded spider obtained from $S_{k,0}$ by removing j end-vertices.

Observation 1. *For paths, cycles, complete multipartite graphs and spiders,*

(i) $\gamma^d_c(P_n) = \begin{cases} 1, & \text{if } n = 1, 2; \\ n-2, & \text{if } n \geq 3. \end{cases}$

(ii) $\gamma^d_c(C_n) = \begin{cases} 1, & \text{if } n = 3; \\ 2, & \text{if } n = 4; \\ n-3, & \text{if } n \geq 5. \end{cases}$

(iii) If $k \geq 2$ and $n_1 \leq n_2 \leq \ldots \leq n_k$, then

$$\gamma^d_c(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} 1, & \text{if } n_1 = 1 \\ 2, & \text{otherwise.} \end{cases}$$

(iv) *For $n \geq 2$ and $k \in \{0, 1, \ldots, n\}$,*

$$\gamma_c^d(S_{n,k}) = \begin{cases} n+1, & \text{if } k = 0 \\ n-k+1, & \text{otherwise.} \end{cases}$$

Proposition 1. *Let G be a connected graph. Then*

$$\gamma^d(G) \leq \gamma_c^d(G) \leq \min\{\gamma_c(G), 5\gamma^d(G) - 4\}. \tag{1}$$

Proof. Since connected disjunctive dominating sets are disjunctive dominating sets, the left-hand inequality in (1) follows immediately. Also, since connected dominating sets are connected disjunctive dominating sets, $\gamma_c^d(G) \leq \gamma_c(G)$. Let $S \subseteq V(G)$ be a γ^d-set of G, and let m be the number of components in $\langle S \rangle$. We claim that $\gamma_c^d(G) \leq \gamma^d(G)+4(m-1)$. If $m = 1$, then S is a connected disjunctive dominating set of G so that $\gamma_c^d(G) = \gamma^d(G) = |S|$, and the desired inequality holds. Suppose that $m \geq 2$. For distinct components C_i and C_j of $\langle S \rangle$, define $d_G(C_i, C_j) = \min\{d_G(u, v) : u \in V(C_i), v \in V(C_j)\}$. Let C_i and C_j be distinct components of $\langle S \rangle$ for which $d_G(C_i, C_j)$ is minimum. Let $u \in V(C_i)$ and $v \in V(C_j)$ for which $d_G(C_i, C_j) = d_G(u, v)$. Suppose that $d_G(u, v) \geq 6$, and let P be a u-v geodesic $[u = x_1, x_2, x_3, x_4, x_5, x_6, x_7, \ldots, x_n = v]$ in G. Since S is a disjunctive dominating set of G, in particular, there exists $w \in S$ such that $d_G(w, x_4) \leq 2$. If $w \in C_i$, then there exists a w-v geodesic joining C_i and C_j of length less than the length of P. If $w \notin C_i$, then there is a w-u geodesic that joins two distinct components of $\langle S \rangle$ with length shorter than the length of P. Either case is a contradiction to the definitions of C_i and C_j. Thus, $d_G(u, v) \leq 5$. Put $S_1 = S \cup (V(P) \setminus \{u, v\}$. Then S_1 is a disjunctive dominating set of G with $|S_1| \leq |S| + 4$ and $\langle S_1 \rangle$ having at most $m - 1$ components. Repeating the same process in at most $(m-1)$ times yields a connected disjunctive dominating set S_{m-1} with $|S_{m-1}| \leq |S| + 4(m-1)$. Thus, $\gamma_c^d(G) \leq \gamma^d(G) + 4(m-1)$, and the claim is established. Since $m \leq \gamma^d(G)$, we have $\gamma_c^d(G) \leq \gamma^d(G) + 4(\gamma^d(G) - 1) = 5\gamma^d(G) - 4$. □

If $\alpha = \min\{\gamma_c(G), 5\gamma^d(G) - 4\}$, then for $G = C_5$, $\alpha = \gamma_c(G)$. For the graph G given in Figure 1, $\alpha = 5\gamma^d(G) - 4$.

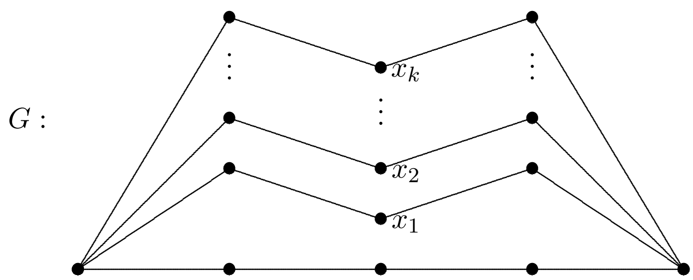

Figure 1: A graph G with $\gamma_c(G) = 5 + k$, $5\gamma^d(G) - 4 = 6$ and $\gamma_c^d(G) = 5$

Proposition 2. *Let G be a connected graph. Then* *(i)* $\gamma_c^d(G) = 1$ *if and only if* $\gamma(G) = 1$.

(ii) $\gamma_c^d(G) = 2$ *if and only if* $\gamma(G) \neq 1$ *and there exist* $u, v \in V(G)$ *for which* $uv \in E(G)$ *and* $d_G(u, z) \leq 2$ *and* $d_G(v, z) \leq 2$ *for all* $z \in V(G)$. *In particular, if* $\gamma_c(G) = 2$, *then* $\gamma_c^d(G) = 2$.

(iii) *If* G *is a tree, then* $\gamma_c^d(G) = \gamma_c(G)$.

Proof. The proof of statement (i) utilizes (1) in Proposition 1. If $\gamma_c^d(G) = 1$, then $\gamma^d(G) = 1$ so that $\gamma(G) = 1$. Conversely, if $\gamma(G) = 1$, then $\gamma_c(G) = 1$ so that $\gamma_c^d(G) = 1$, and (i) holds. Suppose that $\gamma_c^d(G) = 2$. By (i), $\gamma(G) > 1$. Let $\{u, v\}$ be a γ_c^d-set of G, and let $z \in V(G)$. If $z \in N_G[u]$, then $d_G(u, z) \leq 1$ so that $d_G(v, z) \leq 2$. Similarly, if $z \in N_G[v]$, then $d_G(v, z) \leq 1$ so that $d_G(u, z) \leq 2$. Suppose that $z \notin N_G[u] \cup N_G[v]$. Then since $\{u, v\}$ is a disjunctive dominating set, $d_G(z, u) = 2 = d_G(v, z)$. Conversely, by (i), $\gamma_c^d(G) \geq 2$. Further, since u and v constitute a connected disjunctive dominating set, $\gamma_c^d(G) = 2$, and (ii) holds. To prove (iii), we only have to show that $\gamma_c(G) \leq \gamma_c^d(G)$. Let $S \subseteq V(G)$ be a γ_c^d-set of G. Suppose $N_G[S] \neq V(G)$, and let $x \in V(G) \setminus N_G[S]$. Since S is a disjunctive dominating set, there exist distinct vertices $u, v \in S$ for which $d_G(x, u) = 2 = d_G(x, v)$. Since $\langle S \rangle$ does not contain a u-v path, $\langle S \rangle$ is not connected, a contradiction. This shows that S is a connected dominating set of G. Consequently, $\gamma_c(G) \leq |S| = \gamma_c^d(G)$. □

Remark 1. *The bound given for* $\gamma_c^d(G)$ *in Proposition 1 is tight. Indeed, if* $\gamma(G) = 1$, *then* $\gamma_c^d(G) = 5\gamma^d(G) - 4 = \gamma_c(G) = 1$.

Proposition 3. *For every pair of positive integers* a *and* b *with* $2 \leq a \leq b \leq 2a - 1$, *there exists a connected graph* G *for which* $\gamma^d(G) = a$ *and* $\gamma_c^d(G) = b$.

Proof. If $a = b = 2$, then we take $G = P_4$. Suppose that $a = b \geq 3$. Let $P_a = [x_1, x_2, \ldots, x_a]$ be a path on a vertices. Obtain G as the graph G_1 in Figure 2 by adding

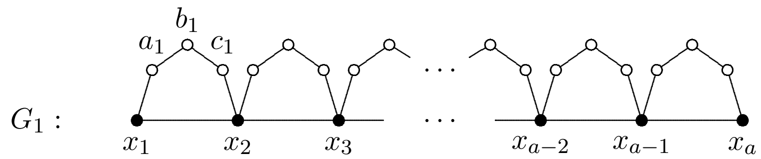

Figure 2: Graph G for which $\gamma^d(G) = \gamma_c^d(G)$

to P_a the path $P^j = [x_j, a_j, b_j, c_j, x_{j+1}]$ for each $j = 1, 2, \ldots, a-1$ such that two distinct P^js may intersect but only at a vertex on P_a. Then $V(P_a)$ is both a γ^d-set and γ_c^d-set of G. Thus, $\gamma^d(G) = \gamma_c^d(G) = a$.

Now, assume that $a < b$, and let $b = a + k$, where $k \leq a - 1$. If $a = k + 1$, then $b = 2k + 1$. In this case, obtain G from P_{2k+1} as the graph G_2 in Figure 3 by adding to $P_{2k+1} = [x_1, x_2, \ldots, x_{2k+1}]$ the path $P^j = [x_j, a_j, b_j, c_j, x_{j+2}]$ for each $j = 1, 2, \ldots, 2k-1$ such that two distinct P^js may intersect but only at a vertex on P_{2k+1}. Then $\gamma^d(G) =$

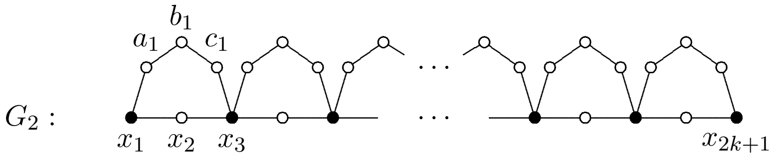

Figure 3: Graph G with $\gamma^d(G) = k+1$ and $\gamma_c^d(G) = \gamma^d(G) + k$

$k+1 = a$, which is determined by the γ^d-set $\{x_1, x_3, x_5, \ldots, x_{2k+1}\}$. Also, $\gamma_c^d(G) = 2k+1 = b$, and $V(P_{2k+1})$ is a γ_c^d-set of G. Suppose that $a = k+2$. Extend the graph G_2 in Figure 3 to a graph $G = G_3$ as in Figure 4 by adding $P_3 = [u, v, x_1]$ through

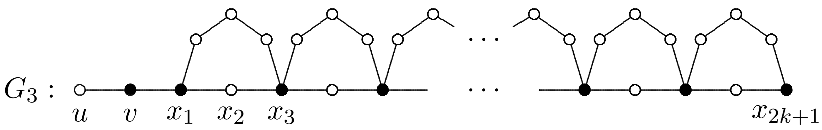

Figure 4: Graph G with $\gamma^d(G) = k+2$ and $\gamma_c^d(G) = \gamma^d(G) + k$

x_1. Then $\{v, x_1, x_3, x_5, \ldots, x_{2k+1}\}$ is a γ^d-set of G and $V(P_{2k+1}) \cup \{v\}$ is a γ_c^d-set of G. Thus, $\gamma^d(G) = k+2 = a$ and $\gamma_c^d(G) = 2k+2 = b$. Finally, suppose that $a \geq k+3$.

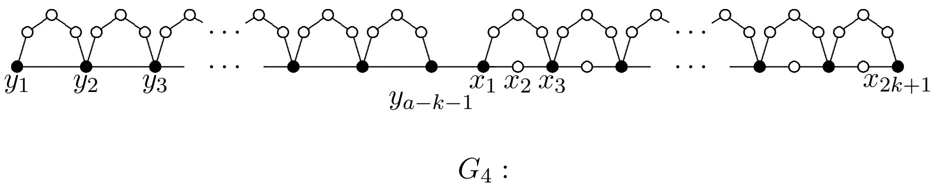

Figure 5: Graph G with $\gamma^d(G) \geq k+3$ and $\gamma_c^d(G) = \gamma^d(G) + k$

Then $a - (k+1) \geq 2$. Form a graph G_1^* similar to the graph G_1 in Figure 2 but using P_{a-k-1} and obtain G as the graph G_4 in Figure 5 by combining G_1^* and the graph G_2 in Figure 3 by adding the edge $y_{(a-k-1)}x_1$. Then $\gamma^d(G) = (a-k-1) + (k+1) = a$ and $\gamma_c^d(G) = (a-k-1) + (2k+1) = a+k = b$. □

Corollary 1. *The difference $\gamma_c^d(G) - \gamma^d(G)$ can be made arbitrarily large.*

Proposition 4. *For every pair of positive integers a and b with $a \leq b$, there exists a connected graph G for which $\gamma_c^d(G) = a$ and $\gamma_c(G) = b$.*

Proof. If $a = b$, then we take, in particular, $G = P_{a+2}$. By Observation 1 and Proposition 2(iii), $\gamma_c^d(G) = \gamma_c(G) = a = b$. Suppose that $a < b$, say $b = a+k$, where $k \geq 1$. Obtain the graph G from $P_a = [x_1, x_2, \ldots, x_a]$, as provided in Figure 6, by adding to P_a the path $[x_1, a_j, b_j, c_j, x_2]$ for all $j = 1, 2, \ldots, k$. Then $V(P_a)$ and $V(P_a) \cup \{c_j : j = 1, 2, \ldots, k\}$ are a γ_c^d-set and a γ_c-set, respectively, of G. Thus, $\gamma_c^d(G) = a$ and $\gamma_c(G) = a+k = b$. □

Corollary 2. *The difference $\gamma_c(G) - \gamma_c^d(G)$ can be made arbitrarily large.*

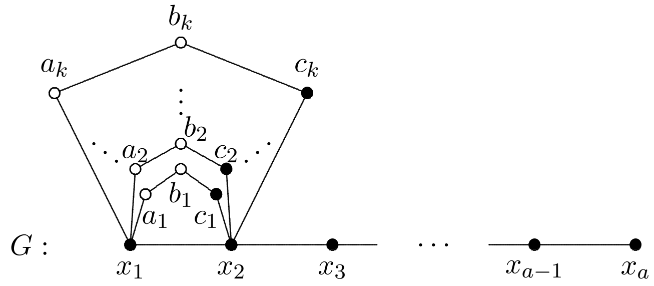

Figure 6: Graph G for which $\gamma_c(G) = \gamma_c^d(G) + k$

4. In the Join of Graphs

Observation 2. *For any graphs G and H, if $S \subseteq V(G+H)$ intersects both $V(G)$ and $V(H)$, then S is a connected dominating set, hence is a connected disjunctive dominating set of $G+H$.*

Theorem 1. *Let G and H be any graphs, and $S \subseteq V(G+H)$. Then S is a connected disjunctive dominating set of $G+H$ if and only if one of the following holds:*

(*i*) *$S \subseteq V(G)$ (resp. $S \subseteq V(H)$) for which either $|S| \geq 2$ and $\langle S \rangle$ is connected or $S = \{x\}$ where $N_G[x] = V(G)$ (resp. $N_H[x] = V(H)$).*

(*ii*) *$S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$.*

Proof. Let S be a connected disjunctive dominating set of $G+H$. Suppose that (ii) does not hold for S, say $S \subseteq V(G)$. The conclusion is obvious for a trivial graph G. Suppose that $|V(G)| \geq 2$. If $|S| \geq 2$, then $\langle S \rangle$ is connected subgraph of G. Suppose that $|S| = 1$, say $S = \{x\}$. Since S is a disjunctive dominating set, $V(G) = N_G[x]$. This proves (i).

Conversely, suppose that $S \subseteq V(G)$ satisfying property (i). Then $V(H) \subseteq N_{G+H}(S)$. If $S = \{x\}$ for which $N_G[x] = V(G)$, then S is a dominating set, hence a connected disjunctive dominating set of $G+H$. Suppose that $|S| \geq 2$. Let $x \in V(G) \setminus S$, and suppose that $x \notin N_G(S)$. Pick any distinct $u, v \in S$. Then $d_{G+H}(x,u) = 2 = d_{G+H}(x,v)$. Thus, S is a disjunctive dominating set of $G+H$. Since $\langle S \rangle$ is connected, the conclusion follows. Similarly, if $S \subseteq V(H)$ for which either $|S| \geq 2$ and $\langle S \rangle$ is connected or $S = \{x\}$ where $N_H[x] = V(H)$, then S is a connected disjunctive dominating set of $G+H$. Observation 2 finally establishes the necessity proof. □

Corollary 3. *For nontrivial graphs G and H,*

$$\gamma_c^d(G+H) = \min\{\gamma(G), \gamma(H), 2\}.$$

5. In the Corona of Graphs

In a corona $G \circ H$, we denote by H^v that copy of H which is being joined to the vertex v of G. We also denote by $H^v + v$ that subgraph $\langle \{v\} \cup V(H^v) \rangle$ of $G \circ H$ induced by

$\{v\} \cup V(H^v)$. Thus,

$$V(G \circ H) = V(G) \cup \left(\cup_{v \in V(G)} V(H^v)\right).$$

Theorem 2. [4] *Let G be a nontrivial connected graph and H any graph, and let $S \subseteq V(G \circ H)$. Then S is a disjunctive dominating set of $G \circ H$ if and only if each of the following holds for S:*

(i) $|S \cap N_G(v)| \geq 2$ *for all* $v \in V(G) \setminus S$ *with* $S \cap V(H^v) = \varnothing$;

(ii) $|S \cap V(H^v)| \geq 1$ *for all* $v \in V(G) \setminus S$ *with* $|S \cap N_G(v)| = 1$; *and*

(iii) $S \cap V(H^v)$ *is a disjunctive dominating set of* $H^v + v$ *for all* $v \in V(G) \setminus S$ *with* $S \cap N_G(v) = \varnothing$. *In particular, if* $\gamma(H) > 1$, *then* $|S \cap V(H^v)| \geq 2$.

Proposition 5. *Let G be a nontrivial connected graph and let H be any graph. Then a set $S \subseteq V(G \circ H)$ is a connected disjunctive dominating set of $G \circ H$ if and only if*

$$S = A \cup \left(\cup_{v \in A} S_v\right), \tag{2}$$

where A is a connected 2-dominating set of G and $S_v \subseteq V(H^v)$ for all $v \in A$.

Proof. Let $S \subseteq V(G \circ H)$ be a connected disjunctive dominating set of $G \circ H$. Put $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then

$$S = A \cup \left(\cup_{v \in A} S_v\right) \cup \left(\cup_{v \in V(G) \setminus A} S_v\right).$$

Since G is nontrivial and $\langle S \rangle$ is connected, $A \neq \varnothing$ and $\langle A \rangle$ is connected. We claim that $S_v = \varnothing$ for all $v \in V(G) \setminus A$, and Equation (2) holds. Suppose $v \in V(G) \setminus A$ for which $S_v \neq \varnothing$. Since $\langle S \rangle$ is connected, $S = S_v$. This is impossible because $A \neq \varnothing$. Hence, $S_v = \varnothing$. Consequently, $|A \cap N_G(v)| \geq 2$ by Theorem 2(i). Thus, $S = A \cup (\cup_{v \in A} S_v)$ and A is a (connected) 2-dominating set of G.

Conversely, assume that S has the form given in Equation 2, where A is a connected 2-dominating set of G and $S_v \subseteq V(H^v)$ for each $v \in A$. Then $|S \cap N_G(v)| = |A \cap N_G(v)| \geq 2$ for each $v \in V(G) \setminus S$. By Theorem 2, S is a disjunctive dominating set of $G \circ H$. Since $S_w = \varnothing$ for all $w \in V(G) \setminus A$ and $\langle A \rangle$ is connected, it follows that $\langle S \rangle$ is connected. Therefore, S is a connected disjunctive dominating set of $G \circ H$. □

Lemma 1. *Let G be a connected graph of order $n \geq 2$.*

(i) $\gamma_{\times 2,c}(G) = 2$ *if and only if* $G = K_2$ *or* $G = K_2 + H$ *for some graph* H.

(ii) $\gamma_{\times 2,c}(G) = n$ *if and only if for every* $v \in V(G)$, v *is either an end-vertex or a cut-vertex.*

(iii) $\gamma_{\times 2,c}(G) = n - 1$ *if and only if*

  (a) G *has a non-cut vertex with degree at least two, and*

(*b*) *for each non-cut vertex $v \in V(G)$ with $deg_G(v) \geq 2$, if w is a non-cut vertex of $H = \langle V(G) \setminus \{v\} \rangle$ with $deg_H(w) \geq 2$, then $deg_G(v) = 2$ and $w \in N_G(v)$.*

Proof. (*i*) Suppose $\gamma_{\times 2,c}(G) = 2$. If $n = 2$, then $G = K_2$. Suppose $n \geq 3$. Let $S = \{a, b\}$ be a $\gamma_{\times 2,c}$-set of G. Since $\langle S \rangle$ is connected, it follows that $ab \in E(G)$. Let $G_1 = \langle \{a, b\} \rangle$ and $H = \langle V(G) \setminus \{a, b\} \rangle$. Let $h \in V(H)$. Since S is a 2-dominating set, $h \in N_G(a) \cap N_G(b)$. Thus, $G = G_1 + H \cong K_2 + H$. The converse is clear.

(*ii*) Suppose $\gamma_{\times 2,c}(G) = n$. Let $v \in V(G)$. Suppose v is neither an end-vertex nor a cut-vertex. Then $S = V(G) \setminus \{v\}$ is a connected 2-dominating set of G. This implies that $\gamma_2^c(G) \leq |S| = n - 1$, contrary to our assumption. Thus, v is end-vertex or a cut-vertex of G.

For the converse, suppose that every $v \in V(G)$ is either an end-vertex or a cut-vertex. Let $End(G)$ denote the set of all end-vertices of G and let S be a $\gamma_{\times 2,c}$-set of G. Since S is a 2-dominating set, $End(G) \subseteq S$. Suppose $S \neq V(G)$, say $x \in V(G) \setminus S$. Then x is a cut-vertex of G. Since $S \subseteq V(G) \setminus \{x\}$ and $\langle S \rangle$ is connected, $S \subseteq V(C_S)$ for some component C_S of $\langle V(G) \setminus S \rangle$. Let C be a component of $\langle V(G) \setminus S \rangle$ different from C_S and let $z \in V(C)$. Then $N_G(z) \cap S = \varnothing$, contrary to the assumption that S is a 2-dominating set. Thus, $S = V(G)$ and $\gamma_{\times 2,c}(G) = n$, showing that (*ii*) holds.

(*iii*) Suppose $\gamma_{\times 2,c}(G) = n - 1$ and let $S = V(G) \setminus \{v\}$ be a $\gamma_{\times 2,c}$-set of G. Then v is a non-cut vertex and $deg_G(v) \geq 2$. Let $H = \langle V(G) \setminus \{v\} \rangle$ and suppose there exists a non-cut-vertex $w \in V(H)$ with $deg_H(w) \geq 2$. Suppose that $w \notin N_G(v)$. Then $S' = V(G) \setminus \{v, w\}$ is a connected 2-dominating set of G, contrary to the assumption that S is a $\gamma_{\times 2,c}$-set. Hence, $w \in N_G(v)$. Suppose $deg_G(v) \geq 3$. Then, again, $V(G) \setminus \{v, w\}$ is a connected 2-dominating set of G, a contradiction. Therefore, $deg_G(v) = 2$.

For the converse, suppose G satisfies (*a*) and (*b*). If $n = 3$, then $G = C_3 = K_3$ and $\gamma_{\times 2,c}(G) = 2$. Suppose $n \geq 4$ and let S be a $\gamma_{\times 2,c}$-set of G. By (*a*), $|S| \leq n - 1$. Let $v \in V(G) \setminus S$. Since $\langle S \rangle$ is a connected 2-dominating set, it follows that v is a non-cut-vertex of G and $deg_G(v) \geq 2$. Suppose there exists $w \in V(G) \setminus (S \cup \{v\})$. Then w is also non-cut-vertex of G and $|N_G(w) \cap S| \geq 2$. Hence, w is a non-cut-vertex of $H = \langle V(G) \setminus \{v\} \rangle$ and $deg_(w) \geq 2$. By (*b*), it follows that $vw \in E(G)$ and $deg_G(v) = 2$. Thus, $|N_G(v) \cap S| \leq 1$, contrary to the assumption that S is a 2-dominating set. Therefore, $S = V(G) \setminus \{v\}$ and $\gamma_{\times 2,c}(G) = |S| = n - 1$. □

The next result is immediate from Proposition 5 and Lemma 1.

Corollary 4. *Let G be a connected graph of order n and let H be any graph. Then*

$$\gamma_c^d(G \circ H) = \gamma_{\times 2,c}(G).$$

In particular, the following hold:

(*i*) *If $G = K_n$ with $n \geq 2$, then $\gamma_c^d(G \circ H) = 2$.*

(*ii*) *If G is a tree, then $\gamma_c^d(G \circ H) = n$.*

(*iii*) *If $G = C_n$ with $n \geq 3$, then $\gamma_c^d(G \circ H) = n - 1$.*

6. In the lexicographic product of graphs

For any graphs G and H and for any $C \subseteq V(G[H])$, there exists $S \subseteq V(G)$ for which $C = \cup_{x\in S} (\{x\} \times T_x)$, where $T_x \subseteq V(H)$ for each $x \in S$. Moreover, if $\langle C\rangle$ is connected, then so is $\langle S\rangle$. Provided $|S| \geq 2$, the converse is also true.

For convenience, we write $N_G^d[S] = S \cup N_G^d(S)$.

Theorem 3. [4] *Let G and H be nontrivial connected graphs, and let $C = \cup_{x\in S} (\{x\} \times T_x)$. Then C is a disjunctive dominating set of $G[H]$ if and only if one of the following holds:*

(i) *S is a disjunctive total dominating set in G;*

(ii) *S is a distance-two dominating set of G satisfying the following:*

  (a) *For each $x \in V(G) \setminus N_G^d[S]$ there exists $u \in S$ for which $d_G(u, x) = 2$ and $|T_u| \geq 2$.*

  (b) *For each $x \in S \setminus N_G(S, 2)$, either $T_x = \{y\}$ and is a dominating set of H or $|T_x| \geq 2$.*

Theorem 4. *Let G and H be nontrivial connected graphs, and let $C = \cup_{x\in S} (\{x\} \times T_x)$. Then C is a connected disjunctive dominating set of $G[H]$ if and only if one of the following holds for S:*

(i) *$|S| \geq 2$ and S is a connected disjunctive dominating set of G;*

(ii) *S is a connected distance-two dominating set of G satisfying exactly one of the following:*

  (a) *$|S| \geq 2$ and for each $x \in V(G)\setminus N_G^d[S]$ there exists $u \in S$ for which $d_G(u, x) = 2$ and $|T_u| \geq 2$.*

  (b) *$S = \{x\}$ for some $x \in V(G)$, where $\langle T_x\rangle$ is a connected graph satisfying the following:*

    ($b.1$) *If S is a non-dominating set of G, then $|T_x| \geq 2$; and*

    ($b.2$) *If $|T_x| = 1$, then S and T_x are dominating sets of G and H, respectively.*

Proof. Suppose that C is a connected disjunctive dominating set of $G[H]$. As previously remarked, $\langle S\rangle$ is a connected graph. First, if Theorem 3(i) holds for S, then $|S| \geq 2$ and S is a disjunctive dominating set of G. In this case, (i) holds. Next, suppose that Theorem 3(ii) holds for S. Then S is a connected distance-two dominating set of G. If $|S| \geq 2$, then condition (ii)(a) follows immediately from Theorem 3(ii)(a). Assume that $S = \{x\}$ for some $x \in V(G)$. Clearly, $\langle T_x\rangle$ is connected as $\langle C\rangle$ is connected. Suppose S is a nondominating set of G, and choose $u \in V(G) \setminus N_G[x]$. For any $v \in V(H)$, there exist distinct $y, z \in T_x$ such that $d_{G[H]} ((u, v), (x, y)) = 2 = d_{G[H]} ((u, v), (x, z))$. Thus, $|T_x| \geq 2$, and (ii)(b)($b.1$) holds. Now, if $|T_x| = 1$, say $T_x = \{y\}$, then $C = \{(x, y)\}$ is a dominating set of $G[H]$. Necessarily, S and $\{y\}$ are dominating sets of G and H, respectively.

Conversely, if (i) holds for S, then S is a disjunctive total dominating set of G. Consequently, C is a disjunctive dominating set of $G[H]$ by Theorem 3. Since $\langle S\rangle$ is connected, $\langle C\rangle$ is connected. Assume that S is a connected distance-two dominating set of G. Suppose that condition $(ii)(a)$ holds for S. Then $\langle C\rangle$ is connected. Let $(x,y)\in V(G[H])\setminus C$.

Case 1: $x\in N_G^d[S]$

If $u\in S\cap N_G(x)$, then $(x,y)(u,v)\in E(G)$ for any $v\in T_u$. Suppose that $x\notin N_G(S)$. Then there exist distinct $u,w\in S$ for which $d_G(x,u)=2=d_G(x,w)$. Pick $v\in T_u$ and $z\in T_w$. Then $(u,v),(w,z)\in C$ and $d_{G[H]}\left((x,y),(u,v)\right)=2=d_{G[H]}\left((x,y),(w,z)\right)$.

Case 2: $x\notin N_G^d[S]$

By the hypothesis, there exists $u\in S$ for which $d_G(u,x)=2$ and $|T_u|\geq 2$, say $v,z\in T_u$. Then (u,v) and (u,z) are distinct vertices in C and $d_{G[H]}\left((x,y),(u,v)\right)=2=d_{G[H]}\left((x,y),(u,z)\right)$.

The above cases imply that C is a connected disjunctive dominating set of $G[H]$.

Finally, suppose that $|S|=1$, say $S=\{x\}$, satisfying condition $(ii)(b)$. Then $\langle C\rangle=\langle\{x\}\times T_x\rangle$ is connected. Let $(u,v)\in V(G[H])\setminus C$.

Case 1: $u\neq x$

If $ux\in E(G)$, then $(u,v)(x,y)\in E(G[H])$ for any $y\in T_x$. If $d_G(u,x)=2$, then by condition $(b.1)$, $|T_x|\geq 2$, say $y,z\in T_x$. Then $(x,y),(x,z)$ are distinct vertices in C with $d_{G[H]}\left((x,y),(u,v)\right)=2=d_{G[H]}\left((x,z),(u,v)\right)$.

Case 2: $u=x$

Note that since G is nontrivial, $d_{G[H]}\left((u,v),(x,y)\right)\leq 2$ for all $y\in T_x$. If $|T_x|\geq 2$, then $(u,v)\in N_{G[H]}^d(C)$. On the other hand, if $T_x=\{y\}$, then by condition $(b.2)$, $vy\in E(H)$ so that $(u,v)(x,y)\in E(G[H])$.

Accordingly, C is a connected disjunctive dominating set of $G[H]$. □

Let $S\subseteq V(G)$ be a distance-two dominating set of G. For each $x\in V(G)\setminus N_G^d[S]$, let $s_x\in S\cap N_G(x,2)$. We define for S, $S^d=\{s_x : x\in V(G)\setminus N_G^d[S]\}$.

Denote by $CD2D(G)$ the families of all connected distance-two dominating sets of G.

Corollary 5. *Let G and H be nontrivial connected graphs. Then*

(i) *Provided $\gamma_{2,c}(G)=1$,*

$$\gamma_c^d(G[H])=\begin{cases}1, & \text{if } \gamma(H)=1=\gamma(G)\\ 2, & \text{if } \gamma(G)\geq 2.\end{cases}$$

(ii) *Provided $\gamma_{2,c}(G)\geq 2$,*

$$\gamma_c^d(G[H])=\min\{\gamma_c^d(G),\alpha(G)\},$$

where $\alpha(G)=\min\{|S\setminus S^d|+2|S^d| : S\in CD2D(G)\}$.

For all nontrivial connected graphs H, $\gamma_c^d(C_4[H])=2$, $\gamma_c^d(C_6[H])=\gamma_c^d(C_6)=3$ and $\gamma_c^d(P_6[H])=\alpha(P_6)=4$.

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