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Vertex-Generator Subgraphs of Complete Bipartite and Tadpole Graphs

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Abstract. Graphs considered in this paper are finite simple undirected graphs. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$, for some positive integer n. The vertex space $\mathscr{V}(G)$ of G, is a vector space over the field $\mathbb{Z}_2 = \{0, 1\}$. The elements of $\mathscr{V}(G)$ are all the subsets of $V(G)$. Vector addition is defined as $A + B = A \triangle B$, the symmetric difference of sets A and B, for all $A, B \in \mathscr{V}(G)$. Scalar multiplication is defined as $1 \cdot A = A$ and $0 \cdot A = \emptyset$, for all $A \in \mathscr{V}(G)$. The subgraph $G\langle S\rangle$ of G induced by a subset S of $V(G)$, is the largest subgraph whose vertex set is S. The vertex-uniform set $V_H(G)$ of a subgraph H with respect to G, is the set of all elements of $\mathscr{V}(G)$ that induces a subgraph isomorphic to H. The span of $V_H(G)$ shall be denoted by $\mathscr{V}_H(G)$. If $V_H(G)$ is a generating set, that is $\mathscr{V}_H(G) = \mathscr{V}(G)$, then H is called a vertex-generator subgraph of G. This study determines some vertex-generator subgraphs of complete bipartite graph $K_{m,n}$ and tadpole graph $T_{n,m}$.

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Key Words and Phrases: Vertex space, induced subgraph, vertex-uniform set, generating set, vertex-generator subgraph

1. Introduction

Graph theory, a fundamental area of mathematics concerned with the study of relationships through networks, has evolved into a vital tool across numerous scientific and technological disciplines. Its capacity to model complex systems has led to significant advancements in fields ranging from computer science to social network analysis.

The integration of algebraic structures has significantly broadened the scope of graph theory. A key development in this direction is the concept of vector spaces, notably edge spaces and vertex spaces [1]. Within this framework, these spaces are treated as vector

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spaces over the finite field $\mathbb{Z}_2 = \{0, 1\}$, with vector addition defined as the symmetric difference of sets and scalar multiplication resulting in either the empty set (multiplication by 0) or the original set (multiplication by 1). The only difference is that, the edge spaces contains all the subsets of the edge set of that graph, while the vertex spaces contains all the subsets of the vertex set of that graph.

The notion of vertex space has been applied in studies such as that by Butenko, Festa, and Pardalos [2], who utilized it to analyze colored vertex sets with adjacency and coloring constraints. Building on the concept of vertex spaces and induced subgraphs, Torino and Mame [3] introduced the vertex-generator subgraph, a concept parallel to the edge-generator subgraph (or generator subgraph) first explored by Gervacio [4]; see also [5]. A key difference is that vertex-generator subgraphs can contain isolated vertices, while generator subgraphs do not allow isolated vertices. While generator subgraphs have been studied for various graph classes, research on vertex-generator subgraphs remains less extensive, with Torino and Mame's work being a significant contribution. A related concept, the even vertex space (elements of the vertex space with even cardinality) offers another perspective for studying graph structure and identifying vertex-generator subgraphs.

A graph G is an ordered pair $(V(G), E(G))$, where the vertex set $V(G)$ is a finite nonempty set of objects called *vertices*, and the edge set $E(G)$ is a set of unordered pairs of vertices called *edges*. The *order* n of G is the number of elements of $V(G)$, and the *size* m of G is the number of elements of $E(G)$. Let $H = (V(H), E(H))$ be another graph. A mapping $\phi : V(G) \mapsto V(H)$ is called a *graph isomorphism* if the following conditions are satisfied: (i) ϕ is bijective; (ii) $[a, b] \in E(G)$ implies that $[\phi(a), \phi(b)] \in E(H)$; and (iii) $[c, d] \in E(H)$ implies that $[\phi^{-1}(c), \phi^{-1}(d)] \in E(G)$. A graph G is *isomorphic* to a graph H, denoted as $G \simeq H$, if there is an isomorphism $\phi : V(G) \mapsto V(H)$ between their vertex sets. A graph H is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a non-empty subset S of $V(G)$, the subgraph of G vertex-induced by S, denoted by $G\langle S\rangle$, is the subgraph of G whose vertex set is $V(G\langle S\rangle) = S$ and whose edge set is $E(G\langle S\rangle) = \{[x, y] \in E(G) \mid x, y \in S\}$. A subgraph H of G is called a *vertex-induced subgraph* or simply *induced subgraph*, if there exists a non-empty subset $S \subseteq V(G)$ such that $H = G\langle S\rangle$.

Torino and Mame's work [3] primarily focused on characterizing vertex-generator subgraphs for several well-known graph classes, including path graphs, cycle graphs, empty graphs, complete graphs, star graphs, and wheel graphs. However, the study of vertex-generator subgraphs in other graph classes, such as complete bipartite graphs and tadpole graphs, remains an open area of research.

A *complete bipartite* graph $K_{m,n}$ is a graph in which $V(G)$ is partitioned into subsets U and V called partite sets, where the cardinality of U and V are m and n, respectively, and every vertex of U is adjacent to every vertex of V [6]. A *tadpole* graph $T_{n,m}$ is the graph obtained by joining a cycle C_n to a path P_m, with a bridge $[x, y]$, where $x \in V(C_n)$ and $y \in V(P_m)$ and $\deg_{P_m}(y)$ is either 0 or 1 [7]. Other graph classes that have been identified as vertex-generator subgraphs of tadpole and complete bipartite graphs are defined in the pertinent part of this work. Readers may refer to the books written by Bollobás [8], Bondy and Murty [9], Chartrand, Lesniak and Zhang [10], and Harary [6], for other basic concepts in graph theory. Also, readers may refer to the books written by Nering [11], and Larson and Falvo [12], for the linear algebra concepts, particularly the vector spaces.

The objective of this research is to determine some vertex-generator subgraphs of complete bipartite and tadpole graphs. The researchers first established a fixed labeling for each of the two graphs, and this labeling was then used to determine some vertex-uniform sets of subgraphs with respect to these two graphs. By applying the symmetric difference to the elements of a vertex-uniform set, it can be shown that a subgraph is a vertex-generator subgraph by applying the existing results of Torino and Mame [3]. The researchers tested several smaller subgraphs to identify patterns and eventually formulated a general result.

2. Preliminaries

2.1. Vertex-Generator Subgraph of a Graph

This section gives the definition of the vertex space, the vertex-uniform set, the span of a vertex-uniform set, and the vertex-generator subgraph. This section also provides results relevant to the said concepts. The main reference for this section is [3].

Definition 1. [1] *Let $G = (V(G), E(G))$ be a graph. The* ***vertex space*** *of G, denoted by $\mathscr{V}(G)$, is a vector space over a field $\mathbb{Z}_2 = \{0, 1\}$, which composes of all subsets of $V(G)$, where for $A, B \in \mathscr{V}(G)$, vector addition and scalar multiplication are given by*

(i) $A + B = A \triangle B$, the symmetric difference of A and B.

(ii) $cA = A$ if $c = 1$ and $cA = \emptyset$ if $c = 0$.

Notably, the basis of the vertex space can be found in the book of Diestel [1], which is presented in the following theorem.

Theorem 1. [1] *Let G be a graph with $V(G) = \{x_1, x_2, x_3, ..., x_n\}$. Then the set $\mathscr{A} = \{\{x_1\}, \{x_2\}, \{x_3\}, ..., \{x_n\}\}$ forms a basis for $\mathscr{V}(G)$. Hence, $\dim \mathscr{V}(G) = n$, the order of G.*

Knowing the structure of the vertex space through its basis, we now explore specific subsets of this space—particularly those associated with induced subgraphs—and how they contribute to generating the entire vertex space. This leads to the notions of vertex-uniform sets, their span, and the concept of vertex-generator subgraphs.

Definition 2. [3] *Let H be a subgraph of a graph G. The* ***vertex-uniform set*** *of H with respect to G, denoted by $V_H(G)$, is the set of all elements of $\mathscr{V}(G)$ that induces a subgraph isomorphic to H.*

Definition 3. [3] *Let $V_H(G)$ be a vertex-uniform set of H with respect to G. The* ***span*** *of $V_H(G)$, denoted by $\mathscr{V}_H(G)$, is the set of all linear combinations of the elements of $V_H(G)$. That is, if $V_H(G) = \{A_1, A_2, A_3, ..., A_k\}$ where $A_i \in \mathscr{V}(G)$, then*

$$\mathscr{V}_H(G) = \left\{ \sum_{i=1}^{k} c_i A_i \mid c_i \in \{0, 1\} \right\}.$$

Definition 4. [3] *Let $\mathscr{V}_H(G)$ be the span of $V_H(G)$. If $\mathscr{V}_H(G) = \mathscr{V}(G)$, then H is a* ***vertex-generator subgraph*** *of G.*

Note that $\mathscr{A} = \{\{x_1\}, \{x_2\}, \{x_3\}, ..., \{x_n\}\}$ forms a basis for $\mathscr{V}(G)$ by Theorem 1. Additionally, $\mathscr{V}_H(G) \subseteq \mathscr{V}(G)$. To show that H is a vertex-generator subgraph of G, it is sufficient to show that $\mathscr{V}(G) \subseteq \mathscr{V}_H(G)$. That is, $\{\{x_1\}, \{x_2\}, \{x_3\}, ..., \{x_n\}\} \subseteq \mathscr{V}_H(G)$. Thus, the following remark gives a necessary and sufficient condition for a subgraph to be a vertex-generator subgraph of a graph.

Remark 1. [3] *Let G be a graph with vertex set $V(G) = \{x_1, x_2, x_3, ..., x_n\}$. Let H be a subgraph of G. Then H is a vertex-generator subgraph of G if and only if $\{x_i\} \in \mathscr{V}_H(G)$ for all $1 \leq i \leq n$.*

For example, consider the cycle graph C_5 in Figure 1, with $V(C_5) = \{a_1, a_2, a_3, a_4, a_5\}$ and $E(C_5) = \{[a_1, a_2], [a_2, a_3], [a_3, a_4], [a_4, a_5], [a_1, a_5]\}$.

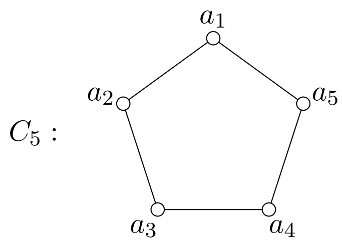

Figure 1: A cycle graph C_5

We show that the path graph P_3 is a vertex-generator subgraph of C_5. First, we determine the vertex-uniform set of P_3 with respect to C_5, as follows.

$$A = \{\{a_1, a_2, a_3\}, \{a_1, a_2, a_5\}, \{a_1, a_4, a_5\}, \{a_2, a_3, a_4\}, \{a_3, a_4, a_5\}\}.$$

It can be observed that each element of A induces a subgraph that is isomorphic to P_3, hence $A \subseteq V_{P_3}(C_5)$. Next, we show that each singleton is a linear combination of the elements of A.

$$\begin{aligned}
&\{a_2, a_3, a_4\} \triangle \{a_3, a_4, a_5\} \triangle \{a_1, a_2, a_5\} = \{a_1\}, \\
&\{a_3, a_4, a_5\} \triangle \{a_1, a_4, a_5\} \triangle \{a_1, a_2, a_3\} = \{a_2\}, \\
&\{a_1, a_4, a_5\} \triangle \{a_1, a_2, a_5\} \triangle \{a_2, a_3, a_4\} = \{a_3\}, \\
&\{a_1, a_2, a_5\} \triangle \{a_1, a_2, a_3\} \triangle \{a_3, a_4, a_5\} = \{a_4\}, \text{ and} \\
&\{a_1, a_2, a_3\} \triangle \{a_2, a_3, a_4\} \triangle \{a_1, a_4, a_5\} = \{a_5\}.
\end{aligned}$$

Hence, $\{a_i\} \in \mathscr{V}_{P_3}(C_5)$ for all $1 \leq i \leq 5$. Therefore, by Remark 1, P_3 is a vertex-generator subgraph of C_5.

2.1.1. Some Known Results

The first theorem shows that the trivial graph is a vertex-generator subgraph of any graph.

Theorem 2. [3] *The trivial graph K_1 is a vertex-generator subgraph of any graph G.*

The trivial graph is clearly a vertex-generator subgraph, because it is always isomorphic to each vertex of any graph, so its uniform set always consists of singletons.

The next theorem is very useful in finding a vertex-generator subgraph of a certain graph, which is based on the order of that graph.

Theorem 3. [3] *Let H be a subgraph of G. If H is a vertex-generator subgraph of G, then $|V(H)|$ is odd.*

The next theorem tells us the relationship of the cardinality of the sets $V(G)$ and $V_H(G)$, where H is a subgraph of any graph G.

Theorem 4. [3] *Let H be a subgraph of the graph G. If H is a vertex-generator subgraph of G, then $|V_H(G)| \geq |V(G)|$.*

The next theorem gives a necessary and sufficient condition for a vertex-generator subgraph of any graph G, where $|V(G)| \leq 3$.

Theorem 5. [3] *Let H be a subgraph of G, where $|V(G)| \leq 3$. Then H is a vertex-generator subgraph of G if and only if $H \simeq K_1$.*

Let $\mathscr{V}^*(G)$ be the set of all elements of $\mathscr{V}(G)$ with even cardinality. Torino and Mame called $\mathscr{V}^*(G)$ as the *even vertex space* of G [3].

The following theorem presents the relevance of the even vertex space of a graph, to the vertex space of the graph.

Theorem 6. [3] *Let G be a graph of order n. Then, $\mathscr{V}^*(G)$ is a subspace of $\mathscr{V}(G)$. Moreover, $\dim \mathscr{V}^*(G) = n - 1$.*

A basis formed from the even vertex space of any graph is presented in the theorem below, which is parallel to Theorem 1.

Theorem 7. [3] *Let G be a graph with $V(G) = \{x_1, x_2, x_3, ...x_n\}$. Then the set $\mathscr{B} = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, ..., \{x_1, x_n\}\}$ forms a basis for $\mathscr{V}^*(G)$.*

The next theorem is also important in the concept of the even vertex space, which is presented below.

Theorem 8. [3] *Let G and H be graphs such that $H \subseteq G$, and $|V(H)|$ is odd. If $\mathscr{V}^*(G) \subseteq \mathscr{V}_H(G)$, then H is vertex-generator subgraph of G.*

Theorem 8 tells us that we only need to show the basis for the even vertex space of the graph is a subset of the span of the the vertex-uniform set, in order to show that a subgraph is a vertex-generator subgraph of a graph. This follows a useful remark, which is parallel to Remark 1.

Remark 2. [3] *Let G be a graph with vertex set $V(G) = \{x_1, x_2, x_3, ..., x_n\}$. Let H be a subgraph of G. Then H is a vertex-generator subgraph of G if and only if $\{x_1, x_i\} \in \mathscr{V}_H(G)$ for all $2 \leq i \leq n$.*

The following theorem provides the necessary and sufficient conditions for a subgraph H to be a vertex generator of an empty graph $\overline{K_n}$. This particular result will be used in following discussion.

Theorem 9. [3] *Let n and t be postive integers. Let H be a subgraph of $\overline{K_n}$, where $|V(H)| = t$. Then H is a vertex-generator subgraph of $\overline{K_n}$ if and only if the following conditions are satisfied:*

(i) *t is odd;*

(ii) *$1 \leq t \leq n-1$; and*

(iii) *$H \simeq \overline{K_t}$.*

3. Main Results

3.1. Vertex-Generator Subgraph of Complete Bipartite Graph $K_{m,n}$

This section provides some vertex-generator subgraphs of complete bipartite graph $K_{m,n}$.

Let $K_{m,n}$ be the complete bipartite graph with vertex set $V(K_{m,n}) = M \cup N$, where $M = \{x_1, x_2, x_3, ..., x_{m-1}, x_m\}$ and $N = \{y_1, y_2, y_3, ..., y_{n-1}, y_n\}$ are the partite sets, and edge set $E(K_{m,n}) = \{[x_i, y_j]\}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Presented in Figure 2 is the labeling of a complete bipartite graph, which will be considered in the discussion of this section.

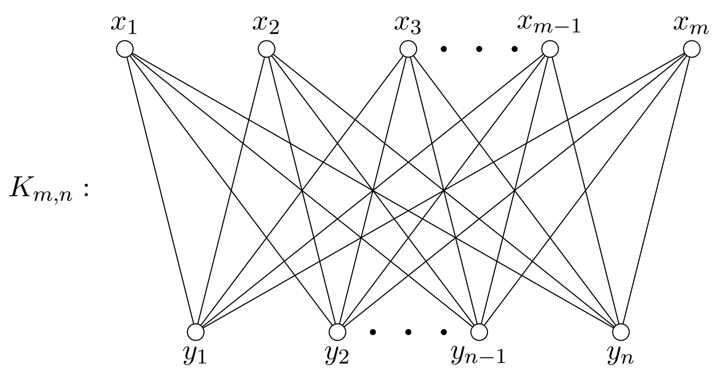

Figure 2: The Labeling of $K_{m,n}$

A complete bipartite graph $K_{m,n}$ has order $m+n$ and size mn for all positive integers m and n. By Definition 1, the vertex space of $K_{m,n}$ is given by $\mathscr{V}(K_{m,n}) = \{S \mid S \subseteq V(K_{m,n})\}$. Given the vertex set $V(K_{m,n})$, the set

$$\mathscr{A} = \{\{x_1\}, \{x_2\}, \{x_3\}, ..., \{x_{m-1}\}, \{x_m\}, \{y_1\}, \{y_2\}, \{y_3\}, \ldots, \{y_{n-1}\}, \{y_n\}\}$$

forms a basis for $\mathscr{V}(K_{m,n})$, thus $\dim \mathscr{V}(K_{m,n}) = m + n$ by Theorem 1.

Furthermore, the even vertex space of $K_{m,n}$ is given by $\mathscr{V}^*(K_{m,n}) = \{S \in \mathscr{V}(K_{m,n}) \mid |S| \text{ is even}\}$. In view of Theorem 7, the set

$$\mathscr{B} = \{\{x_1, x_2\}, \{x_1, x_3\}, ..., \{x_1, x_m\}, \{x_1, y_1\}, \{x_1, y_2\}, ..., \{x_1, y_n\}\}$$

forms a basis for $\mathscr{V}^*(K_{m,n})$. Hence, $\dim \mathscr{V}^*(K_{m,n}) = m + n - 1$ by Theorem 6.

By Theorem 2, we know that trivial subgraph K_1 is a vertex-generator subgraph of $K_{m,n}$. Additionally, We can observe that $K_{m,n}$ exhibits some well-known isomorphisms for small values of m and n, such as $K_{1,1} \simeq P_2$, $K_{2,1} \simeq S_2 \simeq P_3$, and $K_{2,2} \simeq C_4$. By Theorem 5, the vertex-generator subgraph of P_2 and P_3 is the trivial graph. The vertex-generator subgraph of the cycle graph C_4 was also studied. Hence, will now focus our investigation on determining the vertex-generator subgraphs of $K_{m,n}$ where $\min\{m, n\} \geq 2$ and $m+n \geq 5$.

The following theorem presents a necessary condition for an empty graph $\overline{K_t}$ to be a vertex-generator subgraph of $K_{m,n}$.

Theorem 10. *Let m, n and t be positive integers such that $\min\{m, n\} \geq 2$, $m + n \geq 5$, and t is odd. If $t < \min\{m, n\}$, then $\overline{K_t}$ is a vertex-generator subgraph of $K_{m,n}$.*

Proof. From the labeling of complete bipartite graph, it can be observed that the subgraphs of $K_{m,n}$ induced by the partite sets M and N, denoted by $K_{m,n}\langle M\rangle$ and $K_{m,n}\langle N\rangle$, are isomorphic to the empty graphs $\overline{K_m}$ and $\overline{K_m}$, respectively. This is shown in Figure 3.

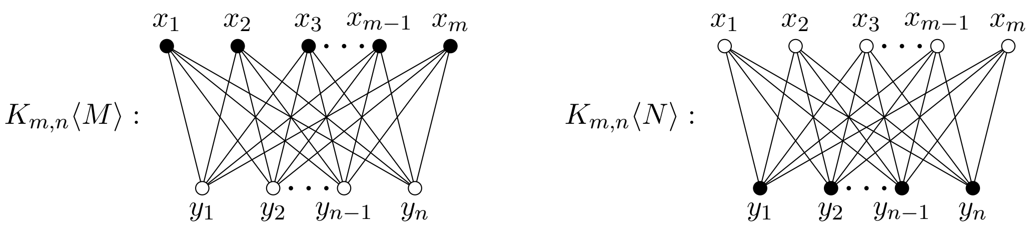

Figure 3: Illustrating the subgraphs of $K_{m,n}$ induced by M and N

Let $\overline{K_t}$ be a subgraph of both $\overline{K_m}$ and $\overline{K_n}$. It was given that t is odd, so condition (i) of Theorem 9 is sastified. Next, since $t < \min\{m, n\}$, it follows that $t < m$ and $t < n$, or $t \leq m - 1$ and $t \leq n - 1$, which satisfies condition (ii) of Theorem 9. Lastly, since $\overline{K_t}$ is isomorphic to itself, condition (iii) of Theorem 9 is also satisfied. All the conditions have met, hence $\overline{K_t}$ is a vertex-generator subgraph of both $\overline{K_m}$ and $\overline{K_n}$. It follows that $\{x_i\} \in \mathscr{V}_{\overline{K_t}}(\overline{K_m})$ for $1 \leq i \leq m$, and $\{y_j\} \in \mathscr{V}_{\overline{K_t}}(\overline{K_n})$ for $1 \leq j \leq n$, by Remark 1. Now, we know that

$$\mathscr{V}_{\overline{K_t}}(\overline{K_m}) \cup \mathscr{V}_{\overline{K_t}}(\overline{K_n}) = \mathscr{V}_{\overline{K_t}}(K_{m,n}\langle M\rangle) \cup \mathscr{V}_{\overline{K_t}}(K_{m,n}\langle N\rangle) = \mathscr{V}_{\overline{K_t}}(K_{m,n}),$$

this means that $\{x_i\}, \{y_j\} \in \mathscr{V}_{\overline{K_t}}(K_{m,n})$. Therefore, by Remark 1, $\overline{K_t}$ is a vertex-generator subgraph of $K_{m,n}$. □

The following theorem will give a necessary condition for the path graph P_t to be a vertex-generator subgraph of $K_{m,n}$.

Theorem 11. *Let m and n be positive integers such that $\min\{m,n\} \geq 2$ and $m+n \geq 5$. If $t=1$ or $t=3$, then P_t is a vertex-generator subgraph of $K_{m,n}$.*

Proof. Suppose $t = 1$ or $t = 3$. By Theorem 2, P_1 is a trivial vertex-generator subgraph of $K_{m,n}$. Now, consider the labeling of $K_{m,n}$. For any $1 \leq i \leq m$, let x_i be an arbitrary vertex of partite set M in the complete bipartite graph $K_{m,n}$, and let A, B, and C be defined as follows:

$$\begin{aligned} A &= \{x_1, x_2, y_1\}, \\ B &= \{x_1, x_2, y_2\}, \text{ and} \\ C &= \{x_i, y_1, y_2\} \text{ where } 1 \leq i \leq m. \end{aligned}$$

It can be verified that $A, B, C \in V_{P_3}(K_{m,n})$, as shown in Figure 4.

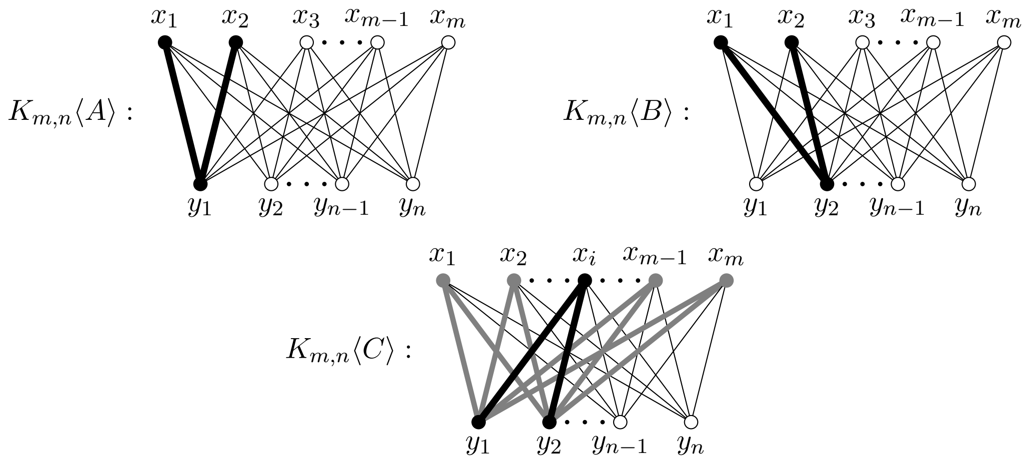

Figure 4: Illustrating the subgraphs of $K_{m,n}$ induced by A, B, and C

Hence, for any $i = 1, 2, 3, ..., m$, we get

$$\begin{aligned} A \triangle B \triangle C &= \{x_1, x_2, y_1\} \triangle \{x_1, x_2, y_2\} \triangle \{x_i, y_1, y_2\} \\ &= \{y_1, y_2\} \triangle \{x_i, y_1, y_2\} \\ &= \{x_i\}. \end{aligned}$$

Hence, $\{x_i\} \in \mathscr{V}_{P_3}(K_{m,n})$ for all $1 \leq i \leq m$. Similarly, for any $1 \leq j \leq n$, let y_j be an arbitrary vertex of partite set N in the complete bipartite graph $K_{m,n}$, and let D, E, and F be defined as follows:

$$\begin{aligned} D &= \{x_1, y_1, y_2\}, \\ E &= \{x_2, y_1, y_2\}, \text{ and} \\ F &= \{x_1, x_2, y_j\} \text{ where } 1 \leq j \leq n. \end{aligned}$$

It can be verified that $D, E, F \in V_{P_3}(K_{m,n})$, as shown in Figure 5.

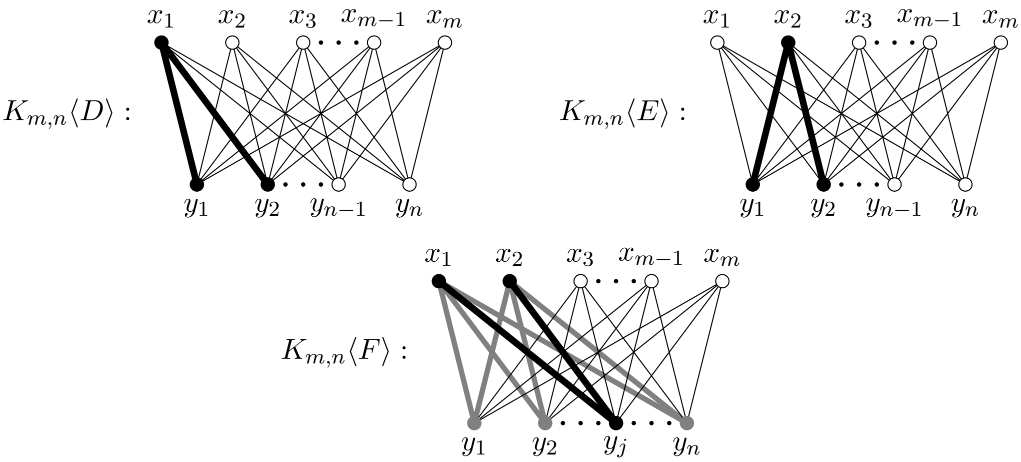

Figure 5: Illustrating the subgraphs of $K_{m,n}$ induced by D, E, and F

Consequently, for any $j = 1, 2, 3, ..., n$, we obtain

$$\begin{aligned} D \triangle E \triangle F &= \{x_1, y_1, y_2\} \triangle \{x_2, y_1, y_2\} \triangle \{x_1, x_2, y_j\} \\ &= \{x_1, x_2\} \triangle \{x_1, x_2, y_j\} \\ &= \{y_j\}. \end{aligned}$$

Hence, $\{y_j\} \in \mathscr{V}_{P_3}(K_{m,n})$ for all $1 \leq j \leq n$. Therefore, P_3 is a vertex-generator subgraph of $K_{m,n}$ by Remark 1. □

It is notable that the path graph P_3, as a vertex-generator subgraph of $K_{m,n}$, is isomorphic to a star graph S_2. We can extend this up to S_t of order $t + 1$, hence t must be even. With this, the following theorem gives us a necessary condition for S_t to be a vertex-generator subgraph of $K_{m,n}$.

Theorem 12. *Let m, n, and t be positive integers such that* $\min\{m, n\} \geq 2$, $m + n \geq 5$, *and t is even. If $t \leq min\{m, n\}$, then S_t is a vertex-generator subgraph of $K_{m,n}$.*

Proof. Let $t \leq min\{m, n\}$. For any $1 \leq i \leq m$, let x_i be an arbitrary vertex of partite set M in the complete bipartite $K_{m,n}$, and for any $1 \leq p \leq t$, let A_p and A be defined as follows:

$$\begin{aligned} A_p &= \{x_1, x_2, x_3, ..., x_t, y_p\} \text{ where } 1 \leq p \leq t, \text{ and} \\ A &= \{x_i, y_1, y_2, y_3, ..., y_t\} \text{ where } 1 \leq i \leq m. \end{aligned}$$

It can be verified that $A_p, A \in V_{S_t}(K_{m,n})$, as shown in Figure 6.
Hence, for any $i = 1, 2, 3, ..., m$, we have

$$\sum_{p=1}^{t} A_p + A = (A_1 + A_2 + A_3 + \cdots + A_t) + A$$

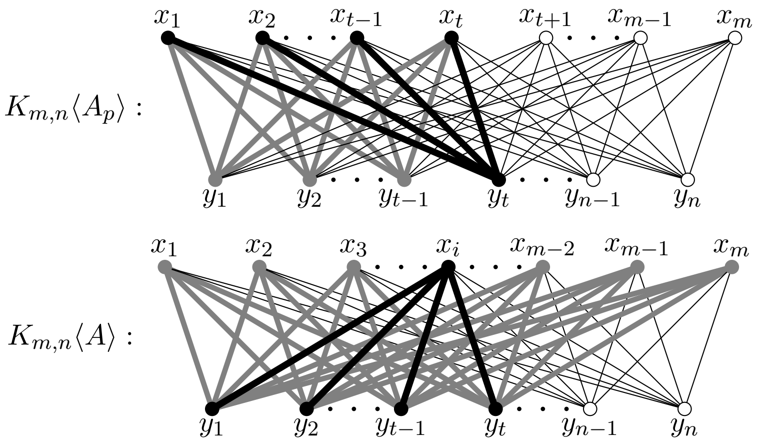

Figure 6: Illustrating the subgraphs of $K_{m,n}$ induced by A_p and A

$$\begin{aligned} &= (A_1 \triangle A_2 \triangle A_3 \triangle \cdots \triangle A_t) \triangle A \\ &= \{y_1, y_2, y_3, ..., y_t\} \triangle \{x_i, y_1, y_2, y_3, ..., y_t\} \\ &= \{x_i\}. \end{aligned}$$

Thus, $\{x_i\} \in \mathscr{V}_{S_t}(K_{m,n})$ for all $1 \leq i \leq m$. In a similar manner, for any $1 \leq j \leq n$, let y_j be an arbitrary vertex of partite set N in the complete bipartite graph $K_{m,n}$, and for any $1 \leq q \leq t$, let B_q and B be defined as follows:

$$B_q = \{x_q, y_1, y_2, y_3, ..., y_t\} \text{ where } 1 \leq q \leq t, \text{ and}$$
$$B = \{x_1, x_2, x_3, ..., x_t, y_j\} \text{ where } 1 \leq j \leq n.$$

It can be verified that $B_q, B \in V_{S_t}(K_{m,n})$, as shown in Figure 7.

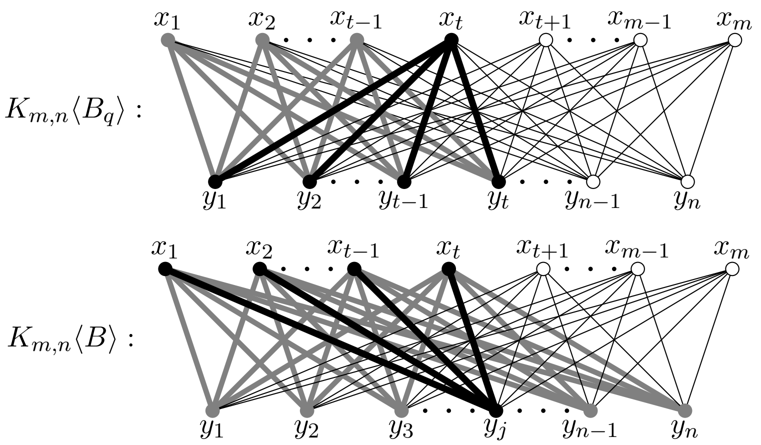

Figure 7: Illustrating the subgraphs of $K_{m,n}$ induced by B_q and B

Hence, for any $j = 1, 2, 3, ..., n$, we get

$$\begin{aligned}\sum_{q=1}^{t} B_q + B &= (B_1 + B_2 + B_3 + \cdots + B_t) + B \\ &= (B_1 \triangle B_2 \triangle B_3 \triangle \cdots \triangle B_t) \triangle B \\ &= \{x_1, x_2, x_3, ..., x_t\} \triangle \{x_1, x_2, x_3, ..., x_t, y_j\} \\ &= \{y_j\}.\end{aligned}$$

Thus, $\{y_j\} \in \mathscr{V}_{S_t}(K_{m,n})$ for all $1 \leq j \leq n$. Therefore, by Remark 1, S_t is a vertex-generator subgraph of $K_{m,n}$. □

The following theorem provides a necessary condition for the complete bipartite graph $K_{r,r+1}$ to be a vertex-generator subgraph of $K_{m,n}$.

Theorem 13. *Let m, n, and r be positive integers such that $\min\{m, n\} \geq 2$ and $m+n \geq 5$. If $r + 1 \leq \min\{m, n\}$, then $K_{r,r+1}$ is a vertex-generator subgraph of $K_{m,n}$.*

Proof. Let $r + 1 \leq \min\{m, n\}$. Then, we have the following cases:

Case 1. If $r = 1$, then we have a subgraph $K_{1,2}$ of $K_{m,n}$. It can be observed that $K_{1,2}$ is isomorphic to the path graph P_3. Since we have shown in Theorem 11 that P_3 is a vertex-generator subgraph of $K_{m,n}$, it follows that $K_{1,2}$ is a vertex-generator subgraph of $K_{m,n}$.

Case 2. We let $r \geq 2$. Then, for any $2 \leq p \leq m - r$ and for any $m - r + 2 \leq q \leq m$, let x_p and x_q be arbitrary vertices of partite set M in the complete bipartite graph $K_{m,n}$, and let sets A, B, C, and D be defined as follows:

$$\begin{aligned}A &= \{x_1, \overbrace{x_{m-r+2}, x_{m-r+3}, ..., x_m}^{r-1 \text{ vertices}}, \overbrace{y_1, y_2, ..., y_r, y_{r+1}}^{r+1 \text{ vertices}}\}, \\ B &= A \setminus \{x_1\} \cup \{x_p\} \text{ where } 2 \leq p \leq m - r, \\ C &= \{\overbrace{x_{m-r+1}, x_{m-r+2}, x_{m-r+3}, ..., x_m}^{r \text{ vertices}}, \overbrace{y_1, y_2, ..., y_r, y_{r+1}}^{r+1 \text{ vertices}}\}, \text{ and} \\ D &= C \setminus \{x_q\} \cup \{x_1\} \text{ where } m - r + 2 \leq q \leq m.\end{aligned}$$

It can be verified that $A, B, C, D \in V_{K_{r,r+1}}(K_{m,n})$, as shown in Figure 8.
Consequently, for $p = 2, 3, 4, ..., m - r$ and for $q = m - r + 2, m - r + 3, ..., m$, we obtain

$$\begin{aligned}A \triangle B &= \{x_1, x_p\}, \\ A \triangle C &= \{x_1, x_{m-r+1}\}, \text{ and} \\ C \triangle D &= \{x_1, x_q\}.\end{aligned}$$

Hence, $\{x_1, x_p\}, \{x_1, x_{m-r+1}\}, \{x_1, x_q\} \in \mathscr{V}_{K_{r,r+1}}(K_{m,n})$ for all $2 \leq p \leq m - r$ and for all $m - r + 2 \leq q \leq m$, or equivalently, $\{x_1, x_i\} \in \mathscr{V}_{K_{r,r+1}}(K_{m,n})$ for all $2 \leq i \leq m$. By similar argument, for any $1 \leq s \leq n - r - 1$ and for any $n - r + 1 \leq t \leq n$, let y_s and y_t be

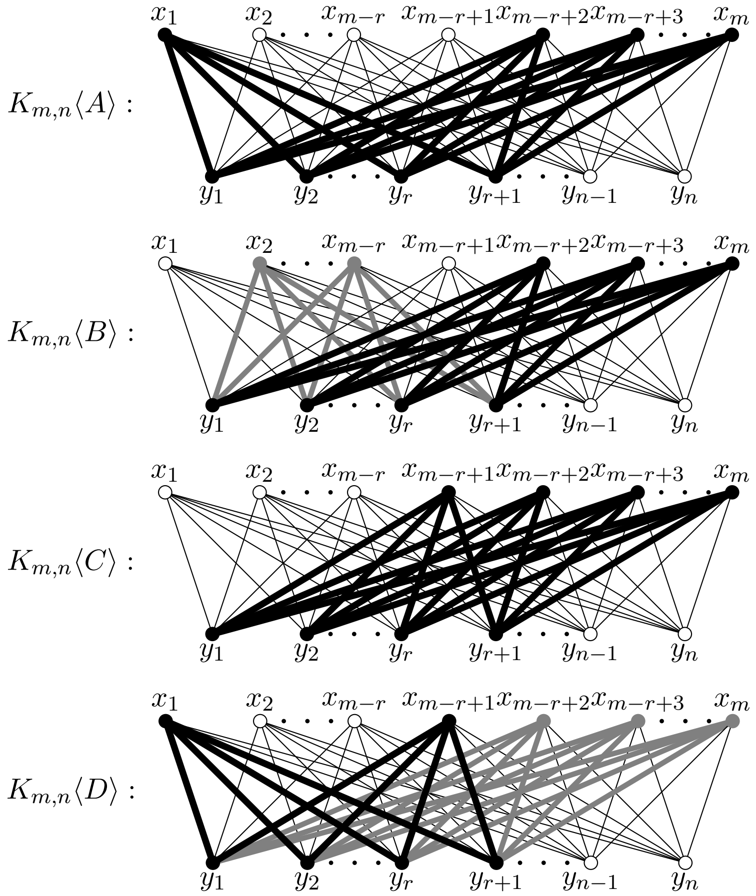

Figure 8: Illustrating the subgraphs of $K_{m,n}$ induced by A, B, C, and D

arbitrary vertices of partite set N in the complete bipartite graph $K_{m,n}$, and let sets W, X, Y, and Z be defined as follows:

$$
\begin{aligned}
W &= \{\overbrace{x_1, x_2, x_3, ..., x_r, x_{r+1}}^{r+1 \text{ vertices}}, \overbrace{y_{n-r+1}, y_{n-r+2}, ..., y_n}^{r \text{ vertices}}\},\\
X &= W \setminus \{x_1\} \cup \{y_s\} \text{ where } 1 \leq s \leq n-r-1,\\
Y &= \{\overbrace{x_2, x_3, ..., x_r, x_{r+1}}^{r \text{ vertices}}, \overbrace{y_{n-r}, y_{n-r+1}, y_{n-r+2}, ..., y_n}^{r+1 \text{ vertices}}\}, \text{ and}\\
Z &= Y \setminus \{y_t\} \cup \{x_1\} \text{ where } n-r+1 \leq t \leq n.
\end{aligned}
$$

It can be verified that $W, X, Y, Z \in V_{K_{r,r+1}}(K_{m,n})$, as shown in Figure 9.
As a consequence, for $s = 1, 2, 3, ..., n-r-1$ and for $t = n-r+1, n-r+2, ..., n$, we get

$$
\begin{aligned}
W \triangle X &= \{x_1, y_s\},\\
W \triangle Y &= \{x_1, y_{n-r}\}, \text{ and}\\
Y \triangle Z &= \{x_1, y_t\}.
\end{aligned}
$$

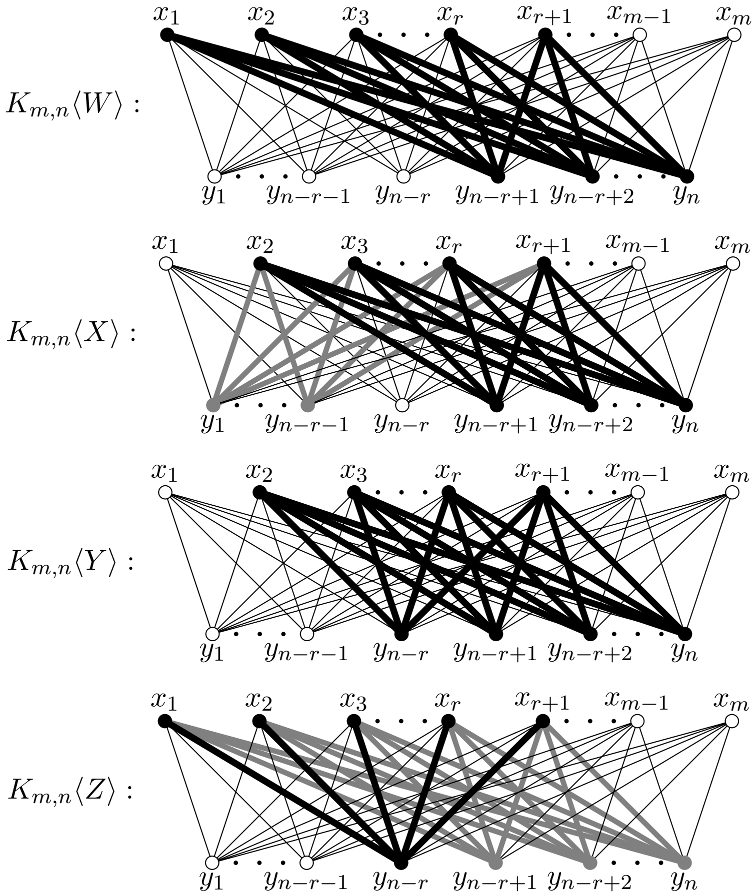

Figure 9: Illustrating the subgraphs of $K_{m,n}$ induced by W, X, Y, and Z

Hence, $\{x_1, y_s\}, \{x_1, y_{n-r}\}, \{x_1, y_t\} \in \mathscr{V}_{K_{r,r+1}}(K_{m,n})$ for all $1 \leq s \leq n-r-1$ and for all $n-r+1 \leq t \leq n$, or equivalently, $\{x_1, y_j\} \in \mathscr{V}_{K_{r,r+1}}(K_{m,n})$ for all $1 \leq j \leq n$. Thus, by Remark 2, $K_{r,r+1}$ is a vertex-generator subgraph of $K_{m,n}$ if $r \geq 2$.

Therefore, in all cases, $K_{r,r+1}$ is a vertex-generator subgraph of $K_{m,n}$. □

3.2. Vertex-Generator Subgraph of Tadpole Graph $T_{n,m}$

This section provides some vertex-generator subgraphs of tadpole graph $T_{n,m}$.

Let $T_{n,m}$ be a tadpole graph whose vertex set is given by $V(T_{n,m}) = V(C_n) \cup V(P_m)$, where $V(C_n) = \{x_1, x_2, x_3, ..., x_{n-1}, x_n\}$ and $V(P_m) = \{y_1, y_2, y_3, ..., y_{m-1}, y_m\}$, and the edge set is given by $E(T_{n,m}) = E(C_n) \cup E(P_m) \cup \{[x_1, y_1]\}$ where $[x_1, y_1]$ is a *bridge*. Presented in Figure 10 is the labeling of a tadpole graph, which will be considered in the discussion of this section.

A tadpole graph $T_{n,m}$ has order $n+m$ and size $n+m$ for all positive integers $n \geq 3$ and m. By Definition 1, the vertex space of $T_{n,m}$ is given by $\mathscr{V}(T_{n,m}) = \{S \mid S \subseteq V(T_{n,m})\}$.

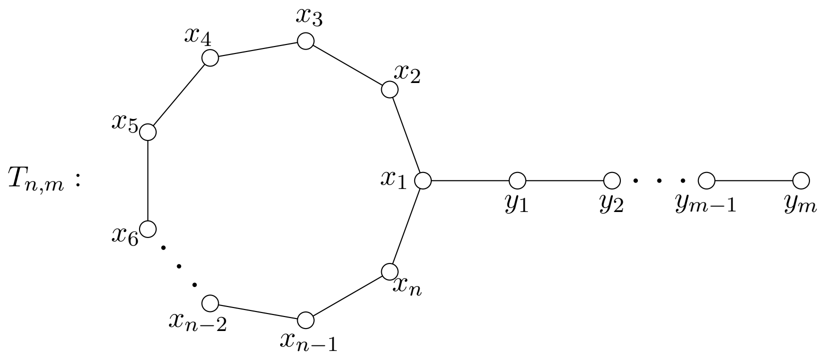

Figure 10: The Labeling of $T_{n,m}$

Given the vertex set $V(T_{n,m})$, the set

$$\mathscr{A} = \{\{x_1\}, \{x_2\}, \{x_3\}, \ldots, \{x_{n-1}\}, \{x_n\}, \{y_1\}, \{y_2\}, \{y_3\}, \ldots, \{y_{m-1}\}, \{y_m\}\}$$

forms a basis for $\mathscr{V}(T_{n,m})$. It follows that $\dim \mathscr{V}(T_{n,m}) = n + m$ by Theorem 1.

By Theorem 2, the trivial graph is a vertex-generator subgraph of $T_{n,m}$, so we are interested in the finding the nontrivial vertex-generator subgraph of $T_{n,m}$.

The following theorem gives us a necessary condition for the disjoint union of a path graph P_t of order t, so t should be even, and a trivial graph K_1, denoted by $P_t \sqcup K_1$, to be a vertex-generator subgraph of $T_{n,m}$.

Theorem 14. *Let $n, m \geq 3$ and $t \geq 2$ be positive integers such that t is even. If $t + 1 \leq \min\{n, m\}$, then $P_t \sqcup K_1$ is a vertex-generator subgraph of $T_{n,m}$.*

Proof. Let $t < \min\{n, m\}$. For any $1 \leq i \leq m$, let x_i be an arbitrary vertex of C_n in the tadpole graph $T_{n,m}$, and for any $1 \leq p \leq t$, let A_p and A be defined as follows:

$$A_p = \{\overbrace{x_2, x_3, x_4, ..., x_{t+1}}^{t \text{ vertices}}, y_{p+1}\} \text{ where } 1 \leq p \leq t, \text{ and}$$
$$A = \{x_i, \overbrace{y_2, y_3, y_4, ..., y_{t+1}}^{t \text{ vertices}}\} \text{ where } 1 \leq i \leq n.$$

It can be verified that $A_p, A \in V_{P_t \sqcup K_1}(T_{n,m})$, as shown in Figure 11.
Thus, for $i = 1, 2, 3, ..., n$, we obtain

$$\begin{aligned}\sum_{p=1}^{t} A_p + A &= (A_1 + A_2 + A_3 + \cdots + A_t) + A \\ &= (A_1 \triangle A_2 \triangle A_3 \triangle \cdots \triangle A_t) \triangle A \\ &= \{y_2, y_3, y_4, ..., y_{t+1}\} \triangle \{x_i, y_2, y_3, y_4, ..., y_{t+1}\} \\ &= \{x_i\}.\end{aligned}$$

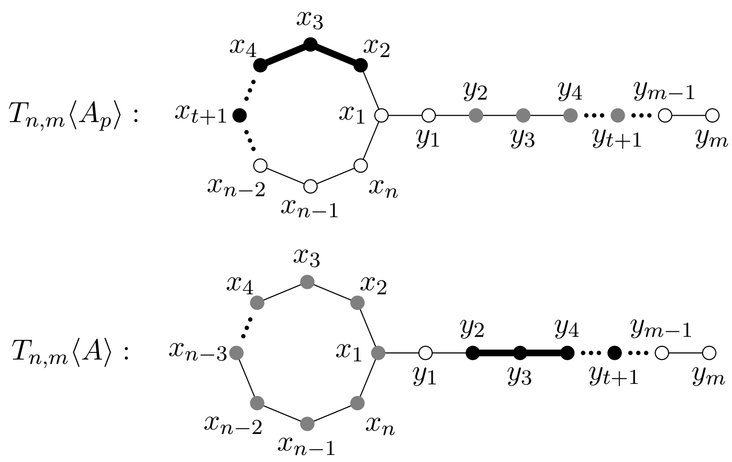

Figure 11: Illustrating the subgraphs of $T_{n,m}$ induced by A_p and A

Hence, $\{x_i\} \in \mathscr{V}_{P_t \sqcup K_1}(T_{n,m})$ for all $1 \leq i \leq n$. Similarly, for any $1 \leq j \leq m$, let y_j be an arbitrary vertex of P_m in the tadpole graph $T_{n,m}$, and for any $1 \leq q \leq t$, let B_q and B be defined as follows:

$$B_q = \{x_{q+1}, \overbrace{y_2, y_3, y_4, ..., y_{t+1}}^{t \text{ vertices}}\} \text{ where } 1 \leq q \leq t, \text{ and}$$
$$B = \{\overbrace{x_2, x_3, x_4, ..., x_{t+1}}^{t \text{ vertices}}, y_j\} \text{ where } 1 \leq j \leq m.$$

It can be verified that $B_q, B \in V_{P_t \sqcup K_1}(T_{n,m})$, as shown in Figure 12.

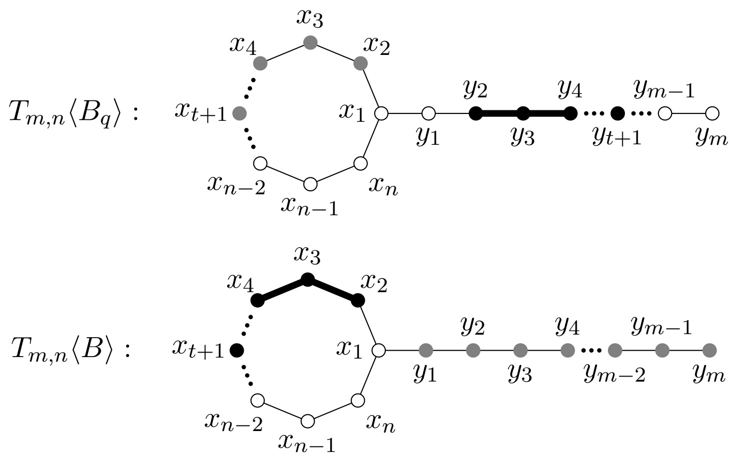

Figure 12: Illustrating the subgraphs of $T_{n,m}$ induced by B_q and B

Hence, for $j = 1, 2, 3, ..., m$, we get

$$\sum_{q=1}^{t} B_q + B = (B_1 + B_2 + B_3 + \cdots + B_t) + B$$

$$= (B_1 \triangle B_2 \triangle B_3 \triangle \cdots \triangle B_t) \triangle B$$
$$= \{x_2, x_3, x_4, ..., x_{t+1}\} \triangle \{x_2, x_3, x_4, ..., x_{t+1}, y_j\}$$
$$= \{y_j\}.$$

Thus, $\{y_j\} \in \mathscr{V}_{P_t \sqcup K_1}(T_{n,m})$ for all $1 \leq j \leq m$. Therefore, by Remark 1, $P_t \sqcup K_1$ is a vertex-generator subgraph of $T_{n,m}$. □

The next theorem gives us a necessary condition for the empty graph $\overline{K_t}$ of order t, so t should be odd, to be a vertex-generator subgraph of $T_{n,m}$.

Theorem 15. *Let $n, m \geq 4$ and $t \geq 3$ be positive integers where t is odd. If $2t - 2 \leq \min\{n, m\}$, then $\overline{K_t}$ is a vertex-generator subgraph of $T_{n,m}$.*

Proof. Let $2t - 2 \leq \min\{n, m\}$. For any $1 \leq i \leq n$, let x_i be an arbitrary vertex of C_n in the tadpole graph $T_{n,m}$, and for any $1 \leq p \leq t - 1$, we define A_p and A as follows:

$$A_p = \{\overbrace{x_2, x_4, x_6, ..., x_{2t-2}}^{t-1 \text{ vertices}}, y_{2p}\} \text{ where } 1 \leq p \leq t - 1, \text{ and}$$
$$A = \{x_i, \overbrace{y_2, y_4, y_6, ..., y_{2t-2}}^{t-1 \text{ vertices}}\} \text{ where } 1 \leq i \leq n.$$

It can be verified that $A_p, A \in V_{\overline{K_t}}(T_{n,m})$, as shown in Figure 13.

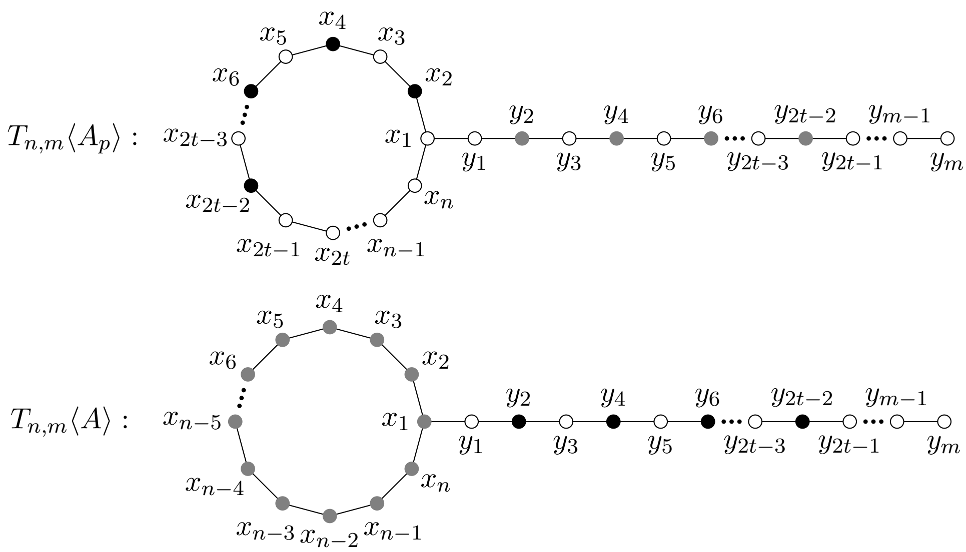

Figure 13: Illustrating the subgraphs of $T_{n,m}$ induced by A_p and A

Hence, for $i = 1, 2, 3, ..., n$, we have

$$\sum_{p=1}^{t-1} A_p + A = (A_1 + A_2 + A_3 + \cdots + A_{t-1}) + A$$

$$
\begin{aligned}
&= (A_1 \triangle A_2 \triangle A_3 \triangle \cdots \triangle A_{t-1}) \triangle A \\
&= \{y_2, y_4, y_6, ..., y_{2t-2}\} \triangle \{x_i, y_2, y_4, y_6, ..., y_{2t-2}\} \\
&= \{x_i\}.
\end{aligned}
$$

Thus, $\{x_i\} \in \mathscr{V}_{\overline{K_t}}(T_{n,m})$ for all $1 \leq i \leq n$. In similar manner, for any $1 \leq j \leq m$, let y_j be an arbitrary vertex of P_m in the tadpole graph $T_{n,m}$, and for any $1 \leq q \leq t-1$, we define B_q and B as follows:

$$
\begin{aligned}
B_q &= \{x_{2q}, \overbrace{y_2, y_4, y_6, ..., y_{2t-2}}^{t-1 \text{ vertices}}\} \text{ where } 1 \leq q \leq t-1, \text{ and} \\
B &= \{\overbrace{x_2, x_4, x_6, ..., x_{2t-2}}^{t-1 \text{ vertices}}, y_j\} \text{ where } 1 \leq j \leq m.
\end{aligned}
$$

It can be verified that $B_q, B \in V_{\overline{K_t}}(T_{n,m})$, as shown in Figure 14.

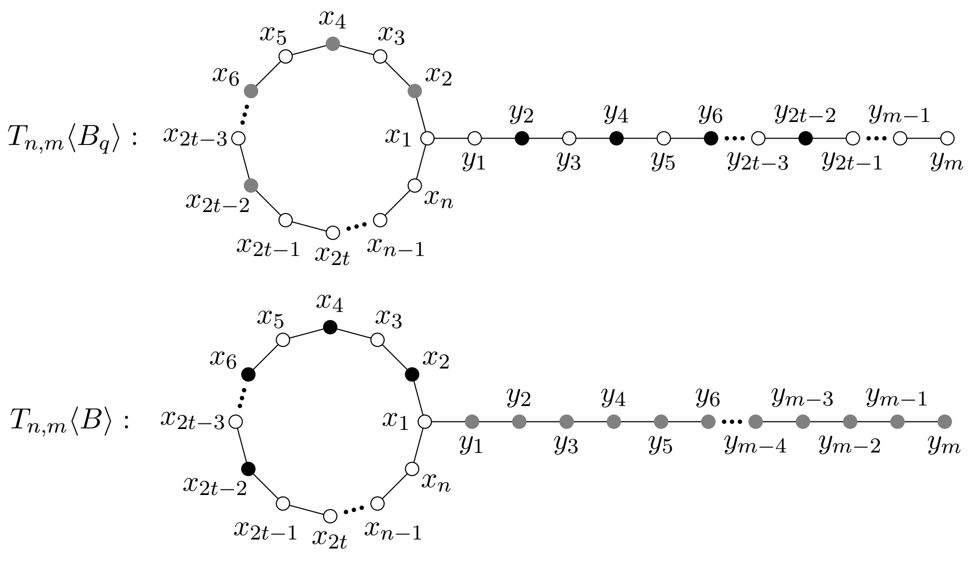

Figure 14: Illustrating the subgraphs of $T_{n,m}$ induced by B_q and B

Thus, for $j = 1, 2, 3, ..., m$, we get

$$
\begin{aligned}
\sum_{q=1}^{t-1} B_q + B &= (B_1 + B_2 + B_3 + \cdots + B_{t-1}) + B \\
&= (B_1 \triangle B_2 \triangle B_3 \triangle \cdots \triangle B_{t-1}) \triangle B \\
&= \{x_2, x_4, x_6, ..., x_{2t-2}\} \triangle \{x_2, x_4, x_6, ..., x_{2t-2}, y_j\} \\
&= \{y_j\}.
\end{aligned}
$$

Hence, $\{y_j\} \in \mathscr{V}_{\overline{K_t}}(T_{n,m})$ for all $1 \leq j \leq m$. Therefore, by Remark 1, $\overline{K_t}$ is a vertex-generator subgraph of $T_{n,m}$. □

The next theorem gives us a necessary condition for the graph $kP_2 \sqcup K_1$, which is the disjoint union of the graph kP_2 and a trivial graph K_1, to be a vertex-generator subgraph of $T_{n,m}$.

Theorem 16. *Let $n, m \geq 3$ and k be positive integers. If $3k \leq \min\{n, m\}$, then $kP_2 \sqcup K_1$ is a vertex-generator subgraph of $T_{n,m}$.*

Proof. Let $3k \leq \min\{n, m\}$. Then, for any $1 \leq i \leq n$, let x_i be an arbitrary vertex of C_n in the tadpole graph $T_{n,m}$, and for any $1 \leq p \leq k$, let A_{2p-1}, A_{2p} and A be defined as follows:

$$\begin{aligned} A_{2p-1} &= \{\overbrace{\overbrace{x_2, x_3}^{P_2}, \overbrace{x_5, x_6}^{P_2}, \ldots, \overbrace{x_{3k-1}, x_{3k}}^{P_2}}^{k \text{ copies of } P_2}, y_{3p-1}\} \text{ where } 1 \leq p \leq k, \\ A_{2p} &= A_{2p-1} \setminus \{y_{3p-1}\} \cup \{y_{3p}\} \text{ where } 1 \leq p \leq k, \text{ and} \\ A &= \{x_i, \overbrace{\overbrace{y_2, y_3}^{P_2}, \overbrace{y_5, y_6}^{P_2}, \ldots, \overbrace{y_{3k-1}, y_{3k}}^{P_2}}^{k \text{ copies of } P_2}\} \text{ where } 1 \leq i \leq n. \end{aligned}$$

It can be verified that that $A_{2p-1}, A_{2p}, A \in V_{kP_2 \sqcup K_1}(T_{n,m})$, as shown in Figure 15.
Thus, for $i = 1, 2, 3, ..., n$, we obtain

$$\begin{aligned} \sum_{p=1}^{k} A_{2p-1} + \sum_{p=1}^{k} A_{2p} + A &= (A_1 + A_3 + \cdots + A_{2k-1}) + (A_2 + A_4 + \cdots + A_{2k}) + A \\ &= (A_1 + A_2 + A_3 + A_4 + \cdots + A_{2k-1} + A_{2k}) + A \\ &= (A_1 \triangle A_2 \triangle A_3 \triangle A_4 \triangle \cdots \triangle A_{2k-1} \triangle A_{2k}) \triangle A \\ &= \{y_2, y_3, y_5, y_6, ..., y_{3k-1}, y_{3k}\} \triangle \{x_i, y_2, y_3, y_5, y_6, ..., y_{3k-1}, y_{3k}\} \\ &= \{x_i\}. \end{aligned}$$

Hence, $\{x_i\} \in \mathscr{V}_{kP_2 \sqcup K_1}(T_{n,m})$ for all $1 \leq i \leq n$. Similarly, for any $1 \leq j \leq m$, let y_j be an arbitrary vertex of P_m in the tadpole graph $T_{n,m}$, and for any j where $1 \leq q \leq k$, let B_{2q-1}, B_{2q} and B be defined as follows:

$$\begin{aligned} B_{2q-1} &= \{x_{3q-1}, \overbrace{\overbrace{y_2, y_3}^{P_2}, \overbrace{y_5, y_6}^{P_2}, \ldots, \overbrace{y_{3k-1}, y_{3k}}^{P_2}}^{k \text{ copies of } P_2}\} \text{ where } 1 \leq q \leq k, \\ B_{2q} &= B_{2q-1} \setminus \{x_{3q-1}\} \cup \{x_{3q}\} \text{ where } 1 \leq q \leq k, \text{ and} \\ B &= \{\overbrace{\overbrace{x_2, x_3}^{P_2}, \overbrace{x_5, x_6}^{P_2}, \ldots, \overbrace{x_{3k-1}, x_{3k}}^{P_2}}^{k \text{ copies of } P_2}, y_j\} \text{ where } 1 \leq j \leq m. \end{aligned}$$

It can be verified that $B_{2q-1}, B_{2q}, B \in V_{kP_2 \sqcup K_1}(T_{n,m})$, as shown in Figure 16.

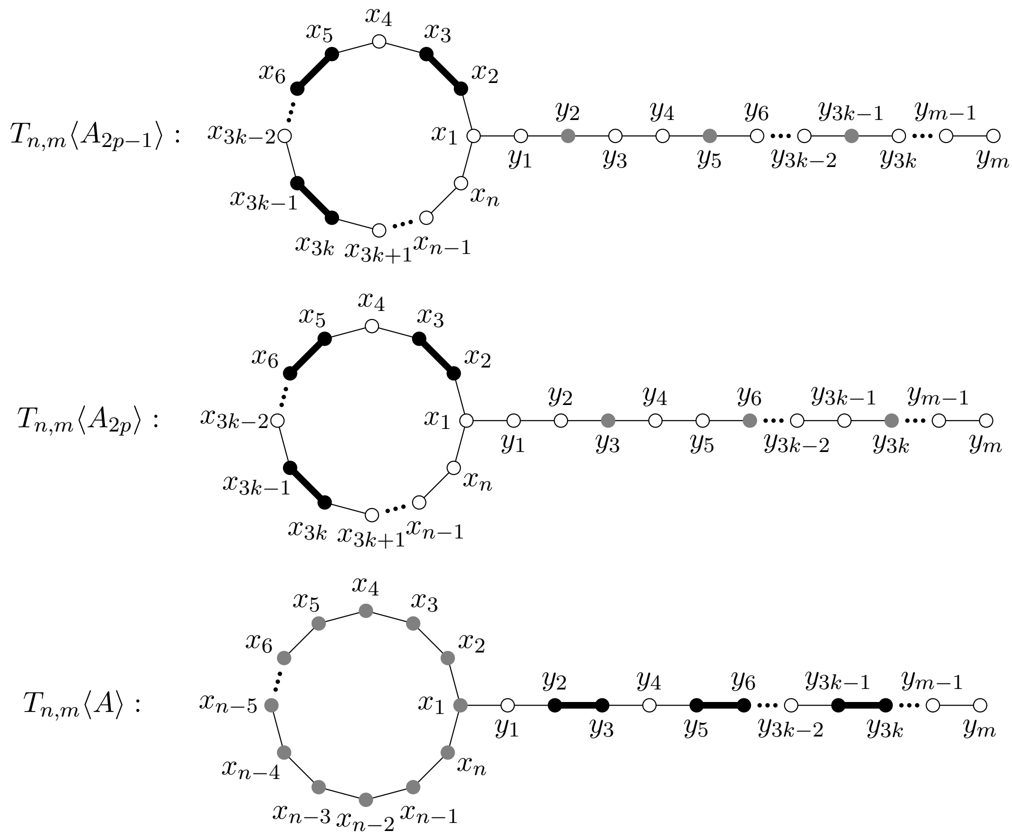

Figure 15: Illustrating the subgraphs of $T_{n,m}$ induced by A_{2p-1}, A_{2p}, and A

Thus, for $j = 1, 2, 3, ..., m$, we have

$$\begin{aligned}\sum_{q=1}^{k} B_{2q-1} + \sum_{q=1}^{k} B_{2q} + B &= (B_1 + B_3 + \cdots + B_{2k-1}) + (B_2 + B_4 + \cdots + B_{2k}) + B \\ &= (B_1 + B_2 + B_3 + B_4 + \cdots + B_{2k-1} + B_{2k}) + B \\ &= (B_1 \triangle B_2 \triangle B_3 \triangle B_4 \triangle \cdots \triangle B_{2k-1} \triangle B_{2k}) \triangle B \\ &= \{x_2, x_3, x_5, x_6, ..., x_{3k-1}, x_{3k}\} \triangle \{x_2, x_3, x_5, x_6, ..., x_{3k-1}, x_{3k}, y_j\} \\ &= \{y_j\}.\end{aligned}$$

Hence, $\{y_j\} \in \mathscr{V}_{kP_2 \sqcup K_1}(T_{n,m})$ for all $1 \leq j \leq m$. Therefore, by Remark 1, $kP_2 \sqcup K_1$ is a vertex-generator subgraph of $T_{n,m}$. □

The following theorem gives us a necessary condition for the graph $P_2 \sqcup \overline{K_t}$, which is the disjoint union of a path graph P_2, and an empty graph $\overline{K_t}$ of order t, so t should be odd, to be a vertex-generator subgraph of $T_{n,m}$.

Theorem 17. *Let $n, m \geq 3$ and t be positive integers where t is odd. If $2t+1 \leq \min\{n, m\}$, then $P_2 \sqcup \overline{K_t}$ is a vertex-generator subgraph of $T_{n,m}$.*

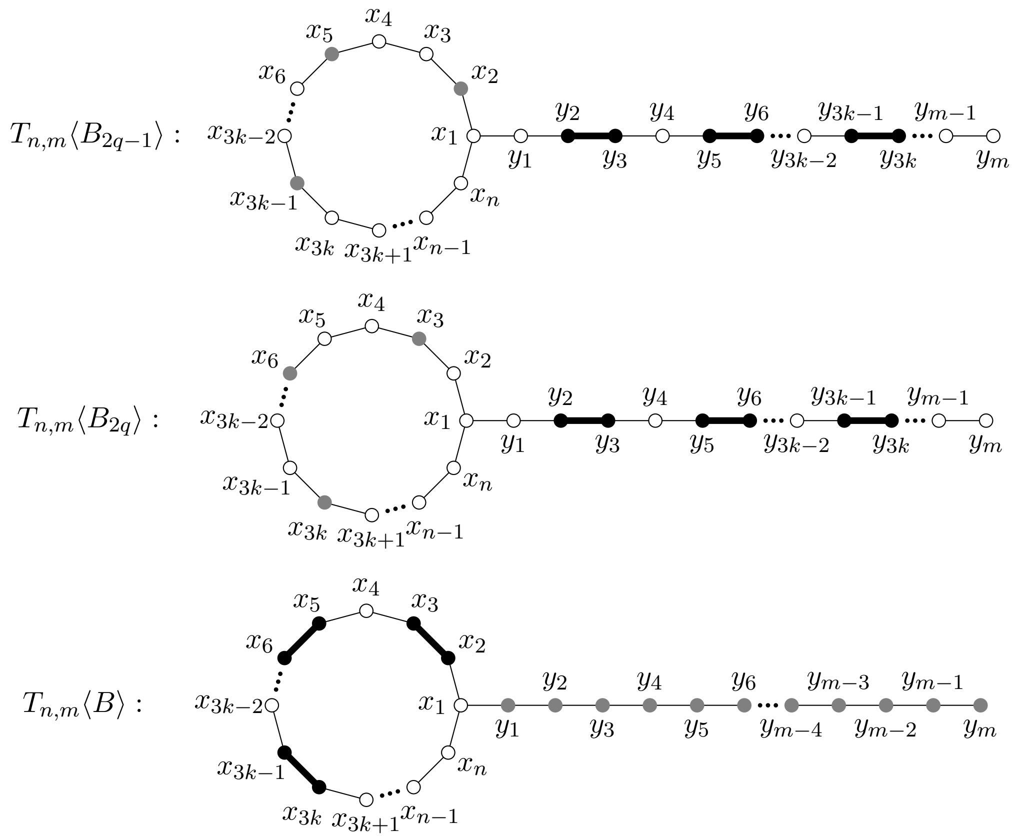

Figure 16: Illustrating the subgraphs of $T_{n,m}$ induced by B_{2q-1}, B_{2q}, and B

Proof. Let $2t+1 \leq \min\{n,m\}$. Then, for any $1 \leq i \leq n$, let x_i be an arbitrary vertex of C_n in the tadpole graph $T_{n,m}$, and for any $1 \leq p \leq t$, let A_1, A_{p+1}, and A be defined as follows:

$$A_1 = \{\overbrace{x_2, x_3}^{P_2}, \overbrace{x_5, x_7, ..., x_{2t+1}}^{t-1 \text{ vertices}}, y_2\},$$
$$A_{p+1} = A_1 \setminus \{y_2\} \cup \{y_{2p+1}\} \text{ where } 1 \leq p \leq t, \text{ and}$$
$$A = \{x_i, \overbrace{y_2, y_3}^{P_2}, \overbrace{y_5, y_7, ..., y_{2t+1}}^{t-1 \text{ vertices}}\} \text{ where } 1 \leq i \leq n.$$

It can be verified that that $A_1, A_{p+1}, A \in V_{P_2 \sqcup \overline{K_t}}(T_{n,m})$, as shown in Figure 17.
Hence, for $i = 1, 2, 3, ..., n$, we get

$$\begin{aligned} A_1 + \sum_{p=1}^{t} A_{p+1} + A &= A_1 + (A_2 + A_3 + A_4 + \cdots + A_{t+1}) + A \\ &= (A_1 + A_2 + A_3 + A_4 + \cdots + A_{t+1}) + A \\ &= (A_1 \triangle A_2 \triangle A_3 \triangle A_4 \triangle \cdots \triangle A_{t+1}) \triangle A \end{aligned}$$

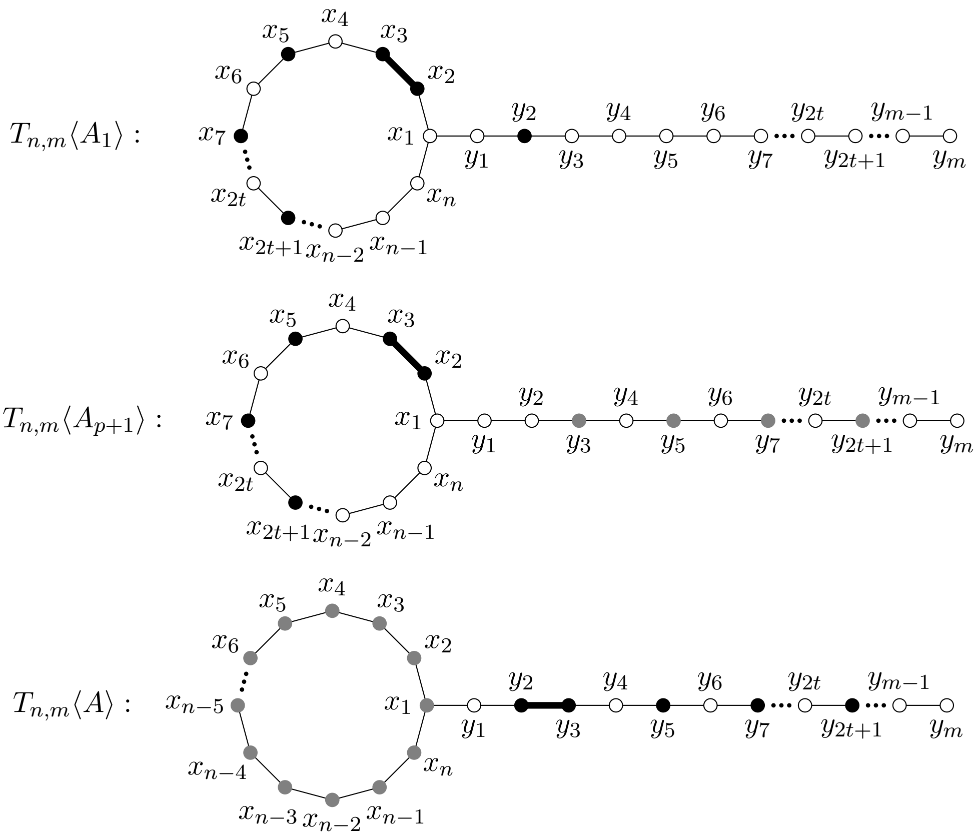

Figure 17: Illustrating the subgraphs of $T_{n,m}$ induced by A_1, A_{p+1}, and A

$$= \{y_2, y_3, y_5, y_7, ..., y_{2t+1}\} \triangle \{x_i, y_2, y_3, y_5, y_7, ..., y_{2t+1}\}$$
$$= \{x_i\}.$$

Consequently, $\{x_i\} \in \mathscr{V}_{P_2 \sqcup \overline{K_t}}(T_{n,m})$ for all $1 \leq i \leq n$. Similarly, for any $1 \leq j \leq m$, let y_j be an arbitrary vertex of P_m in the tadpole graph $T_{n,m}$, and for any $1 \leq q \leq t$, let B_1, B_{q+1} and B be defined as follows:

$$B_1 = \{x_2, \overbrace{y_2, y_3}^{P_2}, \overbrace{y_5, y_7, ..., y_{2t+1}}^{t-1 \text{ vertices}}\},$$
$$B_{q+1} = B_1 \setminus \{x_2\} \cup \{x_{2q+1}\} \text{ where } 1 \leq q \leq t, \text{ and}$$
$$B = \{\overbrace{x_2, x_3}^{P_2}, \overbrace{x_5, x_7, ..., x_{2t+1}}^{t-1 \text{ vertices}}, y_j\} \text{ where } 1 \leq j \leq m.$$

It can be verified that $B_1, B_{q+1}, B \in V_{P_2 \sqcup \overline{K_t}}(T_{n,m})$, as shown in Figure 18.
Thus, for $j = 1, 2, 3, ..., m$, we get

$$B_1 + \sum_{q=1}^{t} B_{q+1} + B = B_1 + (B_2 + B_3 + B_4 + \cdots + B_{t+1}) + B$$

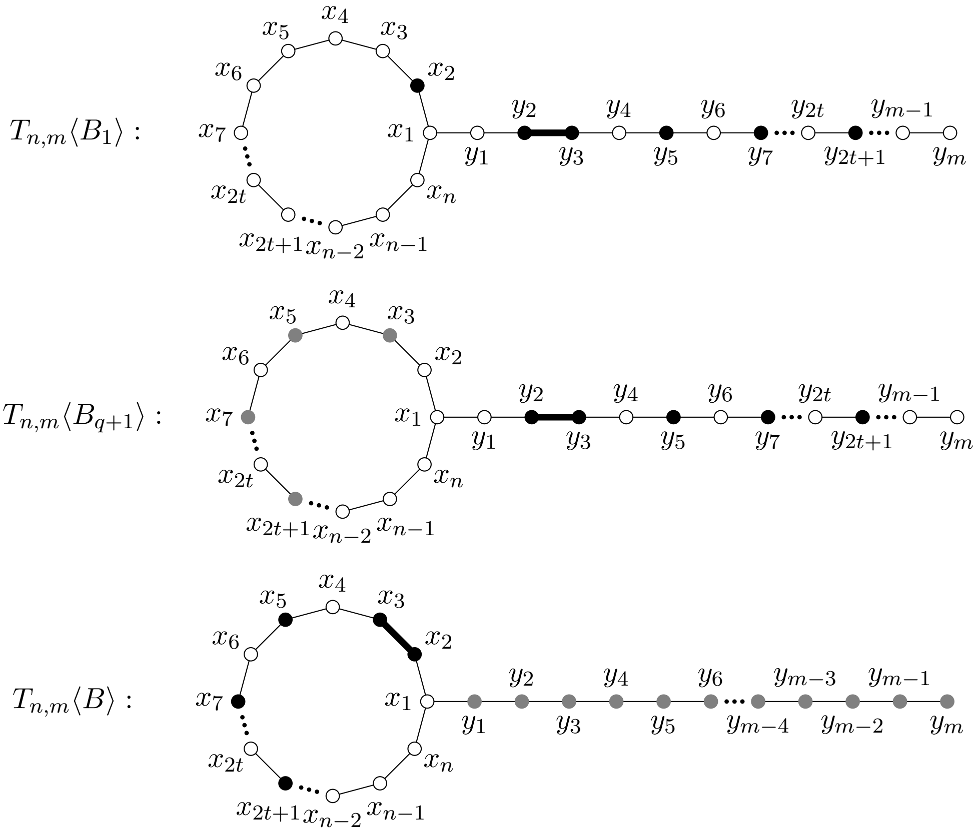

Figure 18: Illustrating the subgraphs of $T_{n,m}$ induced by B_1, B_{q+1}, and B

$$\begin{aligned}
&= (B_1 + B_2 + B_3 + B_4 + \cdots + B_{t+1}) + B \\
&= (B_1 \triangle B_2 \triangle B_3 \triangle B_4 \triangle \cdots \triangle B_{t+1}) \triangle B \\
&= \{x_2, x_3, x_5, x_7, ..., x_{2t+1}\} \triangle \{x_2, x_3, x_5, x_7, ..., x_{2t+1}, y_j\} \\
&= \{y_j\}.
\end{aligned}$$

Hence, $\{y_j\} \in \mathscr{V}_{P_2 \sqcup \overline{K_t}}(T_{n,m})$ for all $1 \leq j \leq m$. Therefore, by Remark 1, $P_2 \sqcup \overline{K_t}$ is a vertex-generator subgraph of $T_{n,m}$. □

4. Conclusions

This paper provides some vertex-generator subgraphs of $K_{m,n}$, such as the empty graph, path graph, star graph, and complete bipartite graph $K_{r,r+1}$. This study also provides some vertex-generator subgraphs of $T_{n,m}$, such as the empty graph and the disjoint union of graphs $P_t \sqcup K_1$, $kP_2 \sqcup K_1$, and $P_2 \sqcup \overline{K_t}$. It is recommended to find the other vertex-generator subgraphs of $K_{m,n}$ and $T_{n,m}$. Furthermore, characterization for the vertex-generator subgraphs of the two graphs remains open for research.

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References

[1] R. Diestel. *Graph Theory*, volume 173 of *Graduate Text in Mathematics*. Springer-Verlag, New York, second edition, 2000.

[2] S. Butenko, P. Festa, and P.M. Pardalos. On the chromatic number of graphs. *Journal of Optimization Theory and Application*, 109:69–82, 2001.

[3] J. Torino and N. Mame. On the vertex-generator subgraph of a graph. Contributed paper, The Asian Mathematical Conference, Chiang Mai, Thailand, August 3-7, 2025.

[4] N. Mame and S. Gervacio. A Note on the Generator Subgraph of a Graph. *Electronic Journal of Graph Theory and Applications*, 8(1):17–27, 2020.

[5] R. Mame. Some Generator Subgraphs of the Square of a Cycle. *Euporean Journal of Pure and Applied Mathematics*, 17(4):3815–3825, 2024.

[6] F. Harary. *Graph Theory*. Addison-Wesley Series in Mathematics. Addison-Wesley Publishing Company, Inc., New York, 1969.

[7] H. Komarullah, J. Halilim, and K. Santoso. On the minimum span of cone, tadpole, and barbell graphs. In *Proceedings of International Conference on Mathematics, Geometry, Statistics and Computation*, page 41, 2021.

[8] B. Bollobás. *Random Graphs*. Number 73 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, New York, second edition, 2001.

[9] J. A. Bondy and U. S. R. Murty. *Graph Theory*, volume 244 of *Graduate Text in Mathematics*. Springer-Verlag, London, first edition, 2008.

[10] G. Chartrand, L. D. Lesniak, and P. Zhang. *Graphs and Digraphs*, volume 39 of *Textbooks in Mathematics*. CRC Press, Taylor & Francis Group, Boca Raton, sixth edition, 2016.

[11] E. D. Nering. *Linear Algebra and Matrix Theory*. John Wiley & Sons, Inc., New York, second edition, 1970.

[12] R. Larson and D. C. Falvo. *Elementary Linear Algebra*, chapter Vector Spaces, pages 179–230. Houghton Mifflin Harcourt Publishing Company, Boston, sixth edition, 2009.