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A Chaundy–Bullard Type Identity and Its q-Analogue

Wathek Chammam[1,*], Mongia Khlifi[2,3], Muhammad Gulistan[4]

[1] *Department of Mathematics, College of Science, Majmaah University, Al Majmaah, 11952, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Sciences of Sfax, Sfax University, Sfax, Tunisia*
[3] *Research Laboratory Mathematics and Applications LR17ES11, Gabes University, Erriadh City, 6072 Zrig, Gabes, Tunisia*
[4] *Department of Electrical and Computer Engineering, University of Alberta, Canada*

Abstract. In this paper, we use the Chaundy–Bullard combinatorial identity to prove some identities involving the Pochhammer k–symbol. In fact, these contributions generalize the results given in the paper [O. Kouba, A Chaundy-Bullard type identity involving the Pochhammer symbol, *Indagationes Mathematicae,* 34 (1), 186–198, 2023. We also present some Chaundy–Bullard type identities satisfied by the generalized hypergeometric series.

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1. Introduction

Diaz and ***Pariguan*** introduced the Pochhammer k-symbol [1, p. 180], by

$$(x)_{n,k} = \prod_{j=0}^{n-1}(x+jk), \quad n,k>0.$$

When $k=1$, the quantity $(x)_{n,1}=(x)_n$ is also called the n-th rising factorial of x.
The q-analogues of the Pochhammer k-symbol $(x)_{n,k}$ are given by (see [2])

$$[x]_{q;n,k} = \prod_{j=0}^{n-1}[x+jk]_q, \quad n,k>0, \tag{1}$$

*Corresponding author.
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Email addresses: w.chammam@mu.edu.sa (W. Chammam), Mongia.Khlifi@issatkas.u-kairouan.tn (M. Khlifi), mgulista@ualberta.ca (M. Gulistan)

where

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \tag{2}$$

and

$$\lim_{q\to 1}[x]_q = x. \tag{3}$$

In particular if $k = 1$, we obtain

$$[x]_{q;n,1} = [x]_{q;n} = \frac{(q^x;q)_n}{(1-q)^n}, \tag{4}$$

where the symbol $(x;q)_n$ is the quantum factorial symbol defined by (see [3, 4])

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}\left(1 - xq^k\right), \tag{5}$$

for $n \geq 1$. It is easy to see that

$$\lim_{q\to 1}[x]_{q;n,k} = (x)_{n,k} \quad \text{and} \quad \lim_{q\to 1}[x]_{q;n} = (x)_n.$$

For $x = n \in \mathbb{N} = \{1, 2, \dots\}$ in (2), we have

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{j=0}^{n-1} q^j.$$

The q-analogue of the factorial $n!$ is defined by (see [5])

$$[n]_q! = \begin{cases} \prod_{j=1}^{n}[j]_q, & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Moreover, the relation between the Pochhammer symbol $(x)_n$ and the classical Euler gamma function $\Gamma(z)$ is

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)},$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.$$

The beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt, \tag{6}$$

for $x, y > 0$.

It is clear that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{7}$$

In [6], ***Jackson*** defined the q-analogue of the gamma function $\Gamma(z)$ as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad |q|<1; \quad \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}}, \quad |q|>1,$$

where

$$(x;q)_\infty = \prod_{k=0}^{\infty}\left(1-xq^k\right).$$

Also, ***Jackson*** defined the q-analogue of the beta function defined by

$$B_q(x,y) = \int_0^1 t^{x-1}(1-qt)_q^{y-1}dqt, \quad x,y>0, \tag{8}$$

the relation between the q-analogue of the gamma function and the q-analogue of the beta function is:

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{9}$$

The q-binomial coefficients or the Gaussian polynomials are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

for $0\le k\le n$ and $|q|<1$. It is not difficult to prove that

$$\lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

For n and m nonnegative integers, the Chaundy–Bullard identity [7–9] defined by

$$(1-X)^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}X^k + X^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(1-X)^k = 1. \tag{10}$$

The q-analogue of Chaundy–Bullard identity (10) defined in [10] by the equality:

$$\sum_{k=0}^{m}\begin{bmatrix} n+k \\ k \end{bmatrix}_q X^k\prod_{j=0}^{n}\left(1-Xq^j\right) + \sum_{k=0}^{n}\begin{bmatrix} m+k \\ k \end{bmatrix}_q q^kX^{m+1}\prod_{j=0}^{k-1}\left(1-Xq^j\right) = 1. \tag{11}$$

2. *New results for the Chaundy–Bullard type identity involving the Pochhammer p–symbol*

Theorem 1. *For n,m nonnegative integers and $p\in\mathbb{N}^*$, then*

$$(Y)_{n+1,p}\sum_{k=0}^{m}\binom{n+k}{k}\frac{(X)_{k,p}}{(X+Y)_{n+k+1,p}} + (X)_{m+1,p}\sum_{k=0}^{n}\binom{m+k}{k}\frac{(Y)_{k,p}}{(X+Y)_{m+k+1,p}} = 1. \tag{12}$$

Proof. By the identity (10), we have

$$(1-X)^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}X^k + X^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(1-X)^k = 1$$

then for $\alpha,\beta>0$ and $p\in\mathbb{N}^*$, we obtain

$$\sum_{k=0}^{m}\binom{n+k}{k}X^{\frac{\alpha}{p}+k-1}(1-X)^{\frac{\beta}{p}+n} + \sum_{k=0}^{n}\binom{m+k}{k}X^{\frac{\alpha}{p}+m}(1-X)^{\frac{\beta}{p}+k-1} = X^{\frac{\alpha}{p}-1}(1-X)^{\frac{\beta}{p}-1}$$

Integrating on $[0,1]$ we conclude that for $\alpha,\beta>0$ and $p\in\mathbb{N}^*$, we have

$$\sum_{k=0}^{m}\binom{n+k}{k}B\Big(\frac{\alpha}{p}+k,\frac{\beta}{p}+n+1\Big) + \sum_{k=0}^{n}\binom{m+k}{k}B\Big(\frac{\beta}{p}+k,\frac{\alpha}{p}+m+1\Big) = B\Big(\frac{\alpha}{p},\frac{\beta}{p},\Big)$$

hence

$$\begin{aligned}
B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big) &= \sum_{k=0}^{m}\binom{n+k}{k}\frac{\Gamma(\frac{\alpha}{p}+k)\Gamma(\frac{\beta}{p}+n+1)}{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p}+n+k+1)} + \sum_{k=0}^{n}\binom{m+k}{k}\frac{\Gamma(\frac{\beta}{p}+k)\Gamma(\frac{\alpha}{p}+m+1)}{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p}+m+k+1)} \\
&= \sum_{k=0}^{m}\binom{n+k}{k}\frac{\Gamma(\frac{\alpha}{p}+k)}{\Gamma(\frac{\alpha}{p})}\frac{\Gamma(\frac{\beta}{p}+n+1)}{\Gamma(\frac{\beta}{p})}\frac{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p})}{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p}+n+k+1)}\frac{\Gamma(\frac{\alpha}{p})\Gamma(\frac{\beta}{p})}{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p})} \\
&+ \sum_{k=0}^{n}\binom{m+k}{k}\frac{\Gamma(\frac{\beta}{p}+k)}{\Gamma(\frac{\beta}{p})}\frac{\Gamma(\frac{\alpha}{p}+m+1)}{\Gamma(\frac{\alpha}{p})}\frac{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p})}{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p}+m+k+1)}\frac{\Gamma(\frac{\alpha}{p})\Gamma(\frac{\beta}{p})}{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p})} \\
&= \sum_{k=0}^{m}\binom{n+k}{k}\frac{\Gamma(\frac{\alpha}{p}+k)}{\Gamma(\frac{\alpha}{p})}\frac{\Gamma(\frac{\beta}{p}+n+1)}{\Gamma(\frac{\beta}{p})}\frac{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p})}{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p}+n+k+1)}B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big) \\
&+ \sum_{k=0}^{n}\binom{m+k}{k}\frac{\Gamma(\frac{\beta}{p}+k)}{\Gamma(\frac{\beta}{p})}\frac{\Gamma(\frac{\alpha}{p}+m+1)}{\Gamma(\frac{\alpha}{p})}\frac{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p})}{\Gamma(\frac{\alpha}{p}+\frac{\beta}{p}+m+k+1)}B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big) \\
&= \sum_{k=0}^{m}\binom{n+k}{k}\frac{(\frac{\alpha}{p})_k(\frac{\beta}{p})_{n+1}}{(\frac{\alpha+\beta}{p})_{n+k+1}}B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big) + \sum_{k=0}^{n}\binom{m+k}{k}\frac{(\frac{\beta}{p})_k(\frac{\alpha}{p})_{m+1}}{(\frac{\alpha+\beta}{p})_{m+k+1}}B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big) \\
&= \sum_{k=0}^{m}\binom{n+k}{k}\frac{p^k(\frac{\alpha}{p})_k p^{n+1}(\frac{\beta}{p})_{n+1}}{p^{n+k+1}(\frac{\alpha+\beta}{p})_{n+k+1}}B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big) + \sum_{k=0}^{n}\binom{m+k}{k}\frac{p^k(\frac{\beta}{p})_k p^{m+1}(\frac{\alpha}{p})_{m+1}}{p^{m+k+1}(\frac{\alpha+\beta}{p})_{m+k+1}}B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big) \\
&= \sum_{k=0}^{m}\binom{n+k}{k}\frac{(\alpha)_{k,p}(\beta)_{n+1,p}}{(\alpha+\beta)_{n+k+1,p}}B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big) + \sum_{k=0}^{n}\binom{m+k}{k}\frac{(\beta)_{k,p}(\alpha)_{m+1,p}}{(\alpha+\beta)_{m+k+1,p}}B\Big(\frac{\alpha}{p},\frac{\beta}{p}\Big)
\end{aligned}$$

then

$$(\beta)_{n+1,p}\sum_{k=0}^{m}\binom{n+k}{k}\frac{(\alpha)_{k,p}}{(\alpha+\beta)_{n+k+1,p}} + (\alpha)_{m+1,p}\sum_{k=0}^{n}\binom{m+k}{k}\frac{(\beta)_{k,p}}{(\alpha+\beta)_{m+k+1,p}} = 1$$

The required proof is complete.

Example 1. *For n and m nonnegative integers and $p = 1$ in (12), we find the known results [11]*

$$(Y)_{n+1}\sum_{k=0}^{m}\binom{n+k}{k}\frac{(X)_k}{(X+Y)_{n+k+1}} + (X)_{m+1}\sum_{k=0}^{n}\binom{m+k}{k}\frac{(Y)_k}{(X+Y)_{m+k+1}} = 1. \tag{13}$$

Example 2. *For n and m nonnegative integers and $X = 6$, $Y = 4$ and $p = 2$ in (12), we obtain*

$$\sum_{k=0}^{m}\frac{(n+2)(n+1)(k+2)(k+1)}{(n+k+1)_5} + \sum_{k=0}^{n}\frac{(m+3)(m+2)(m+1)(k+1)}{(m+k+1)_5} = \frac{1}{12}. \tag{14}$$

Example 3. *For n and m nonnegative integers and $X = Y = p \in \mathbb{N}^*$ in (12), we have*

$$\sum_{k=0}^{m}\frac{n+1}{(n+k+2)(n+k+1)} + \sum_{k=0}^{n}\frac{m+1}{(m+k+2)(m+k+1)} = 1. \tag{15}$$

Remark 1. *For $(X, Y) = (\lambda X, \lambda(1-X))$ in (12) and then taking the limit as λ tends to infinity we obtain the original Chaundy–Bullard identity (10).*

3. *Identity of Chaundy–Bullard type involving generalized hypergeometric series*

The generalized hypergeometric series [12], defined for complex numbers $a_i \in \mathbb{C}$ and $b_i \in \mathbb{C}\setminus\{0,-1,-2,...\}$, for positive integers $r, s \in \mathbb{N}$ by

$${}_rF_s\left[\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array};z\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n...(a_r)_n}{(b_1)_n...(b_s)_n}\frac{z^n}{n!}. \tag{16}$$

The generalized basic hypergeometric series

$${}_r\phi_s\left[\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array};q,z\right] = \sum_{n=0}^{\infty}\frac{(a_1,\dots,a_r;q)_n}{(q;q)_n(b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \tag{17}$$

is defined in [12, p. 125] for $b_1,...,b_s \neq q^{-m}$, $m \in \mathbb{N}$,
where

$$(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n. \tag{18}$$

The generalized basic hypergeometric series ${}_r\phi_s$ is a q-analogues of the generalized hypergeometric series (16).
For finite sums of generalized hypergeometric series and generalized basic hypergeometric series, we will use the following symbols

$${}_rF_s\left[\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array};z\right]_n = \sum_{k=0}^{n}\frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_s)_k}\frac{z^k}{k!},$$

and

$$ {}_{r+1}\phi_r \left[\begin{array}{c} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array} ; q, z \right]_n = \sum_{n=0}^{n} \frac{(a_1, \ldots, a_{r+1}; q)_n}{(q;q)_n (b_1, \ldots, b_r; q)_n} z^n. \tag{19} $$

Theorem 2. *For n, m nonnegative integers and $p \in \mathbb{N}^*$, we obtain the following equality:*

$$ (1-X)^{n+1} {}_1F_0 \left[\begin{array}{c} n+1 \\ - \end{array} ; X \right]_m + X^{m+1} {}_1F_0 \left[\begin{array}{c} m+1 \\ - \end{array} ; 1-X \right]_n = 1. \tag{20} $$

Proof. For n, m nonnegative integers, we have

$$ \begin{aligned} 1 &= (1-X)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} X^k + X^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1-X)^k \\ &= (1-X)^{n+1} \sum_{k=0}^{m} \frac{\Gamma(n+1+k)}{\Gamma(n+1)\Gamma(k+1)} X^k + X^{m+1} \sum_{k=0}^{n} \frac{\Gamma(m+1+k)}{\Gamma(m+1)\Gamma(k+1)} (1-X)^k \\ &= (1-X)^{n+1} \sum_{k=0}^{m} (n+1)_k \frac{X^k}{k!} + X^{m+1} \sum_{k=0}^{n} (m+1)_k \frac{(1-X)^k}{k!} \\ &= (1-X)^{n+1} {}_1F_0 \left[\begin{array}{c} n+1 \\ - \end{array} ; X \right]_m + X^{m+1} {}_1F_0 \left[\begin{array}{c} m+1 \\ - \end{array} ; 1-X \right]_n . \end{aligned} $$

The required proof is complete.

Theorem 3. *For n, m nonnegative integers, we have*

$$ \prod_{j=0}^{n} \left(1 - Xq^j\right) {}_1\phi_0 \left[\begin{array}{c} q^{n+1} \\ - \end{array} ; q, X \right]_m + X^{m+1} {}_2\phi_1 \left[\begin{array}{c} q^{m+1}, X \\ 0 \end{array} ; q, q \right]_n = 1. \tag{21} $$

Proof. For n, m nonnegative integers, we have

$$ (a;q)_{n+m} = (a;q)_n (aq^n;q)_m $$

then, by (11) we obtain

$$ \begin{aligned} 1 &= \sum_{k=0}^{m} \begin{bmatrix} n+k \\ k \end{bmatrix}_q X^k \prod_{j=0}^{n} \left(1 - Xq^j\right) + \sum_{k=0}^{n} \begin{bmatrix} m+k \\ k \end{bmatrix}_q q^k X^{m+1} \prod_{j=0}^{k-1} \left(1 - Xq^j\right) \\ &= \prod_{j=0}^{n} \left(1 - Xq^j\right) \sum_{k=0}^{m} \frac{(q;q)_{n+k}}{(q;q)_n (q;q)_k} X^k + X^{m+1} \sum_{k=0}^{n} \frac{(q;q)_{m+k} (X;q)_k}{(q;q)_m (q;q)_k} q^k \\ &= \prod_{j=0}^{n} \left(1 - Xq^j\right) \sum_{k=0}^{m} \frac{(q;q)_n (q^{n+1};q)_k}{(q;q)_n (q;q)_k} X^k + X^{m+1} \sum_{k=0}^{n} \frac{(q;q)_m (q^{m+1};q)_k (X;q)_k}{(q;q)_m (q;q)_k} q^k \end{aligned} $$

$$
\begin{aligned}
&= \prod_{j=0}^{n}\left(1 - Xq^j\right)\sum_{k=0}^{m}(q^{n+1};q)_k\frac{X^k}{(q;q)_k} + X^{m+1}\sum_{k=0}^{n}(q^{m+1};q)_k(X;q)_k\frac{q^k}{(q;q)_k}\\
&= \prod_{j=0}^{n}\left(1 - Xq^j\right)\sum_{k=0}^{m}(q^{n+1};q)_k\frac{X^k}{(q;q)_k} + X^{m+1}\sum_{k=0}^{n}\frac{(q^{m+1};q)_k(X;q)_k}{(0;q)_k}\frac{q^k}{(q;q)_k}\\
&= \prod_{j=0}^{n}\left(1 - Xq^j\right)\,{}_1\phi_0\left[\begin{matrix} q^{n+1} \\ - \end{matrix};q,X\right]_m + X^{m+1}{}_2\phi_1\left[\begin{matrix} q^{m+1},X \\ 0 \end{matrix};q,q\right]_n.
\end{aligned}
$$

We find the result.

Now, we are interested in relations between the identity of Chaundy–Bullard involving the Pochhammer p–symbol and hypergeometric series asserted in the following Theorem.

Theorem 4. *For n, m nonnegative integers and $p \in \mathbb{N}^*$, we obtain the following equality:*

$$
\frac{(Y)_{n+1,p}}{(X+Y)_{n+1,p}}\,{}_2F_1\left[\begin{matrix} \frac{X}{p}, n+1 \\ \frac{X+Y}{p}+n+1 \end{matrix};1\right]_m + \frac{(X)_{m+1,p}}{(X+Y)_{m+1,p}}\,{}_2F_1\left[\begin{matrix} \frac{Y}{p}, m+1 \\ \frac{X+Y}{p}+m+1 \end{matrix};1\right]_n = 1. \tag{22}
$$

Proof. For n, m nonnegative integers and $p \in \mathbb{N}^*$, we have

$$
\begin{aligned}
&(Y)_{n+1,p}\sum_{k=0}^{m}\binom{n+k}{k}\frac{(X)_{k,p}}{(X+Y)_{n+k+1,p}}\\
&= (Y)_{n+1,p}\sum_{k=0}^{m}\frac{\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(k+1)}\frac{p^k(\frac{X}{p})_k}{p^{n+k+1}(\frac{X+Y}{p})_{n+k+1}}\\
&= (Y)_{n+1,p}\sum_{k=0}^{m}\left(\frac{X}{r}\right)_k\frac{\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(k+1)}\frac{\Gamma(\frac{X+Y}{p})}{p^{n+1}\Gamma(\frac{X+Y}{p}+n+k+1)}\frac{\Gamma(\frac{X+Y}{p}+n+1)}{\Gamma(\frac{X+Y}{p}+n+1)}\\
&= (Y)_{n+1,p}\sum_{k=0}^{m}\left(\frac{X}{p}\right)_k\frac{\Gamma(n+1+k)}{\Gamma(n+1)}\frac{\Gamma(\frac{X+Y}{p}+n+1)}{\Gamma(\frac{X+Y}{p}+n+1+k)}\frac{\Gamma(\frac{X+Y}{p})}{p^{n+1}\Gamma(\frac{X+Y}{p}+n+1)}\frac{1}{\Gamma(k+1)}\\
&= (Y)_{n+1,p}\sum_{k=0}^{m}\frac{(\frac{X}{p})_k(n+1)_k}{(\frac{X+Y}{p}+n+1)_k p^{n+1}(\frac{X+Y}{p})_{n+1}}\frac{1}{\Gamma(k+1)}\\
&= \frac{(Y)_{n+1,p}}{(X+Y)_{n+1,p}}\sum_{k=0}^{m}\frac{(\frac{X}{p})_k(n+1)_k}{(\frac{X+Y}{p}+n+1)_k}\frac{1}{k!}\\
&= \frac{(Y)_{n+1,p}}{(X+Y)_{n+1,p}}\,{}_2F_1\left[\begin{matrix} \frac{X}{p}, n+1 \\ \frac{X+Y}{p}+n+1 \end{matrix};1\right]_m.
\end{aligned}
$$

Then

$$
(Y)_{n+1,p}\sum_{k=0}^{m}\binom{n+k}{k}\frac{(X)_{k,p}}{(X+Y)_{n+k+1,p}} = \frac{(Y)_{n+1,p}}{(X+Y)_{n+1,p}}\,{}_2F_1\left[\begin{matrix} \frac{X}{p}, n+1 \\ \frac{X+Y}{p}+n+1 \end{matrix};1\right]_m
$$

Consequently, we have the equality

$$(X)_{m+1,p}\sum_{k=0}^{n}\binom{m+k}{k}\frac{(Y)_{k,p}}{(X+Y)_{m+k+1,p}}=\frac{(X)_{m+1,p}}{(X+Y)_{m+1,p}}\;{}_2F_1\left[\begin{matrix}\frac{Y}{p},m+1\\ \frac{X+Y}{p}+m+1\end{matrix};1\right]_n$$

We using (12), we obtain the result.

Corollary 1. *For n, m nonnegative integers, we have*

$$\frac{(Y)_{n+1}}{(X+Y)_{n+1}}\;{}_2F_1\left[\begin{matrix}X,n+1\\ X+Y+n+1\end{matrix};1\right]_m+\frac{(X)_{m+1}}{(X+Y)_{m+1}}\;{}_2F_1\left[\begin{matrix}Y,m+1\\ X+Y+m+1\end{matrix};1\right]_n=1. \tag{23}$$

Proof. For n, m nonnegative integers and $p = 1$ in (22), we obtain the result.

Example 4. *If $X = Y$, $m = n$ in (22) we have*

$${}_2F_1\left[\begin{matrix}\frac{X}{p},n+1\\ \frac{2X}{p}+n+1\end{matrix};1\right]_n=\frac{(2X)_{n+1,p}}{2(X)_{n+1,p}}. \tag{24}$$

Example 5. *If $X = Y$, $m = n$ and $p = 1$ in (22) we obtain*

$${}_2F_1\left[\begin{matrix}X,n+1\\ 2X+n+1\end{matrix};1\right]_n=\frac{(2X)_{n+1}}{2(X)_{n+1}}. \tag{25}$$

4. Conclusion and Perspectives

In this paper, we have established a generalization of the classical Chaundy–Bullard identity together with its q-analogue. Our approach, based on combinatorial manipulations of generalized factorials and hypergeometric-type series, highlights the structural links between binomial identities, q-series, and special functions. Several illustrative examples were provided, showing how known formulas (such as the Beta integral and its q-extension) can be recovered as particular cases of our results.

Beyond the intrinsic combinatorial interest of such identities, these results open several directions for future research:

- exploring further extensions involving multiple parameters, higher-order factorials or multivariate generalizations;
- investigating connections with orthogonal polynomials, especially those arising in the Askey scheme and their q-analogues;
- applying these identities to the study of partition functions, q-series transformations, and related problems in analytic number theory;

- examining possible applications in approximation theory, where Beta-type integrals and their discrete versions naturally arise.

We believe that the framework introduced here provides a unifying point of view for various classical and modern identities, and may stimulate further developments at the intersection of combinatorics, special functions and q-series.

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