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On Edge Q-Algebras

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Abstract. The concept of an edge in Q-algebra is introduced in this work. We explore some properties of edge Q-algebras. The characterization of subsets of an edge Q-algebra to be subalgebras is provided. We show that for an edge Q-algebra X of order n, there are 2^{n-1} subalgebras of X. Moreover, the set of all subalgebras forms a semigroup with a right identity. Beside this, the concept of ideal is discussed. We obtain that the set of all ideals forms a left zero semigroup and a simple semigroup. Finally, we describe all possible structures of edge Q-algebras and enumerate all members of a class of all edge Q-algebras. We prove that there are exactly 2^{n^2-3n+2} edge Q-algebras of order n. Finally, we show a connection between Q-algebras and d-algebras. We obtain that every edge d-algebra is a Q-algebra. Precisely, every edge d-algebra is an edge Q-algebra.

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1. Introduction and Preliminaries

Back in the period of the late 20th century, K. Iseki and Y. Imai introduced two classes of logical algebras which are called BCK-algebra and BCI-algebra in 1966 [1, 2]. The class of BCK-algebras is a proper subclass of the class of BCI-algebras. Later, in 1978 K. Iseki and S. Tanaka discussed the theory of BCK-algebras in [3]. Since then, many new kinds of algebras which are related to BCK/BCI- algebras are introduced. In 1983, Q. P. Hu and X. Li introduced the notion of BCH-algebra which is a generalization of BCK/BCI-algebras [4, 5]. In 1984, a class of BCC-algebras was presented by Y. Komori [6], and then in 1992 W.A. Dudek discussed some properties of this algebra in [7].

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A d-algebra, another generalization of BCK/BCI-algebras, was appeared in 1999 by J. Neggers and H.S. Kim [8]. The authors explored relations between d-algebras and BCK-algebras. The concept of edge in d-algebras is also provided. They showed that every d-transitive edge d-algebra is a BCK-algebra. Moreover, they obtained a correspondence between oriented digraphs and edge d-algebras such that every edge d-algebra produces an oriented digraph. Some years later, Neggers and Kim also introduced the notion of Q-algebras and B-algebras. In 2001, an algebra which related to BCK/BCI-algebras was emerged, so called Q-algebra, by J. Neggers, S. Ahn and H. S. Kim [9]. A Q-algebra is an algebraic system $(X; *, 0)$ consists of a non-empty set X, a constant $0 \in X$ and a binary operation $*$ defined on X that yields the following three conditions $(Q_1), (Q_2)$ and (Q_3) as the following: For any $x, y, z \in X$,
(Q_1) $x * x = 0$,
(Q_2) $x * 0 = x$,
(Q_3) $(x * y) * z = (x * z) * y$.

We omit the symbol $*$ for a convenient reason. Let us mention here, later on we will denote the symbol $\mathbf{X}$ as a Q-algebra $(X; *, 0)$ unless otherwise specified.

In [9], the authors presented some connections of $BCK/BCI/BCH$-algebras and Q-algebras. They showed that a Q-algebra $\mathbf{X}$ satisfying the condition (A): *for all* $x, y \in X$, $xy = 0$ *and* $yx = 0$ *implies* $x = y$, is a BCH-algebra. A Q-algebra $\mathbf{X}$ satisfying the conditions (A) and (B): $(xy)(xz) = zy$ *for all* $x, y, z \in X$, is a BCI-algebra. A Q-algebra $\mathbf{X}$ is a BCK-algebra if the conditions $(A), (B)$ and (C): *for all* $x, y, z \in X$, $(xy)x = 0$, are hold. The concepts of subalgebra, G-part and ideal were also offered in [9]. A non-empty subset S of X is a subalgebra if $ab \in S$ for any a and b in S. It is easy to see that a subset $\{0\}$ is a subalgebra of a Q-algebra $\mathbf{X}$ since $00 = 0$ by (Q_1). A non-empty subset I of X is an ideal of $\mathbf{X}$ if the following conditions (I_1) and (I_2) are hold:
(I_1) $0 \in I$;
(I_2) for $x, y \in X$, if $xy \in I, y \in I$, then $x \in I$.
The subsets $\{0\}$ and X are obviously ideals of $\mathbf{X}$. An ideal I of $\mathbf{X}$ is called a zero ideal if $I = \{0\}$, otherwise I is a non-zero ideal of $\mathbf{X}$. The G-part of $\mathbf{X}$, denoted by $G(X)$, is defined by

$$G(X) = \{a \in X \mid 0a = a\}.$$

It is easy to see that $0 \in G(X)$ since $00 = 0$. The authors in [9] obtained the characterization of $G(X)$ which is an ideal of $\mathbf{X}$ when $|X| \leq 3$. They also provided that every subalgebra S of $\mathbf{X}$, $G(X) \cap S = G(S)$ where $G(S) = \{x \in S \mid 0x = x\}$.

Example 1. *Let* $X = \{0, a, b, c, d, f\}$. *Define a binary operations* $*$ *on* X *as the following table:*

$*$	0	a	b	c	d	f
0	0	a	c	b	c	b
a	a	0	b	c	b	c
b	b	c	0	a	0	a
c	c	d	a	0	a	0
d	d	c	0	a	0	a
f	f	d	a	0	a	0

It is a routine to check that $\boldsymbol{X}$ is a Q-algebra. A subset $S = \{0, a, b, c, d\}$ is a subalgebra of $\boldsymbol{X}$ but a subset $T = \{0, f\}$ is not a subalgebra since $0 * f = b \notin T$. Moreover, a subset T is not an ideal since $c * f = 0 \in T$ and $f \in T$ but $c \notin T$. Let consider a subset $I = \{0, a\}$. It is not difficult to check that I is an ideal and $I = G(X)$.

In 2004, S. S. Ahn, H. Kim and H. D. Lee discussed the homomorphisms of Q-algebra in [10]. They introduced the self-maps of Q-algebras which are called a right map and a left map. They obtained that a right map is an endomorphism whenever $\mathbf{X}$ is a positive implicative Q-algebra, i.e. $(xy)(xz) = xyz$ for all $x, y, z \in X$. In 2011, another self-map of a Q-algebra $\mathbf{X}$, called a right fixed map, is discussed and examined their properties in [11] by S. M. Lee. The author showed that the set of all right fixed maps of $\mathbf{X}$ is a Q-algebra under a binary operation which is induced from a binary operation on X. In 2010, S. S. Ahn and S. E. Kang proposed the concept of atom in Q-algebras. An element w of X is an atom if $xw = 0$ implies $x = w$ for all $x \in X$. They showed that any subalgebra of $\mathbf{X}$ is an ideal of $\mathbf{X}$ whenever every non-zero element of $\mathbf{X}$ is an atom [12]. In 2024 the authors in [13] examined some properties of atoms in Q-algebras. They showed some relations between atoms and the set G-part which is related to the concept of ideal. They proved that every element of $G(X)$ is an atom whenever $G(X)$ is an ideal. The notion of strong atoms was also offered in [13], which was inspired from the concept of strong atom in BCK-algebra (see [14]). They proved that $\mathbf{X}$ does not contain a strong atom whenever $\mathbf{X}$ contains a non-zero ideal $G(X)$. The concept of fuzzy set on Q-algebra can be found in [15] and [16]. The authors provided some properties of fuzzy Q-ideals, fuzzy prime ideals and fuzzy relations of Q-algebras. Recently, in 2025 the authors in [17] discussed the concept of ideal in Q-algebra. They provided a characterization of ideals which is related to the G-part. They showed that every G-part that is an ideal, is an abelian group.

In this work, we introduce the notion of edge Q-algebras. We describe all possible edge Q-algebras of order n. We examine some properties of edge Q-algebras. The connection between d-algebras and edge Q-algebras is provided. The concepts of subalgebras and ideals are discussed in edge Q-algebras. We give necessary and sufficient conditions for subsets of an edge Q-algebra to be subalgebras. We also provide some properties in edge Q-algebras which are related to the concept of ideals. We obtain that the set of all ideals forms a right zero semigroup and a simple semigroup. Finally, we describe all possible structures of edge Q-algebras and enumerate all members of a class of all edge Q-algebras.

2. Edge Q-algebras

Let A and B be non-empty subsets of a Q-algebra $\mathbf{X}$. We define a product AB in a usual way as follow:

$$AB = \{ab \mid a \in A, b \in B\}.$$

If $B = \{x\}$, we denote AB and BA by Ax and xA, respectively.

Proposition 1. *Let A, B and C be non-empty subsets of a Q-algebra $\boldsymbol{X}$. Then the following properties are valid:*

(i) $(AB)C = (AC)B$.

(ii) $A\{0\} = A0 = A$.

(iii) If $A \subseteq B$, *then* $AC \subseteq BC$ *and* $CA \subseteq CB$.

(iv) If $0 \in B$, *then* $A \subseteq AB$.

(v) If $A \cap B \neq \emptyset$, *then* $0 \in AB$.

Proof. (i) Follows directly from the condition (Q_3).

(ii) By the condition (Q_2), we get $A\{0\} = \{a0 \mid a \in A\} = \{a \mid a \in A\} = A$.

(iii) It is obvious.

(iv) By (ii), (iii) and since $0 \in B$, then $A = A\{0\} \subseteq AB$.

(v) Let $x \in A \cap B$. Then by the condition (Q_1), $0 = xx \in AB$.

Concerning to the concept of subalgebra in algebras, in general a subalgebra is a non-empty subset which is closed. In Q-algebras, we get a sharp condition as seen in the following proposition:

Proposition 2. *Let A and B be non-empty subsets of a Q-algebra $\boldsymbol{X}$. Then*

(i) A is a subalgebra of $\boldsymbol{X}$ if and only if $AA = A$.

(ii) If $A \subseteq B$ and B is a subalgebra, then $AB \subseteq B$.

Proof. (i) Assume that A is a subalgebra of $\mathbf{X}$, then $AA \subseteq A$. Since $0 \in A$, then by Proposition 1(iv), $A \subseteq AA$. Thus, $AA = A$. The converse direction is clear.

(ii) Assume that B is a subalgebra of $\mathbf{X}$ and $\emptyset \neq A \subseteq B$. Then by (i) and Proposition 1(iii), we get $AB \subseteq BB \subseteq B$.

Next, we will introduce the notion of an edge Q-algebra. In d-algebra, J. Neggers and H. S. Kim introduced the concept of edge d-algebras in 1999 [8]. A d-algebra consists of a non-empty set X with a constant $0 \in X$ together with a binary operation on X satisfying the following axioms: for all $x, y, z \in X$,

(d_1) $xx = 0$,

(d_2) $x0 = 0$,

(d_3) $xy = 0$ and $yx = 0$ imply $x = y$.

Both d-algebras and Q-algebras are generalizations of $BCI/BCK/BCH$-algebras. But both of them are independent, i.e. a Q-algebra need not be a d-algebra and vice versa. The following example shows this fact:

Example 2. *Let* $X = \{0, a, b, c\}$, $Y = \{0, x, y, z\}$ *and* $T = \{0, \gamma, \beta, \nu\}$. *Define binary operations* $*$ *on* X, $\bullet$ *on* Y *and* $\circ$ *on* T *as the following tables:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	c	b
b	c	0	0	a
c	c	a	c	0

$\bullet$	0	x	y	z
0	0	x	z	y
x	x	0	y	z
y	y	z	0	x
z	z	y	x	0

$\circ$	0	γ	β	ν
0	0	0	0	0
γ	γ	0	γ	0
β	β	0	0	β
ν	ν	ν	ν	0

Then $(X; *, 0)$ *is a* d*-algebra and* $(Y; \bullet, 0)$ *is a* Q*-algebra. Since* $0 * b = c \neq 0$, *then the condition* (Q_2) *is not satisfied in* X. *Then* $(X; *, 0)$ *is not a* Q*-algebra. Moreover,* $(Y; \bullet, 0)$ *is not a* d*-algebra since* $0 \bullet y = z \neq 0$, *i.e. the condition* (d_2) *is not valid. The algebra* $(T; \circ, 0)$ *is a* d*-algebra and a* Q*-algebra.*

A d-algebra $\mathbf{X}$ is said to be edge if $xX = \{0, x\}$ for all $x \in X$. From Example 2, a d-algebra $(T; \circ, 0)$ is an edge d-algebra. In [8] the authors obtained some properties of edge d-algebras as the following:

Proposition 3. *[8] Let* X *be an edge* d*-algebra. Then* $x0 = x$ *for all* $x \in X$.

Now we describe the concept of edge in Q-algebra motivated by [8].

Definition 1. *A* Q*-algebra* $\boldsymbol{X}$ *is said to be edge if* $aX = \{0, a\}$ *for all* $a \in X$.

Example 3. *Let* $X = \{0, x, y, z\}$ *and* $Y = \{0, a, b, c, d, f\}$. *Define binary operations* $*$ *on* X *and* $\bullet$ *on* Y *as the following tables:*

$*$	0	x	y	z
0	0	0	0	0
x	x	0	0	0
y	y	y	0	y
z	z	0	0	0

$\bullet$	0	a	b	c	d	f
0	0	a	f	b	b	b
a	a	0	b	f	f	f
b	b	c	0	a	a	a
c	c	b	a	0	0	0
d	d	b	a	0	0	0
f	f	b	a	0	0	0

Then $(X; *, 0)$ *and* $(Y; \bullet, 0)$ *are* Q*-algebras. Since* $0X = \{0\}, xX = \{0, x\}, yX = \{0, y\}$ *and* $zX = \{0, z\}$, *then* X *is an edge* Q*-algebra. Since* $aY = \{0, a, b, f\} \neq \{0, a\}$, *then* Y *is not edge.*

Next proposition provides a connection between d-algebras and Q-algebras. We show a sufficient condition for a d-algebra to be a Q-algebra.

Proposition 4. *Every edge d-algebra is a Q-algebra, more specific it is an edge Q-algebra.*

Proof. Let $\mathbf{X}$ be an edge d-algebra. We want to show that $\mathbf{X}$ is a Q-algebra. The condition (Q_1) and (Q_2) follow from the condition (d_1) and Proposition 3, respectively. Let $x, y, z \in X$. We calculate $(xy)z$ and $(xz)y$. Since $\mathbf{X}$ is an edge d-algebra, then $xy \in xX = \{0, x\}$, i.e. $xy = 0$ or $xy = x$. If $xy = 0$, then by (d_2) $(xy)z = 0z = 0$. Since $xz \in xX = \{0, x\}$, there follows $(xz)y \in \{0y, xy\} = \{0\}$. Thus, $(xy)z = (xz)y$. Now we assume $xy = x$. Then $(xy)z = xz \in xX = \{0, x\}$. If $xz = 0$, then by (d_2) $(xy)z = xz = 0 = 0y = (xz)y$. If $xz = x$, then $(xy)z = xz = x = xy = (xz)y$. Therefore, $(xy)z = (xz)y$ for all $x, y, z \in X$. Hence, the condition (Q_3) is fulfilled. Thus, $\mathbf{X}$ is a Q-algebra. Since $\mathbf{X}$ is edge, then $\mathbf{X}$ is an edge Q-algebra.

The converse of Proposition 4 is not true, i.e. there is an edge Q-algebra which is not an edge d-algebra as shown in the following example.

Example 4. *Let $X = \{0, \alpha, \eta, \mu\}$ and a binary operation $*$ be defined on X as the following table.*

$*$	0	α	η	μ
0	0	0	0	0
α	α	0	α	0
η	η	0	0	η
μ	μ	0	μ	0

Then $(X; *, 0)$ is an edge Q-algebra. Since $\alpha * \mu = 0$ and $\mu * \alpha = 0$ but $\alpha \neq \mu$, then the condition (d_3) is not satisfied. Thus, $(X; *, 0)$ is not a d-algebra.

Next, we examine some properties of edge Q-algebras.

Proposition 5. *Let $\mathbf{X}$ be an edge Q-algebra and let A and B be non-empty subsets of X. Then the following properties are valid.*

(i) $0X = \{0\}$.

(ii) *If* $0 \in A$, *then* $0 \in AB$.

Proof. (i) It is clear.

(ii) Let $0 \in A$ and $b \in B$, then by (i) there follows that $0 = 0b \in 0B \subseteq AB$.

For any $a \in X$, a subset aX normally is not necessarily a subalgebra of $\mathbf{X}$. For example, a set $aY = \{0, a, b, f\}$ in Example 3 is not a subalgebra of $(Y; \bullet, 0)$ since $b \bullet a = c \notin aY$. But for edge Q-algebras we get the following positive result.

Proposition 6. *If $\mathbf{X}$ is an edge Q-algebra, then aX is a subalgebra of $\mathbf{X}$ for all $a \in X$.*

Proof. Let $a \in X$. Then $aX = \{0, a\}$. It is easy to verify that $00, a0$ and aa are elements in aX. Now we calculate $0a$. Since $0a \in 0X$, by Proposition 5(i) we get $0a = 0 \in aX$. Therefore, aX is closed and then aX is a subalgebra of $\mathbf{X}$.

Next, we will examine some conditions that lead the product of subsets of a Q-algebra $\mathbf{X}$ to be a subalgebra of $\mathbf{X}$.

Proposition 7. *Let $\boldsymbol{X}$ be an edge Q-algebra and let $\emptyset \neq A, B \subseteq X$. If $0 \in A$ or $A \cap B \neq \emptyset$ then AB is a subalgebra of $\boldsymbol{X}$.*

Proof. *Assume that $0 \in A$. Then by Proposition 5(ii), $0 \in AB$. Let $x, y \in AB$. Since $\boldsymbol{X}$ is an edge Q-algebra, then $xy \in xX = \{0, x\}$. There follows that $xy = 0$ or $xy = x$. Thus, $xy \in AB$. Therefore, AB is a subalgebra. Assume now that $A \cap B \neq \emptyset$. By Proposition 1(v), $0 \in AB$. Using the same argument, we get AB is closed. Hence, AB is a subalgebra of $\boldsymbol{X}$.*

The converse of Proposition 7 is not true as shown in the following example.

Example 5. *Let $X = \{0, a, b, c\}$ and let a binary operation $*$ be defined on X as follow:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	0	0	0
c	c	c	c	0

It is a routine to check that $(X; *, 0)$ is a Q-algebra. Since $0X = \{0\}, aX = \{0, a\}, bX = \{0, b\}$ and $cX = \{0, c\}$, then X is an edge. Let $A = \{b\}$ and $B = \{c\}$. Then $AB = \{b * c\} = \{0\}$ is a subalgebra. But neither $0 \in A$ nor $A \cap B \neq \emptyset$. Thus, the converse of Proposition 7 is not true.

Corollary 1. *Let $\boldsymbol{X}$ be an edge Q-algebra and let A, B be non-empty subsets of X. Then the following properties are true.*

(i) If A is a subalgebra of $\boldsymbol{X}$, then AB is a subalgebra of $\boldsymbol{X}$.

(ii) If A is an ideal of $\mathbf{X}$, then AB is a subalgebra of $\boldsymbol{X}$.

Proof. (i) Follows directly from Proposition 7 since 0 is a member of A.
(ii) Assume that A is an ideal of $\mathbf{X}$. By (I_1), 0 is a member of I there follows by Proposition 7 that AB is a subalgebra of $\mathbf{X}$.

Corollary 2. *Let X be an edge Q-algebra. Then the following properties are true.*

(i) For any $a, b \in X$, $aX \cup bX$ is a subalgebra of $\boldsymbol{X}$.

(ii) Let $\emptyset \neq \Lambda \subseteq X$. Then $\bigcup_{a \in \Lambda} aX$ is a subalgebra of $\boldsymbol{X}$.

Proof. (i) Let $a, b \in X$. Since $\mathbf{X}$ is an edge Q-algebra, then $aX \cup bX = \{0, a\} \cup \{0, b\} = \{0, a, b\}$. Set $A = \{0, a, b\}$ and $B = \{0\}$, then by Proposition 1(ii), $AB = A = \{0, a, b\} = aX \cup bX$. Since $0 \in A$, then by Proposition 7, $AB = aX \cup bX$ is a subalgebra of $\mathbf{X}$.

(ii) Let $a \in \Lambda \subseteq X$. Since $\mathbf{X}$ is an edge Q-algebra, then $aX = \{0, a\}$. There follows $\bigcup_{a\in\Lambda} aX = \Lambda \cup \{0\}$. Set $A = \Lambda \cup \{0\}$ and $B = \{0\}$. Then $AB = \Lambda \cup \{0\} = \bigcup_{a\in\Lambda} aX$. Since $0 \in A$, by Proposition 7 $AB = \bigcup_{a\in\Lambda} aX$ is a subalgebra of $\mathbf{X}$.

For a Q-algebra $\mathbf{X}$, we denote the set of all subalgebras of $\mathbf{X}$ by $Sub(X)$. Next proposition provides the characterization of subalgebras of an edge Q-algebra and enumerate all of subalgebras of any edge Q-algebra $\mathbf{X}$.

Proposition 8. *Let $\mathbf{X}$ be an edge Q-algebra and let $|X| = n$ for some positive integer n. Then the following conditions are hold:*

(i) For $\emptyset \neq S \subseteq X$, S is a subalgebra of $\mathbf{X}$ if and only if $0 \in S$.

(ii) $|Sub(X)| = 2^{n-1}$.

Proof. (i) $(\Rightarrow)$ It is clear.

$(\Leftarrow)$ Let S be a non-empty subset of X and assume $0 \in S$. By Proposition 1(ii), $S\{0\} = S$. Set $A = S, B = \{0\}$ and then by Proposition 7 there follows $AB = S\{0\} = S$ is a subalgebra of $\mathbf{X}$.

(ii) From (i) we conclude that any subset of X containing a constant 0 is a subalgebra of $\mathbf{X}$. There follows that

$$|Sub(X)| = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}.$$

Proposition 9. *Let $\mathbf{X}$ be an edge Q-algebra. Then $Sub(X)$ is a semigroup with a right identity $\{0\}$.*

Proof. Let $A, B, C \in Sub(X)$. By Corollary 1, AB is a subalgebra and then $Sub(X)$ is closed. Since $0 \in B$, by Proposition 1(iv) we get $A \subseteq AB$. Since $\mathbf{X}$ is edge and $0 \in A$, then $AB \subseteq AX = A \cup \{0\} = A$. Therefore, $A \subseteq AB \subseteq A$. Thus, $AB = A$. There follows $(AB)C = AC = A = AB = A(BC)$. This gives an associative law. Hence, $Sub(X)$ is a semigroup. Moreover, by Proposition 1(ii) $A\{0\} = A$ this shows that $\{0\}$ is a right identity.

From Proposition 6, we get an information that a set aX is a subalgebra for any element a in an edge Q-algebra $\mathbf{X}$. But this kind of subset need not be an ideal. For example, a subset aX of X in Example 5 is not an ideal since $b * a = 0 \in aX$, $a \in aX$ but $b \notin aX$. Next, we will show a condition for a subset of an edge Q-algebra $\mathbf{X}$ in the form aX, $a \in X$ to be an ideal of $\mathbf{X}$.

Proposition 10. *Let* $\boldsymbol{X}$ *be an edge* Q*-algebra and let* $a \in X$. *Then* aX *is an ideal if and only if* $wa = w$ *for all* $w \in X \backslash \{a\}$.

Proof. *Let* $a \in X$, *and assume that* aX *is an ideal of* $\boldsymbol{X}$. *Suppose that there is an element* $w \in X, w \neq a$ *such that* $wa \neq w$. *Since* $wa \in wX = \{0, w\}$ *and* $wa \neq w$, *then* $wa = 0$. *There follows* $wa \in aX$. *Since* $wa \in aX, a \in aX$ *and* aX *is an ideal, then* $w \in aX$. *This gives a contradiction. Hence, for all* $w \in X \backslash \{a\}$, $wa = w$. *For the converse direction, assume* $wa = w$ *for all* $w \in X \backslash \{a\}$. *We want to show that* $aX = \{0, a\}$ *is an ideal. If* $a = 0$, *then it is obvious that* aX *is an ideal of* X. *Assume now* $a \neq 0$. *Let* $xy \in aX$ *and* $y \in aX$. *If* $y = 0$, *then* $x = x0 = xy \in aX$. *If* $y = a$, *then* $xa \in aX$. *There follows* $xa = 0$ *or* $xa = a$. *Suppose* $xa = a$. *Then* $x \neq a$ *otherwise* $a = xa = aa = 0$, *a contradiction. Hence,* $xa = 0$. *Then by assumption* $x \notin X \backslash \{a\}$. *Therefore,* $x = a \in aX$. *Altogether,* aX *is an ideal of* $\boldsymbol{X}$.

In a Q-algebra $\mathbf{X}$, a product $I_1 I_2$ of ideals I_1 and I_2 of X is not necessarily an ideal as can be seen in the following example.

Example 6. *Let* $X = \{0, a, b, c, d, f\}$. *Define a binary operation* $*$ *on* X *as the following table:*

$*$	0	a	b	c	d	f
0	0	a	c	d	c	c
a	a	0	d	c	d	d
b	b	c	0	a	0	0
c	c	b	a	0	a	a
d	d	c	0	a	0	0
f	f	c	0	a	0	0

Then $(X; *, 0)$ *is a* Q*-algebra. The subsets* $I_1 = \{0, a\}$ *and* $I_2 = \{0, b, d, f\}$ *are ideals of* $(X; *, 0)$. *Let consider a product* $I_1 I_2 = \{0, c, d\}$. *Since* $a * c = c \in I_1 I_2$ *and* $c \in I_1 I_2$ *but* $a \notin I_1 I_2$, *then* $I_1 I_2$ *is not an ideal of* $(X; *, 0)$. *Therefore, the set of all ideals of a* Q*-algebra is not necessarily closed. In an edge* Q*-algebra, we get a good outcome, i.e. the product of ideals is again an ideal as obtained in the following proposition.*

Proposition 11. *Let* $\boldsymbol{X}$ *be an edge* Q*-algebra. If* A *and* B *are ideals of* $\boldsymbol{X}$, *then* AB *is an ideal.*

Proof. *Let* A *and* B *be ideals of* $\boldsymbol{X}$. *Let* $a \in A$. *Since* $0 \in B$, *by Proposition 1(iv),* $A \subseteq AB$. *Since* X *is an edge,* $aB \subseteq aX = \{0, a\}$. *Thus, we can conclude that* $AB \subseteq A \cup \{0\} = A$. *Hence,* $AB = A$ *so that* AB *is an ideal of* $\boldsymbol{X}$.

As a direct consequence of Proposition 11 we have the subsequence corollary.

Corollary 3. *If* A *and* B *are ideals of an edge* Q*-algebra* $\boldsymbol{X}$, *then* $AB = A$.

For a Q-algebra $\mathbf{X}$, we denote $I(X)$ as the set of all ideals of X. By Proposition 11 and Corollary 3, we obtain that the set of all ideals of an edge Q-algebra forms a semigroup.

Proposition 12. *Let* $\boldsymbol{X}$ *be an edge* Q*-algebra. Then* $I(X)$ *is a semigroup.*

Proposition 13. *Let* $\boldsymbol{X}$ *be an edge* Q*-algebra. Then* $I(X)$ *is a subsemigroup of* $Sub(X)$*.*

Proof. Let $I \in I(X)$. By (I_1), $0 \in I$, so that by Proposition 8(i), I is a subalgebra. Thus, $I(X) \subseteq Sub(X)$. Since $I(X)$ is a semigroup and $I(X) \subseteq Sub(X)$, then $I(X)$ is a subsemigroup of $Sub(X)$.

We will recall some concepts of semigroup theory. A semigroup S is a left zero semigroup if $za = z$ for all $z, a \in S$. A non-empty subset I of S is a left semigroup ideal (right semigroup ideal) if $SI \subseteq I$ ($IS \subseteq I$, respectively). If I is both a left semigroup ideal and a right semigroup ideal, then I is a semigroup ideal. A semigroup ideal (left semigroup ideal, right semigroup ideal) I such that $I \neq S$ is called a proper semigroup ideal (left semigroup ideal, right semigroup ideal). A semigroup S is a left simple semigroup if S has no proper left semigroup ideals. A right simple semigroup and a simple semigroup are defined in an analogous way. For more intensive details in semigroup theory we refer to [18].

Proposition 14. *Let* $\boldsymbol{X}$ *be an edge* Q*-algebra. Then* $I(X)$ *is a left zero semigroup.*

Proof. By Proposition 12, $I(X)$ *is a semigroup. Let* $A, B \in I(X)$*. Then by Corollary 3,* $AW = A$*. Therefore,* $I(X)$ *is a left zero semigroup.*

Moreover, $I(X)$ is a simple semigroup.

Proposition 15. *Let* $\boldsymbol{X}$ *be an edge* Q*-algebra. Then* $I(X)$ *is a left simple semigroup.*

Proof. Let P *be a left semigroup ideal of a semigroup* $I(X)$*. Then* $I(X)P \subseteq P$*. Let* $A \in I(X)$ *and* $B \in P$*. Then* $AB = A$ *by Corollary 3. There follows that* $I(X) \subseteq I(X)P \subseteq P$*. Hence,* $I(X) = P$*. Therefore,* $I(X)$ *does not contain a proper semigroup left ideal. Thus,* $I(X)$ *is a left simple semigroup.*

Corollary 4. *Let* $\boldsymbol{X}$ *be an edge* Q*-algebra. Then* $I(X)$ *is a simple semigroup.*

Proof. It follows from Proposition 15.

3. Enumeration of Edge Q-algebras

In this section, we describe all possible structures of edge Q-algebras of order n, for any positive integer n. To do this we need to construct a Q-algebra as follows:

Construction$(*)$: Let $X_n = \{x_1, x_2, x_3, \ldots, x_n\}$ be a set of order n. We define a binary operation on X_n as follow: For $x_i, x_j \in X_n$,

$$x_i x_j = \begin{cases} x_1 & \text{if } i = j, \\ x_1 & \text{if } i = 1, \\ x_i & \text{if } j = 1, \\ a \in \{x_1, x_i\} & \text{if otherwise.} \end{cases}$$

From the Construction $(*)$, in the case $i \neq j$ and $i, j \neq 1$, the product $x_i x_j \in \{x_1, x_i\}$, i.e. the product $x_i x_j$ is either x_1 or x_i. We denote here $x \vee y$ by " either x or y". Then we obtain the following Cayley table:

	x_1	x_2	x_3	...	x_{i-1}	x_i	x_{i+1}	...	x_n
x_1	x_1	x_1	x_1	...	x_1	x_1	x_1	...	x_1
x_2	x_2	x_1	$x_1 \vee x_2$	...	$x_1 \vee x_2$	$x_1 \vee x_2$	$x_1 \vee x_2$	...	$x_1 \vee x_2$
x_3	x_3	$x_1 \vee x_3$	x_1	...	$x_1 \vee x_3$	$x_1 \vee x_3$	$x_1 \vee x_3$	...	$x_1 \vee x_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	x_i	$x_1 \vee x_i$	$x_1 \vee x_i$	...	$x_1 \vee x_i$	x_1	$x_1 \vee x_i$	...	$x_1 \vee x_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	x_n	$x_1 \vee x_n$	$x_1 \vee x_n$	...	$x_1 \vee x_n$	$x_1 \vee x_n$	$x_1 \vee x_n$	...	x_1

For the set X_n, the Construction $(*)$ allows us to get 2^{n^2-3n+2} algebraic structures. Let $EQ(X_n)$ be the set of all algebras which obtained from the Construction $(*)$.

Example 7. *Let $X_3 = \{x_1, x_2, x_3\}$. The following algebras A, B, C and D are obtained from the Construction $(*)$:*

	x_1	x_2	x_3
x_1	x_1	x_1	x_1
x_2	x_2	x_1	x_1
x_3	x_3	x_1	x_1

A

	x_1	x_2	x_3
x_1	x_1	x_1	x_1
x_2	x_2	x_1	x_1
x_3	x_3	x_3	x_1

B

	x_1	x_2	x_3
x_1	x_1	x_1	x_1
x_2	x_2	x_1	x_2
x_3	x_3	x_1	x_1

C

	x_1	x_2	x_3
x_1	x_1	x_1	x_1
x_2	x_2	x_1	x_2
x_3	x_3	x_3	x_1

D

It is not difficult to check that all above tables A, B, C and D are Q-algebras with x_1 acts as a constant 0. Let consider the table A. Since $x_1 X_3 = \{x_1\}$, $x_2 X_3 = \{x_1, x_2\}$ and $x_3 X_3 = \{x_1, x_3\}$, X_3 is an edge Q-algebra. Similarly, we get that tables B, C and D are edge Q-algebras. Moreover, $|EQ(X_3)| = 2^{3^2-3(3)+2} = 2^2 = 4$ and $EQ(X_3) = \{A, B, C, D\}$.

Next proposition reveals that any algebraic structure obtained from the Construction $(*)$ is an edge Q-algebra.

Proposition 16. *Let $X_n = \{x_1, x_2, \ldots, x_n\}$. For any $\mathcal{A} \in EQ(X_n)$, $\mathcal{A}$ is an edge Q-algebra.*

Proof. *Let $\mathcal{A}$ be any algebra in $EQ(X_n)$. Let $x_i, x_j, x_k \in X_n$. Then we get $x_i x_1 = x_i$ and $x_i x_i = x_1$. Therefore, the conditions (Q_1) and (Q_2) hold and x_1 acts as a constant 0. Next, we calculate $(x_i x_j)x_k$ and $(x_i x_k)x_j$. Observe that $x_i x_j \in \{x_1, x_i\}$ and $x_i x_k \in \{x_1, x_i\}$.*

If $x_ix_j = x_1$, then $(x_ix_j)x_k = x_1x_k = x_1$ and $(x_ix_k)x_j \in \{x_1x_j, x_ix_j\} = \{x_1\}$. It follows that $(x_ix_j)x_k = x_1 = (x_ix_k)x_j$. If $x_ix_j = x_i$, then $(x_ix_j)x_k = x_ix_k$. If $x_ix_k = x_1$, then $(x_ix_j)x_k = x_ix_k = x_1 = x_1x_j = (x_ix_k)x_j$. If $x_ix_k = x_i$, then $(x_ix_j)x_k = x_ix_k = x_i = x_ix_j = (x_ix_k)x_j$. Altogether, (Q_3) is fulfilled. Since $x_iX_n = \{x_1, x_i\}$ for all $x_i \in X_n$, then the edge property is hold. Altogether, $\mathcal{A}$ is an edge Q-algebra.

From Example 7, if we replace x_1 by 0, then we get the following edge Q-algebras:

	0	x_2	x_3
0	0	0	0
x_2	x_2	0	0
x_3	x_3	0	0

A

	0	x_2	x_3
0	0	0	0
x_2	x_2	0	0
x_3	x_3	x_3	0

B

	0	x_2	x_3
0	0	0	0
x_2	x_2	0	x_2
x_3	x_3	0	0

C

	0	x_2	x_3
0	0	0	0
x_2	x_2	0	x_2
x_3	x_3	x_3	0

D

Let consider an edge Q algebra $\mathbf{X}$ of order n, for any positive integer n. For any element $a \in X$, $aX = \{0, a\}$. There follows that $ax \in \{0, a\}$ for all $x \in X$. Hence, there is an algebraic structure $Y \in EQ(X_n)$ which is coinciding to X.

Proposition 17. *Let n be a positive integer and let $\mathbf{Y}$ be an edge Q-algebra of order n. Then Y is isomorphic to X for some $X \in EQ(X_n)$.*

Combining Proposition 16 and Proposition 17 we get:

Theorem 1. *$EQ(X_n)$ is the set of all edge Q-algebras of order n and hence, there are precisely 2^{n^2-3n+2} different edge Q-algebras of order n.*

4. Conclusion

We have introduced the concept of edge Q-algebras and explored their properties. We obtained some results related to the concepts of subalgebras and ideals. The product of subalgebras of an edge Q-algebra is again a subalgebra, offered in Corollary 1. Similarly, the product of ideals of edge Q-algebra is also an ideal. Moreover, the set of all subalgebras of an edge Q-algebra $\mathbf{X}$, $Sub(X)$, and the set of all ideals, $I(X)$ form a semigroup as shown in Proposition 9 and Proposition 12, respectively. We enumerated all of subalgebras of edge Q-algebra $\mathbf{X}$. Proposition 8 shows the total number of all subalgebras of $\mathbf{X}$ and $|Sub(X)| = 2^{|X|-1}$. The construction of edge Q-algebras is presented. We also showed the total number of all structures of edge Q-algebras, as in Theorem 1. There are 2^{n^2-3n+2} structures of edge Q-algebras of order n. For further study, one can

consider edge Q-algebras based on the following concepts:
- Hyper algebras;
- Filters and another kinds of ideals;
- Fuzzy subalgebras, fuzzy ideals;
- Homomorphisms and isomorphisms;
- Connections with related algebras.

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References

[1] Y. Imai and K. Iseki. On axiom system of proposaitional calculi. xiv. *Proceedings of the Japan Academy*, 42:19–22, 1966.

[2] K. Iseki. An algebra related with a propositional calculus. *Proceedings of the Japan Academy*, 42:26–29, 1966.

[3] K. Iseki and S. Tanaka. An introduction to theory of *bck*-algebra. *Mathematica Japonica*, 23:1–26, 1978.

[4] Q. P. Hu and X. Li. On *bch*-algebras. *Mathematics Seminar Notes Kobe University*, 2:313–320, 1983.

[5] Q. P. Hu and X. Li. On proper *bch*-algebras. *Mathematica Japonica*, 4:659–661, 1983.

[6] Y. Komori. The class of *bcc*-algebras is not a variety. *Mathematica Japonica*, 29:391–394, 1984.

[7] W. Dudek. On proper *bcc*-algebras. *Bulletin of the Institute of Mathematics Academia Sinica*, 20:137–150, 1992.

[8] J. Neggers and H. S. Kim. On d-algebras. *Mathematica Slovaca*, 49:19–26, 1999.

[9] J. Neggers, S. Ahn, and H. S. Kim. On q-algebras. *International Journal of Mathematics and Mathematical Sciences*, 27:749–757, 2001.

[10] S. Ahn, H. S. Kim, and H. D. Lee. r-maps and l-map in q-algebras. *International Journal of Pure and Applied Mathematics*, 12:419–425, 2004.

[11] S. M. Lee and K. H. Kim. On right fixed maps of q-algebras. *International Mathematical Forum*, 6:31–37, 2011.

[12] S. Ahn and S. E. Kang. The role of $t(x)$ in the ideal theory of q-algebras. *Honam Mathematical Journal*, 32:515–523, 2010.

[13] A. Anantayasethi, T. Kunawat, and P. Moonipa. Relations between g-part and atoms in q-algebras. *European Journal of Pure and Applied Mathematics*, 17:3268–3276, 2024.

[14] D. Sun. On atoms of bck-algebras. *Scientiae Mathematicae Japonicae Online*, 2:115–124, 2001.

[15] S. M. Mostafa, M. A. Naby, and O. R. Elgendy. Fuzzy q-ideals in q-algebras. *World Applied Programming*, 2:69–80, 2012.

[16] H. K. Abdullah and M. Tach. Intuitionistic fuzzy prime ideal on q-algebras. *International Journal of Academic and Applied Research*, 4:66–78, 2020.

[17] A. Anantayasethi and J. Koppitz. Characterization of ideals of q-algebras related to its g-part. *Journal of Discrete Mathematical Sciences and Cryptography*, 28:131–141, 2025.

[18] J. Howie. *Fundamentals of semigroup theory.* Oxford University Press Inc., UAS, 1995.