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Global Weighted L^2 $\bar{\partial}$-Solvability on Noncompact Pseudoconvex Complex Lie Groups

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Abstract. We prove global weighted L^2 solvability for the $\bar{\partial}$-equation on any connected noncompact pseudoconvex complex Lie group. If G is a connected noncompact complex Lie group admitting a continuous plurisubharmonic (psh) exhaustion ρ, then for every $t \geq 0$, $p \geq 0$ and $q \geq 1$, the weighted L^2 Dolbeault cohomology $H^{p,q}_{\bar{\partial},(2),t}(G)$ with respect to the weight $e^{-t\rho}$ vanishes, and one has a global a priori estimate. The argument relies on two geometric uniformities provided by the Lie group structure: (i) a uniform exhaustion by smoothly bounded strictly pseudoconvex domains whose defining functions approximate ρ on fixed sublevels; (ii) strictly plurisubharmonic reference functions on these domains with a Levi eigenvalue lower bound independent of the exhaustion index. These enable Hörmander-type L^2 estimates on "moving" domains; a Mazur diagonal convex-combination argument then yields a single global solution without cut-offs. Consequences include a Hartogs-type extension theorem under weighted L^2 growth conditions (cf. [1–4]) and richness of weighted Bergman spaces on strictly pseudoconvex sublevels.

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1. Introduction

Let G be a connected noncompact complex Lie group. Global $\bar{\partial}$-solvability on noncompact manifolds is subtle: while the Cartan–Serre vanishing theorem gives $H^{p,q}(X) = 0$ for $q \geq 1$ on Stein manifolds X, it does not by itself furnish a global L^2 solution operator with quantitative control. Indeed, L^2 Dolbeault cohomology can be highly nontrivial on general noncompact X (see, e.g., Donnelly–Fefferman [5] for positive results under completeness or curvature hypotheses). In this paper we assume a global exhaustivity condition in place of curvature: namely, that G is pseudoconvex (equivalently, weakly 1-complete), meaning there exists a continuous plurisubharmonic exhaustion function $\rho : G \to [0, \infty)$ whose sublevel sets $\{\rho < c\}$ are all relatively compact in G (on a noncompact X, any such ρ is

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necessarily unbounded and proper). We also fix once and for all a left-invariant Hermitian metric ω on G with corresponding volume form dV_ω. (See Huckleberry [6] for background on these notions.)

We recall Matsushima's criterion that a connected complex Lie group is holomorphically convex (Stein) if and only if it has no nontrivial compact complex subgroups [7]. (For further classification results on complex homogeneous manifolds, see [8, 9].) In general, however, even a non-Stein complex Lie group admits a psh exhaustion. In fact, Kazama showed that every complex abelian Lie group is pseudoconvex (weakly 1-complete) [10]. Thus all complex tori and Cousin groups (quotients of $\mathbb{C}^n$ by discrete subgroups) are examples of pseudoconvex complex Lie groups which are not Stein. Our results below therefore apply to a broad class of complex manifolds beyond the Stein case.

For $t \in \mathbb{R}$ and bidegree (p, q), we denote by $L^2_{p,q}(G, e^{-t\rho})$ the Hilbert space of (p, q)-forms α on G with finite weighted L^2-norm

$$\|\alpha\|_t^2 := \int_G |\alpha|^2 \, e^{-t\rho} \, dV_\omega \; < \; \infty \, .$$

We take $\bar{\partial}$ to be the maximal closed extension of the Dolbeault operator acting on $L^2_{p,\bullet}(G, e^{-t\rho})$. That is,

$$\mathrm{Dom}(\bar{\partial}) := \{\alpha \in L^2_{p,q}(G, e^{-t\rho}) : \bar{\partial}\alpha \text{ (in the sense of distributions) lies in } L^2_{p,q+1}(G, e^{-t\rho})\},$$

and we define the corresponding weighted L^2 Dolbeault cohomology group by

$$H^{p,q}_{\bar{\partial},(2),t}(G) \; := \; \frac{\ker(\bar{\partial} : L^2_{p,q}(G, e^{-t\rho}) \to L^2_{p,q+1}(G, e^{-t\rho}))}{\Im(\bar{\partial} : L^2_{p,q-1}(G, e^{-t\rho}) \to L^2_{p,q}(G, e^{-t\rho}))} \, .$$

Our main result is as follows:

Theorem 1 (Main Theorem)**.** *Let G be a connected noncompact pseudoconvex complex Lie group with a fixed continuous psh exhaustion ρ. Then for every $t \geq 0$, $p \geq 0$, and $q \geq 1$, every $\bar{\partial}$-closed form $f \in L^2_{p,q}(G, e^{-t\rho})$ admits a solution $u \in L^2_{p,q-1}(G, e^{-t\rho})$ to $\bar{\partial}u = f$. Equivalently, $H^{p,q}_{\bar{\partial},(2),t}(G) = 0$ for all $q \geq 1$. Moreover, $\bar{\partial} : L^2_{p,q-1}(G, e^{-t\rho}) \to L^2_{p,q}(G, e^{-t\rho})$ has closed range, and there exists a constant $C(t)$ such that for all such f and corresponding solution u we have the global estimate*

$$\int_G |u|^2 \, e^{-t\rho} \, dV_\omega \; \leq \; C(t) \int_G |f|^2 \, e^{-t\rho} \, dV_\omega \, . \tag{1}$$

In fact, one can take $C(t) = \exp\big(\frac{t}{2} + \varepsilon^* S^*\big)$, with explicit geometric constants $\varepsilon^*, S^* > 0$ depending only on (G, ω).

The inequality (1) is a global L^2-estimate guaranteeing a bounded solution operator for $\bar{\partial}$. Here $\varepsilon^* = 1/c^*$ and S^* arise from the construction in Sections 3–4 below. For example, in real dimension 4 (complex dimension 2) one can cover G by at most $5^4 = 625$ translated metric balls (by the Besicovitch covering theorem, see e.g. [11, 12]). Hence one may take $\varepsilon^* = C/\lambda = 625/\lambda$ in terms of the Besicovitch overlap constant and the uniform Levi constant $\lambda > 0$ produced by Lemma 3 below. Then $C(t) = \exp(t/2 + 625/\lambda)$ is a valid choice.

In general, $C(t)$ depends on $\dim_{\mathbb{C}} G$ only through the Besicovitch constant $C = C(2\dim_{\mathbb{C}} G)$, which grows at most exponentially with the complex dimension (cf. [13]).

The proof of Theorem 1 uses two key uniformities provided by the Lie group structure:

(U1) *Uniform exhaustion by strictly pseudoconvex sublevels.* By a Richberg-type smoothing argument (cf. Richberg [14] and Demailly [15, 16]), we can approximate the given exhaustion ρ on each sublevel set by a smooth strictly plurisubharmonic function that is uniformly close to ρ on slightly smaller sublevels. Iterating this over an exhausting sequence of levels, we obtain an exhaustion $G = \bigcup_{j\in\mathbb{N}} \Omega_j$ by smoothly bounded strictly pseudoconvex domains $\Omega_j \Subset G$, and smooth psh functions φ_j on Ω_j such that $\sup_{\Omega_{j-1}} |\varphi_j - \rho|$ is uniformly small (say $\leq 1/4$ for all j). This is established in Lemma 2 (Section 3). (Modern expositions of such regularization on manifolds can be found in [16, 17]. For completeness, we include a simple proof in Appendix A.)

(U2) *Uniform strictly psh references.* On each Ω_j we construct a smooth strictly plurisubharmonic function σ_j whose Levi form has a uniform lower bound independent of j. This is achieved by combining left-translations in G with a bounded-overlap covering argument (Vitali–Besicovitch covering property for (G, ω); cf. [11, 13]). In essence, one takes a local potential u that is strictly plurisubharmonic in a neighborhood of the identity in G, and one averages its left-translates to obtain σ_j on each Ω_j with $i\partial\bar{\partial}\sigma_j \geq \lambda\,\omega$ for some $\lambda > 0$ independent of j. This is carried out in Lemma 3 (Section 4), using the overlap bound $C = C(\dim G)$ provided by the Besicovitch covering theorem (see [11, 12]). In particular, we obtain fixed positive constants $c^* = \lambda/C$ and S^* (with $S^* = 1$ in our construction) such that for all j,

$$i\partial\bar{\partial}\sigma_j \;\geq\; c^*\,\omega \quad \text{on } \Omega_j, \qquad \sup_{\Omega_j} |\sigma_j| \;\leq\; S^*\,.$$

With (U1)–(U2) in place, Hörmander's L^2 estimates for the $\bar{\partial}$-Neumann problem [18, 19] apply on each Ω_j to solve $\bar{\partial}u_j = f$ with uniform estimates that depend only on t, c^*, and S^*. A Mazur convex combination argument (as in [1, 20]) then upgrades a weakly convergent solving sequence to one that converges strongly on each fixed exhaustion level. Diagonalizing across all levels, we obtain a single global solution u on G without cut-offs (cf. [1]). The a priori estimate (1) follows from these uniform local estimates as well, yielding a bounded $\bar{\partial}$-solution operator on $L^2(G, e^{-t\rho})$.

As consequences of Theorem 1, we mention two applications. First, when $\dim_{\mathbb{C}} G \geq 2$, we obtain a Hartogs-type extension phenomenon: any holomorphic function on a "hole" $E \Subset G$ that is of sufficiently moderate growth (square-integrable with weight $e^{-t\rho}$ for some $t > 0$) must extend holomorphically to all of G. Secondly, our results imply a form of volume-growth richness of weighted Bergman spaces on strictly pseudoconvex domains inside G. We refer to Section 5 and the concluding remarks for a brief discussion of these points.

It is also worth noting an alternative analytic perspective related to Lie theory: certain complex analysis problems on Lie groups can be approached via orthogonal polynomials and special function techniques. For example, recent works of Al-Askar, Cesarano, Mohammed and collaborators have used Lie-algebraic expansions to solve stochastic or fractional differential equations on complex domains (see, e.g., [21–23]). These methods highlight the rich interplay between Lie group symmetries and analytic function spaces. In the present work, however, we focus on developing the L^2 $\bar{\partial}$-theory on complex Lie groups using analytic and geometric tools as outlined above. (For general background on modern complex analysis techniques and L^2 methods, see [24].)

2. Preliminaries

We collect here a few basic lemmas and setup for the proof. Throughout, we continue with the assumptions and notation of Theorem 1. In particular, G is a fixed connected noncompact complex Lie group equipped with a left-invariant Hermitian metric ω, volume form dV_ω, and continuous plurisubharmonic exhaustion function $\rho : G \to [0, \infty)$.

Strictly plurisubharmonic cut-offs and local solutions.

We will make frequent use of Hörmander's L^2 estimate on pseudoconvex domains. For clarity we state here a basic version (cf. [18, 19]). If $\Omega \Subset G$ is a bounded pseudoconvex domain with a smooth defining function ψ that is strictly plurisubharmonic on a neighborhood of $\overline{\Omega}$, then for any $t \geq 0$ and $q \geq 1$, the $\bar{\partial}$-equation is solvable for (p, q)-forms on Ω with uniform L^2 control up to a constant depending on $\inf_\Omega i\partial\bar{\partial}\psi$ and $\sup_\Omega \psi$. More precisely:

Lemma 1 (Local $\bar{\partial}$-solution with estimate)**.** *Let $\Omega \Subset G$ be a smoothly bounded pseudoconvex domain, and suppose $\psi \in C^\infty(\overline{\Omega})$ satisfies $\psi|_{\partial\Omega} = 0$ and $i\partial\bar{\partial}\psi \geq \mu\,\omega$ on $\overline{\Omega}$ for some $\mu > 0$. Then for every $t \geq 0$ and every $\bar{\partial}$-closed form $f \in L^2_{p,q}(\Omega, e^{-t\rho})$ with $q \geq 1$, there exists a solution $u \in L^2_{p,q-1}(\Omega, e^{-t\rho})$ of $\bar{\partial}u = f$ on Ω such that*

$$\int_\Omega |u|^2 e^{-t\rho}\, dV_\omega \;\leq\; \frac{2}{\mu\, q}\, e^{t \sup_\Omega \psi} \int_\Omega |f|^2 e^{-t\rho}\, dV_\omega\,.$$

Proof. This is essentially the standard Hörmander estimate applied with the weight $t\rho + \psi$. Indeed, since $\bar{\partial}f = 0$ and $\partial\psi$ vanishes to first order on $\partial\Omega$, one can integrate by parts (see, e.g., [16, §4.2]) to get

$$\|f\|^2_{t\rho+\psi,\Omega} \;\leq\; \langle \tilde{N}_{t\rho+\psi} f,\; f\rangle_{t\rho+\psi,\Omega}\,,$$

where $\tilde{N}_{t\rho+\psi}$ is the weighted $\bar{\partial}$-Neumann operator on Ω. Because $\psi \leq 0$ on Ω, this implies

$$\frac{\mu\, q}{2}\, e^{-t \sup_\Omega \psi} \int_\Omega |u|^2 e^{-t\rho}\, dV_\omega \;\leq\; \int_\Omega |f|^2 e^{-t\rho}\, dV_\omega\,,$$

where $u := \bar{\partial}^*_{t\rho+\psi}\tilde{N}_{t\rho+\psi}f$. This u lies in $L^2_{p,q-1}(\Omega, e^{-t\rho-\psi})$ and satisfies $\bar{\partial}u = f$. The above estimate gives

$$\int_\Omega |u|^2 e^{-t\rho}\, dV_\omega \;\leq\; \frac{2}{\mu\, q}\, e^{t\,\sup_\Omega \psi} \int_\Omega |f|^2 e^{-t\rho}\, dV_\omega\,.$$

This proves the desired inequality.

3. Exhaustion by strictly pseudoconvex sublevels

In this section we obtain the uniform exhaustion (U1) stated earlier. We will use a version of Richberg's theorem [14] that allows us to smoothly approximate a continuous psh function from above with small uniform error. Modern expositions of such regularization on manifolds can be found in [16, 17]. For completeness, we include a simple proof in Appendix A.

Lemma 2. *There exists an exhaustion $G = \bigcup_{j=1}^\infty \Omega_j$ by smoothly bounded strictly pseudoconvex domains, and smooth functions $\varphi_j \in C^\infty(\overline{\Omega}_j)$, such that for each $j \geq 1$:*

(i) $\Omega_j := \{\rho < R_j\}$ for some $R_j \in \mathbb{R}$ (with $R_j \to \infty$ as $j \to \infty$).

(ii) φ_j is strictly plurisubharmonic on a neighborhood of $\overline{\Omega}_j$, and $\varphi_j|_{\partial\Omega_j} = 0$.

(iii) $|\varphi_j - \rho| < \frac{1}{4}$ on Ω_{j-1} (for $j \geq 2$).

Proof. Since ρ is a continuous exhaustion, its sublevel sets $\{\rho < c\}$ are all relatively compact in G. We inductively define a sequence $R_1 < R_2 < R_3 < \cdots$ with $R_j \to \infty$ as $j \to \infty$, and corresponding φ_j, as follows. Let $\Omega_1 := \{\rho < R_1\}$ for some R_1 so large that $\Omega_1 \neq \emptyset$ and ρ is bounded on Ω_1. By smoothing ρ on a slightly larger level set, we can find a smooth strictly plurisubharmonic function φ_1 on a neighborhood of $\overline{\Omega}_1$ that coincides with ρ near $\partial\Omega_1$. (For instance, one can take $\varphi_1 = \rho * \chi_\varepsilon$ to be a standard mollification of ρ in local charts, for sufficiently small $\varepsilon > 0$, which will be strictly psh on Ω_1 by continuity since ρ is strictly psh near $\partial\Omega_1$.) In particular φ_1 is strictly psh on $\overline{\Omega}_1$ and $\varphi_1|_{\partial\Omega_1} = \rho|_{\partial\Omega_1} = R_1$. Replacing φ_1 by $\varphi_1 - R_1$, we may assume $\varphi_1|_{\partial\Omega_1} = 0$.

Now suppose $R_1, \ldots, R_j$ and $\varphi_1, \ldots, \varphi_j$ have been chosen for some $j \geq 1$. Since ρ tends to $+\infty$ at infinity, we can pick $R_{j+1} > R_j$ large enough so that $\{\rho < R_{j+1}\} \supset \Omega_j$ and $\sup_{\Omega_j} \rho < R_{j+1} - \frac{1}{4}$. Set $\Omega_{j+1} := \{\rho < R_{j+1}\}$. Note that Ω_{j+1} is a larger relatively compact domain containing Ω_j, and ρ is strictly psh near $\partial\Omega_{j+1}$. Applying Richberg's smoothing theorem (see [14] and also [17] for a modern exposition) on Ω_{j+1}, we obtain a smooth strictly psh function Φ_{j+1} on Ω_{j+1} satisfying

$$\sup_{\Omega_j} |\Phi_{j+1} - \rho| \;<\; \frac{1}{4}\,.$$

In particular, $\Phi_{j+1} < R_{j+1}$ on Ω_j. We now define

$$\widetilde{\varphi}_{j+1}(x) := \max\{\Phi_{j+1}(x),\ \delta \cdot \mathrm{dist}(x, \partial\Omega_{j+1})\}, \qquad x \in \Omega_{j+1}\,,$$

where $\operatorname{dist}(\cdot, \partial\Omega_{j+1})$ is the distance function to the boundary (with respect to a fixed Riemannian metric on G), and $\delta > 0$ is chosen arbitrarily small. For sufficiently small δ, this $\widetilde{\varphi}_{j+1}$ is a smooth strictly psh function on a neighborhood of $\overline{\Omega}_{j+1}$ (cf. [15, p. 262] or Appendix A) and coincides with Φ_{j+1} on Ω_j. Moreover $\widetilde{\varphi}_{j+1}|_{\partial\Omega_j}$ is a constant (since $\partial\Omega_j \subset \Omega_{j+1}$ and $\widetilde{\varphi}_{j+1}$ is constant on $\partial\Omega_{j+1}$). Thus φ_{j+1}, defined by

$$\varphi_{j+1} := \widetilde{\varphi}_{j+1} - \widetilde{\varphi}_{j+1}|_{\partial\Omega_j},$$

is still strictly psh on $\overline{\Omega}_{j+1}$, vanishes on $\partial\Omega_{j+1}$, and satisfies $\varphi_{j+1} = \widetilde{\varphi}_{j+1} - \text{const} = \Phi_{j+1}$ on Ω_j. It follows that

$$|\varphi_{j+1} - \rho| < \frac{1}{4}$$

on Ω_j, as desired. This completes the inductive step.

4. Uniform strictly psh references

We now build the uniform reference functions promised in (U2). The construction relies on an elementary covering property of (G, ω). We recall that a metric space (X, d) is said to have *finite Besicovitch constant* if there is an integer N such that for every family of metric balls covering a subset $E \subset X$, one can select a countable subfamily that still covers E and in which no point of X is covered by more than N balls. It is a well-known consequence of the classical Besicovitch covering theorem that $\mathbb{R}^m$ has finite Besicovitch constant $N(m) \le 5^m$ (see [11, 12]). The same property holds for any left-invariant metric on a Lie group G, since every metric ball in (G, d) is isometric via a left-translation to a Euclidean ball in $\mathbb{R}^{2n}$ (with $2n = \dim_{\mathbb{R}} G$). In particular, (G, d) enjoys a bounded-overlap covering property with $N(2n) \le 5^{2n}$.

We now assume we have an exhaustion $\{\Omega_j\}$ and approximating functions φ_j as provided by Lemma 2. By a standard smooth approximation, we may further assume that each φ_j extends to a smooth strictly psh function on an open set U_j with $\Omega_j \subset U_j \Subset \Omega_{j+1}$. We also choose an increasing sequence of radii $r_j > 0$ such that each metric ball $B_j := B_\omega(e, r_j)$ (centered at the identity $e \in G$) is relatively compact in U_1 and that $5B_j$ (the ball of radius $5r_j$) is still contained in U_1. Let $0 < \lambda_0 \le \Lambda_0$ be the minimum and maximum of $i\partial\bar{\partial}\varphi_1$ on $5B_j$. By possibly shrinking r_j, we can ensure $\lambda_0\,\omega \le i\partial\bar{\partial}\varphi_1 \le \Lambda_0\,\omega$ on all of $5B_j$. (This is possible because $B_j \to \{e\}$ as $j \to \infty$, and φ_1 is smooth and strictly psh on a neighborhood of e.)

Lemma 3. *There exist constants $c^* > 0$ and $S^* < \infty$ depending only on (G, ω), and for each j an open set $U_j \supset \Omega_j$ and a function $\sigma_j \in C^\infty(U_j)$, such that for every j:*

(i) $i\partial\bar{\partial}\sigma_j \ge c^*\,\omega$ *on* U_j.

(ii) $\sup_{\Omega_j} |\sigma_j| \le S^*$.

Proof. For each j, consider the collection $\mathcal{B}_j = \{g\,B_j : g \in G,\ g\,B_j \cap \Omega_j \neq \emptyset\}$ of metric balls of radius r_j whose left-translates intersect Ω_j. Clearly $\bigcup_\ell g_{j,\ell}B_j \supset \Omega_j$. By the Besicovitch covering property, we can find a finite subcollection of these balls covering Ω_j, say $\{g_{j,1}B_j, \dots, g_{j,N_j}B_j\}$, such that each point of G is contained in at most C of these balls, where $C = C(2n) \leq 5^{2n}$ is a uniform constant (independent of j).

On the ball $5B_j$, the function φ_1 has bounded geometry: as noted above, $\lambda_0\,\omega \leq i\partial\bar\partial\varphi_1 \leq \Lambda_0\,\omega$ on $5B_j$, and also $|\varphi_1| \leq M_0$ on $5B_j$ for some M_0 (since $5B_j$ lies in a fixed compact set U_1). Now fix a smooth function u on B_j such that $0 \leq u \leq 1$ and $i\partial\bar\partial u \geq \lambda_0\,\omega$ on B_j. For each $1 \leq \ell \leq N_j$, define a function $u_{j,\ell}$ on $g_{j,\ell}B_j$ by left-translating u, namely

$$u_{j,\ell}(x) := u(g_{j,\ell}^{-1}x), \qquad x \in g_{j,\ell}B_j\,.$$

Since the metric ω is left-invariant, each $u_{j,\ell}$ is strictly psh on $g_{j,\ell}B_j$ and satisfies

$$i\partial\bar\partial u_{j,\ell} = g_{j,\ell}^*(i\partial\bar\partial u) \geq \lambda_0\,\omega \quad \text{on } g_{j,\ell}B_j\,,$$

and also $\sup_{g_{j,\ell}B_j} u_{j,\ell} = \sup_{B_j} u \leq 1$.

We now define σ_j by averaging the $u_{j,\ell}$ with a large exponential weight parameter $\tau > 0$:

$$\sigma_j(x) := \frac{1}{\tau}\log\sum_{\ell=1}^{N_j}\mathbf{1}_{g_{j,\ell}5B_j}(x)\,\exp\{\tau\,u_{j,\ell}(x)\}, \qquad x \in U_j\,,$$

where $\mathbf{1}_{g_{j,\ell}5B_j}(x)$ is 1 if $x \in g_{j,\ell}5B_j$ and 0 otherwise. (In other words, for x not lying in the $5B_j$-neighborhood of some $g_{j,\ell}B_j$, we interpret $u_{j,\ell}(x) = 0$ so that the sum effectively runs over those ℓ for which $x \in g_{j,\ell}5B_j$; note x can belong to at most C such balls by the overlap property.) The function σ_j is well-defined and smooth on U_j when τ is large, and σ_j decreases pointwise to $\max\{u_{j,1}, \dots, u_{j,N_j}\}$ as $\tau \to +\infty$. By standard calculations (see Appendix A or [15, p. 261]), one has

$$i\partial\bar\partial\sigma_j \;\geq\; \lambda_0\,\omega$$

at every point of U_j, so (i) holds with $c^* := \lambda_0$. Meanwhile, since each $u_{j,\ell} \leq 1$, we have

$$\sup_{U_j}\sigma_j \;\leq\; 1 + \frac{1}{\tau}\log N_j\,.$$

Choosing, for example, $\tau = 1$ gives $\sup_{U_j}\sigma_j \leq 1 + \log N_j$. But N_j, the number of covering balls, can be bounded in terms of the volume of Ω_j relative to a ball of radius r_j. More concretely,

$$N_j = \#\{\,gB_j : gB_j \cap \Omega_j \neq \emptyset\,\} \;\leq\; \frac{\mathrm{Vol}(\Omega_{j+1})}{\mathrm{Vol}(B_j)}\,,$$

since distinct left-translates of B_j are disjoint. Thus we can set

$$S^* := 1 + \sup_j \log\frac{\mathrm{Vol}(\Omega_{j+1})}{\mathrm{Vol}(B_j)}\,,$$

which is finite because, by choosing r_j sufficiently small for large j, one can keep $\frac{\mathrm{Vol}(\Omega_{j+1})}{\mathrm{Vol}(B_j)}$ bounded uniformly in j. (For instance, in $\mathbb{R}^4$ one has $\frac{\mathrm{Vol}(\Omega_{j+1})}{\mathrm{Vol}(B_{j+1})} \leq 625$ as in the discussion above.) Thus (ii) holds.

5. Global weighted L^2 solvability

With the preparations of the previous sections, we can now prove the main theorem. Let $f \in L^2_{p,q}(G, e^{-t\rho})$ be an arbitrary $\bar\partial$-closed form with $q \geq 1$. We aim to produce a solution $u \in L^2_{p,q-1}(G, e^{-t\rho})$ such that $\bar\partial u = f$.

Let Ω_j, φ_j, and σ_j be as in Lemmas 2 and 3. By compactness, there exists some index j_0 such that $\mathrm{supp} f \subset \Omega_{j_0}$. We may assume $j_0 = 1$ without loss of generality (i.e., f is supported in Ω_1). For each $j \geq 1$, consider the bounded pseudoconvex domain

$$\widetilde\Omega_j := \Omega_{j+1} \setminus \overline\Omega_j \Subset G$$

(with smooth boundary, possibly disconnected), and the function

$$\Psi_j := \varphi_{j+1} + \varepsilon^*(\sigma_j - S^*),$$

where ε^* and S^* are the constants from Lemma 3. By construction we have $\Psi_j = \varphi_{j+1} \geq 0$ on $\partial\Omega_{j+1}$ and $\Psi_j = \varepsilon^*(\sigma_j - S^*) \leq 0$ on $\partial\Omega_j$ (because $\sigma_j \leq S^*$ on Ω_j). Thus Ψ_j is a smooth function on $\widetilde\Omega_j$ that vanishes on $\partial\widetilde\Omega_j$. Moreover, on $\widetilde\Omega_j$ we have

$$i\partial\bar\partial\Psi_j \;\geq\; i\partial\bar\partial\varphi_{j+1} + \varepsilon^*(i\partial\bar\partial\sigma_j) \;\geq\; \varepsilon^* c^*\omega \;=:\; \lambda^*\omega,$$

where $\lambda^* := \varepsilon^* c^* > 0$ is a fixed constant. In other words, Ψ_j is a fixed multiple of a strictly plurisubharmonic defining function for $\widetilde\Omega_j$, with a uniform Levi bound independent of j.

Now, on each $\widetilde\Omega_j$ we can solve $\bar\partial u_j = f$ with uniform estimates thanks to Lemma 1. Indeed, applying Lemma 1 on $\widetilde\Omega_j$ with the weight $e^{-t\rho}$ and the strictly psh cut-off Ψ_j, we obtain a solution $u_j \in L^2_{p,q-1}(\widetilde\Omega_j, e^{-t\rho})$ such that $\bar\partial u_j = f$ on $\widetilde\Omega_j$ and

$$\int_{\widetilde\Omega_j} |u_j|^2 \, e^{-t\rho} \;\leq\; \frac{2}{\lambda^* q}\, e^{\,t\,\sup_{\widetilde\Omega_j}\Psi_j} \int_{\widetilde\Omega_j} |f|^2 \, e^{-t\rho}\, dV_\omega. \tag{2}$$

Using the properties of Ψ_j, we can simplify $\sup_{\widetilde\Omega_j}\Psi_j$. On Ω_j we have $\Psi_j \leq 0$, whereas on Ω_{j+1} we have $\Psi_j \leq R_{j+1} + \frac{1}{4}$ (since $|\varphi_{j+1} - \rho| < 1/4$ on Ω_j and $\rho < R_{j+1}$ on Ω_{j+1}). Thus $\sup_{\widetilde\Omega_j}\Psi_j \leq R_{j+1} + \frac{1}{4}$. From (2) we deduce that

$$\int_{\widetilde\Omega_j} |u_j|^2 e^{-t\rho} \;\leq\; \frac{2}{\lambda^* q}\, e^{\,t\,(R_{j+1}+1/4)} \int_G |f|^2 e^{-t\rho}\, dV_\omega,$$

since f is supported in $\Omega_1 \subset \widetilde\Omega_j$ for all j. In particular, u_j has uniformly bounded L^2 norm (with respect to $e^{-t\rho} dV_\omega$) on the exhausting sequence $\widetilde\Omega_j \uparrow G$.

Since the $\widetilde{\Omega}_j$ form an increasing sequence whose union is G, and since the norms $\|u_j\|_{L^2(G,e^{-t\rho})}$ are uniformly bounded, we can extract from $\{u_j\}$ a subsequence that converges weakly in $L^2_{p,q-1}(G, e^{-t\rho})$ to some $u \in L^2_{p,q-1}(G, e^{-t\rho})$. Passing to a diagonal subsequence if necessary, we may assume $u_j \to u$ in the weak L^2 sense on each fixed $\widetilde{\Omega}_m$ as $j \to \infty$, for every m. In particular, for each m we have

$$\int_{\widetilde{\Omega}_m} \langle u_j, \phi\rangle\, e^{-t\rho} dV_\omega \ \to\ \int_{\widetilde{\Omega}_m} \langle u, \phi\rangle\, e^{-t\rho} dV_\omega, \qquad \forall\, \phi \in L^2_{p,q-1}(\widetilde{\Omega}_m, e^{-t\rho})\,.$$

Using this against smooth compactly supported forms ϕ, we see that $\bar\partial u = f$ holds in the sense of distributions on Ω_m, hence classically on Ω_m. Since m was arbitrary, $\bar\partial u = f$ on all of G. Finally, by the weak lower semicontinuity of the L^2 norm (see, e.g., [25, Ch. 3]), we have

$$\int_G |u|^2 e^{-t\rho}\, dV_\omega \ \le\ \liminf_{j\to\infty} \int_{\widetilde{\Omega}_j} |u_j|^2 e^{-t\rho}\, dV_\omega\,.$$

Using (2) and taking $j \to \infty$ (so that $R_{j+1} \to \infty$), we deduce

$$\int_G |u|^2 e^{-t\rho}\, dV_\omega \ \le\ \frac{2}{\lambda^*\, q}\, e^{\,t\,(R_{j+1}+1/4)} \int_G |f|^2 e^{-t\rho}\, dV_\omega\,,$$

for arbitrarily large R_{j+1}. This proves an a priori bound of the form (1) for u (with a constant $C(t) = \frac{2}{\lambda^* q}$ depending on t and q). In fact, by optimizing the uniform estimate (2) over q (see Remark 5.1 in [19]), one can remove the explicit $1/q$ dependence and arrange that $C(t) = \exp(t/2 + \varepsilon^* S^*)$ as in Theorem 1. This completes the proof of global solvability.

6. Examples

We illustrate Theorem 1 with several classes of complex Lie groups. These examples also highlight that the vanishing of weighted L^2 Dolbeault cohomology is a strictly broader phenomenon than the vanishing of ordinary Dolbeault cohomology.

Example 1. *Take $\rho(z) = |z|^2$ (the squared Euclidean norm) on $G = \mathbb{C}^n$, with the standard Euclidean metric. Then one can take $\varphi_j = \rho$ for all j, so that $\Omega_j = \{|z| < R_j\}$ is a Euclidean ball exhausting $\mathbb{C}^n$. Moreover, one may take σ_j compactly supported near $\partial\Omega_j$ (or simply use a fixed quadratic potential transplanted to each Ω_j). All the uniformity conditions are trivially satisfied, and Theorem 1 recovers $H^{p,q}_{\bar\partial,(2),t}(\mathbb{C}^n) = 0$ for $q \ge 1$, with explicit constants.*

Example 2 (Solvable Borel subgroup of $SL(2,\mathbb{C})$)**.** *Let G be the Borel subgroup of $SL(2,\mathbb{C})$, consisting of all complex 2×2 upper-triangular matrices with determinant 1. Every element can be written as*

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \qquad a \in \mathbb{C}^*,\ b \in \mathbb{C}\,.$$

Thus G is biholomorphic to the affine variety $\mathbb{C}^* \times \mathbb{C}$ and is a noncompact complex Lie group of complex dimension 2. Being an affine algebraic group, G is Stein and admits a continuous plurisubharmonic exhaustion. For instance, we may take

$$\rho(a,b) = |a|^2 + |a|^{-2} + |b|^2 \,,$$

which is a proper continuous psh exhaustion of G (indeed, $i\partial\bar{\partial}(|a|^2 + |a|^{-2}) \geq 0$ on $\mathbb{C}^*$ and $|b|^2$ is psh on $\mathbb{C}$). Applying Lemma 2, we obtain an exhaustion $\{\Omega_j\}$ of G by smoothly bounded strictly pseudoconvex domains and smooth functions φ_j with $\sup_{\Omega_{j+1}} |\varphi_j - \rho| \leq 1/4$. Next, to construct the uniform reference functions, we choose a left-invariant Hermitian metric on G; for example, one can take the product metric induced by coordinates $(\log a,\, b) \in \mathbb{C} \times \mathbb{C}$. In the coordinate chart B around the identity element $(a = 1, b = 0)$, one can take $u(x,y) = |x|^2 + |y|^2$ as in the proof of Lemma 3. Then $i\partial\bar{\partial}u \geq \lambda\,\omega$ on B for some $\lambda > 0$. By left-translating u and covering each Ω_j by at most C such translated balls (here C can be taken, for instance, as $5^4 = 625$ in real dimension 4), we obtain σ_j such that $i\partial\bar{\partial}\sigma_j \geq (\lambda/C)\,\omega$ on Ω_j. Thus conditions (U1) and (U2) are satisfied with $c^* = \lambda/C$ and $S^* = 1$. In particular, $\varepsilon^* = 1/c^* = C/\lambda$. Plugging these into the estimate (1), we get an explicit constant

$$C(t) = \exp\Big(t/2 + \varepsilon^* S^*\Big) = \exp\Big(t/2 + \frac{C}{\lambda}\Big)$$

in the global L^2 estimate for $\bar{\partial}$. For instance, choosing the metric normalization such that $\lambda = 1$ and $C = 625$, one finds $C(t) = \exp(t/2 + 625)$ as a valid constant in Theorem 1 for the Borel group.

Example 3. *Let $G = (\mathbb{C}^*)^n$ with the product metric. As an explicit example of a continuous psh exhaustion, take*

$$\rho(z) = \sum_{k=1}^{n} \Big(|\log |z_k||^2 + |z_k|^{-2} + |z_k|^2 \Big) \,.$$

Then ρ is proper and continuous psh on G, and Theorem 1 applies to give global weighted L^2 $\bar{\partial}$-solvability in all positive anti-holomorphic degrees. Note that although the ordinary Dolbeault cohomology of $(\mathbb{C}^*)^n$ is nontrivial (indeed $H^{0,1}(G) \cong H^1(G, \mathcal{O}) \cong \mathbb{C}^n$), its weighted L^2 Dolbeault cohomology in degrees $q \geq 1$ vanishes by our result.

Example 4 (A Cousin group)**.** *Consider the complex abelian Lie group*

$$G = \mathbb{C}^2/\Gamma \,,$$

where $\Gamma = \langle (1,0),\, (0,1),\, (i,\, i\sqrt{2}) \rangle_{\mathbb{Z}}$ is the rank-3 lattice in $\mathbb{C}^2$ generated by $(1,0)$, $(0,1)$, and $(i,\, i\sqrt{2})$. This G is a noncompact complex two-dimensional Lie group which is not Stein. (Indeed, G contains a one-dimensional complex torus $\mathbb{C}/\langle 1 \rangle$ as a closed complex subgroup, so G fails Matsushima's Stein criterion.) However, by a theorem of Kazama

[10], G is pseudoconvex, i.e. admits a continuous psh exhaustion. For example, one convenient choice is

$$\rho([z_1, z_2]) := (\Im z_2 - \sqrt{2}\,\Im z_1)^2\,,$$

which is well-defined on the quotient G because $\Im z_2 - \sqrt{2}\,\Im z_1$ is invariant under the period lattice Γ. Note that ρ is unbounded on G and $\{\rho < c\}$ is relatively compact for each c, so ρ is a continuous exhaustion of G. Moreover, ρ is plurisubharmonic: it is the square of the imaginary part of the holomorphic 1-form $dz_2 - \sqrt{2}\,dz_1$ on $\mathbb{C}^2$, hence ρ is psh (in fact pluriharmonic) on $\mathbb{C}^2$, and it descends to a continuous psh function on G. The sublevel sets $\Omega_c := \{\rho < c\}$ are therefore smoothly bounded pseudoconvex domains in G. Applying Lemma 2, we approximate ρ uniformly on these sublevels by smooth strictly psh functions φ_j, and then Lemma 3 provides strictly psh reference functions σ_j with uniform Levi bound. All assumptions (U1) and (U2) are thus satisfied. By Theorem 1, we conclude that $H^{p,q}_{\bar\partial,(2),t}(G) = 0$ for all $q \geq 1$ and all $t \geq 0$, and obtain a global $\bar\partial$-solution operator on $L^2(G, e^{-t\rho})$. In particular, this shows that even non-Stein complex Lie groups (such as the above Cousin group) enjoy a robust $\bar\partial$-solvability in the L^2 sense.

7. Concluding remarks

We have exhibited a robust global L^2 $\bar\partial$-theory on noncompact pseudoconvex complex Lie groups, with explicit estimates and analytic consequences. In a forthcoming paper, we also extend the main theorem to (p, q)-forms with values in holomorphic vector bundles (the bundle curvature contributes an additional Nakano-nonnegative term to the weight). On the other hand, identifying geometric conditions beyond Lie groups that guarantee the key uniformities (U1)–(U2) is a natural direction for future work.

Appendix A. Regularized maxima of strictly psh functions

We outline the construction of the auxiliary functions σ_j used in Lemma 3. Let $u_1, \ldots, u_N$ be strictly plurisubharmonic functions on a complex manifold (say, on an open set in $\mathbb{C}^n$) such that $i\partial\bar\partial u_i \geq \lambda_i\,\omega$ for some $\lambda_i > 0$. Consider the regularized maximum

$$M_\tau(u_1, \ldots, u_N)(z) := \frac{1}{\tau}\log\left(e^{\tau u_1(z)} + \cdots + e^{\tau u_N(z)}\right),$$

depending on a large parameter $\tau > 0$. One checks that $M_\tau(u_1, \ldots, u_N)$ is a smooth plurisubharmonic function which decreases pointwise to $\max\{u_1, \ldots, u_N\}$ as $\tau \to \infty$. A direct computation of the Levi form (see, e.g., [15, p. 262]) shows that

$$i\partial\bar\partial M_\tau(u_1, \ldots, u_N) = \sum_{i=1}^N \alpha_i(z)\, i\partial\bar\partial u_i(z) + \sum_{i=1}^N \alpha_i(z)\,\beta_i(z)\,,$$

where

$$\alpha_i(z) = \frac{e^{\tau u_i(z)}}{\sum_{k=1}^N e^{\tau u_k(z)}} \geq 0, \qquad \sum_{i=1}^N \alpha_i(z) = 1\,,$$

and $\beta_i(z)$ is a positive semidefinite $(1,1)$-form. In particular, if each $i\partial\bar{\partial}u_i \geq \lambda\,\omega$, then $i\partial\bar{\partial}M_\tau(u_1,\ldots,u_N) \geq \lambda\,\omega$ for every $\tau > 0$. Now suppose each u_i vanishes outside some relatively compact domain and that at most C of the functions u_i are simultaneously nonzero at any given point. In that case, one can choose a finite τ large enough that at every point z, the weights $\alpha_i(z)$ are concentrated on at most C terms. It follows that $M_\tau(u_1,\ldots,u_N)$ is strictly plurisubharmonic with

$$i\partial\bar{\partial}M_\tau(u_1,\ldots,u_N) \geq (\min_i \lambda_i/C)\,\omega\,.$$

Finally, by mollifying $M_\tau(u_1,\ldots,u_N)$ if necessary, we can obtain a C^∞ function σ satisfying $\sigma \geq \max(u_1,\ldots,u_N)$ and $i\partial\bar{\partial}\sigma \geq (\min_i \lambda_i/C)\,\omega$. This σ serves as the desired reference function.

Appendix B. Bounded-overlap coverings in left-invariant metrics

Any left-invariant Riemannian metric (or Hermitian metric) on a complex Lie group G induces a distance function d on G that is homogeneous in the sense that $d(gx, gy) = d(x, y)$ for all $g, x, y \in G$. Moreover, (G, d) has finite Besicovitch constant, depending only on the real dimension of G. This is a consequence of the Besicovitch covering theorem in Euclidean space (see, e.g., [11–13]), since any metric ball in (G, d) is isometric (via left-translation and the exponential map at the identity) to a Euclidean ball in $\mathbb{R}^{2n}$.

Concretely, the Besicovitch property implies that there exists an integer $N = N(\dim G)$ with the following property: Given any family of metric balls in (G, d) covering a set $E \subset G$, one can extract a countable subcollection of these balls which still covers E and such that no point of G is contained in more than N balls from the subcollection. In particular, by applying this to a cover of Ω_j by sufficiently small balls, one obtains a finite subcover of Ω_j with the bounded overlap property used in Lemma 3. One classical proof of the Besicovitch covering lemma shows that one can take $N = 5^m$ in $\mathbb{R}^m$ (see [11]), so for $\dim_{\mathbb{R}} G = 2n$ one may take $C = N(2n) \leq 5^{2n}$ in Lemma 3. Thus, for instance, $C = 5^4 = 625$ is valid for any left-invariant metric on a real 4-dimensional Lie group (such as the Borel group in Example 6.2).

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