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A Comparative Study of Bernoulli Collocation and Hermite-Galerkin Methods for Solving Two-Dimensional Nonlinear Volterra Integral Equations of the Second Kind

M. H. Ahmed[1,*], A. M. Aljabri[1]

[1] *Taibah University, College of Science, Department of Mathematics, P.O. Box 344, Madinah, 3002 Saudi Arabia*

Abstract. This article is devoted to the presentation of two numerical methods which give the solution of a two-dimensional nonlinear Volterra integral equation of the second kind. The first method, Bernoulli collocation, depend on approximating the unknown function using Bernoulli polynomials, while applying the collocation technique at shifted Chebyshev points over the interval [0,1]. The second method, Hermite-Galerkin method, relies on constructing an operational matrices and applying the Galerkin projection, which we have a system of nonlinear algebraic equations from Volterra integral equation. Discussion on the existence and uniqueness of the solution is provided. Finally, the effect of that two numerical methods is described. To illustrate the previously described methods, several numerical examples are provided. Numerical results show that the Bernoulli collocation method consistently provides more accurate and efficient results than the Hermite–Galerkin method for the same number of collocation points. Comparisons with previously published approaches further demonstrate the superiority of the proposed methods in terms of convergence and stability.

2020 Mathematics Subject Classifications: 11B39, 45B05, 45D05, 65R20

Key Words and Phrases: Non-linear two-dimensional Volterra integral equations, Bernoulli polynomials, shifted Chebyshev points, Hermite polynomials and Galerkin method

1. Introduction

Volterra integral equations (VIEs), particularly in two dimensions, serve as an effective tool for modeling various phenomena in physical and engineering sciences. For instance, (VIEs) are used to describe viscoelastic rods and plates, where stress depends on the full strain history [1], and nonlinear viscoelastic solids [2], highlighting their relevance in modeling real-world materials and systems. These equations have attracted considerable

*Corresponding author.
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Email addresses: mramadan@taibahu.edu.sa (M. H. Ahmed),
asmazaljabrii@gmail.com (A. M. Aljabri)

attention in both theoretical analysis and the development of accurate and efficient numerical methods for their solution [3–5]. In the present study, we examine the following two-dimensional nonlinear Volterra integral equation of the second kind:

$$u(\chi,t) = f(\chi,t) + \lambda \int_0^t \int_0^\chi K(\chi,t,s,y,u(s,y))\,ds\,dy, \quad \chi,t \in [0,1], \tag{1}$$

where $u(\chi,t)$ is the unknown function defined on the domain $\mathrm{D} = [0,1] \times [0,1]$, and $f(\chi,t)$, $K(\chi,t,s,y,u(s,y))$ are assumed to be known analytical functions.

Many numerical methods have been developed to solve two-dimensional Volterra integral equations due to the difficulty of obtaining analytical solutions. For example, several numerical techniques have been proposed for these equations, including the Chebysheve polynomials as the basis in the collocation method [6], differential transformation method [7], reproducing kernel function method [8], Legendre polynomials method [9], rationalized Haar functions [10] and two-dimensional block-pulse functions [11, 12]. In [13], a numerical method was presented for solving a class of nonlinear two-dimensional integral equations using Bernoulli polynomials. The approach relies on constructing an operational matrix of Bernoulli polynomials, resulting in an efficient and accurate approximation scheme. In [14], a numerical scheme based on shifted Jacobi operational matrices combined with the collocation method was developed to solve two-dimensional nonlinear fractional Volterra and Fredholm integral equations. This method effectively handles fractional orders while maintaining high accuracy. The authors of [15] introduced a Laguerre wavelet-based method for solving two-dimensional nonlinear integral equations. Their work included a detailed convergence analysis that confirmed the method's accuracy and efficiency. Moreover, in [16], a numerical method using radial basis functions (RBFs) was developed to solve nonlinear two-dimensional Volterra integral equations of the second kind on non-rectangular domains. The method exhibits strong flexibility in dealing with complex geometries while preserving computational accuracy.

However, the field of integral equations is closely related to the solution of a wide class of differential models, including fractional ones, as they share both challenges and advanced numerical techniques. Several numerical [17], analytical [18], and semi-analytical techniques have been developed for solving differential equations, particularly focusing on accurate differentiation [19–21]. These studies provide a foundational background for extending such methods to more complex problems, including Volterra integral equations.

In this work, we propose two numerical methods for solving a nonlinear two-dimensional Volterra integral equation. The first method, Bernoulli collocation, is based on approximating the unknown function using Bernoulli polynomials, while applying the collocation technique at shifted Chebyshev points over the interval [0,1]. The integral equation is then transformed into a system of nonlinear algebraic equations. The second method, Hermite-Galerkin, relies on approximating the solution using Hermite functions and applying the Galerkin projection, which also reduces the integral equation to a system of nonlinear algebraic equations.

The outline of this paper is as follows: In Section 2, we review some fundamental formulations and properties of Bernoulli and Hermite polynomials. Section 3 is devoted to the existence and uniqueness of the solution of the integral equation. In Section 4, we present computational methods for solving the two-dimensional Volterra integral equation (1) using both Bernoulli collocation method and Hermite-Galerkin method. Comparison of results are found in some numerical examples which provided in Section 5. The results are discussed in Section 6. Finally, the paper is concluded in Section 7. These previously proposed methods provide a foundation and motivation for the present study, which focuses on Bernoulli collocation and Hermite–Galerkin approaches as efficient alternatives for solving two-dimensional nonlinear Volterra integral equations.

2. Some Definitions and Properties

2.1. Bernoulli polynomials

Bernoulli polynomials were introduced by Jacob Bernoulli and later generalized by Euler to arbitrary values of the variable [22–24]. The generating function of Bernoulli polynomials $B_N(\chi)$ is given by

$$\frac{z_1 e^{\chi z_1}}{e^{z_1}-1} = \sum_{N=0}^{\infty} B_N(\chi)\frac{z_1^N}{N!}, \quad |z_1| < 2\pi, \tag{2}$$

The Bernoulli numbers $B_N := B_N(0)$ are defined via the generating function

$$\frac{z_1}{e^{z_1}-1} = \sum_{N=0}^{\infty} B_N\frac{z_1^N}{N!}, \quad |z_1| < 2\pi. \tag{3}$$

From this, the Bernoulli polynomials $B_N(\chi)$ can be expressed as

$$B_N(\chi) = \sum_{k=0}^{N} \binom{N}{k} B_k \chi^{N-k}. \tag{4}$$

They satisfy the identity

$$B_N(\chi+1) - B_N(\chi) = N\chi^{N-1}, \quad N \in \mathbb{N}_0, \tag{5}$$

which implies that

$$B_N(0) = B_N(1), \quad N \in \mathbb{N}\setminus\{1\}. \tag{6}$$

Substituting $\chi = 1$ into (4), and using (6), yields

$$B_N = \sum_{k=0}^{N} \binom{N}{k} B_k. \tag{7}$$

These polynomials satisfy several important identities, such as

$$B'_N(\chi) = NB_{N-1}(\chi), \quad N \geq 1,$$

$$\int_0^1 B_N(\chi)\, d\chi = 0, \quad N \geq 1,$$

$$B_N(\chi + 1) - B_N(\chi) = N\chi^{N-1}, \quad N \geq 1,$$

and

$$B_N(1-\chi) = (-1)^N B_N(\chi).$$

The first five Bernoulli polynomials are

$$\begin{aligned}
B_0(\chi) &= 1, \\
B_1(\chi) &= \chi - \frac{1}{2}, \\
B_2(\chi) &= \chi^2 - \chi + \frac{1}{6}, \\
B_3(\chi) &= \chi^3 - \frac{3}{2}\chi^2 + \frac{1}{2}\chi, \\
B_4(\chi) &= \chi^4 - 2\chi^3 + \chi^2 - \frac{1}{30}.
\end{aligned}$$

2.2. Hermite polynomials

Hermite polynomials were first introduced in 1810 by Pierre-Simon Laplace. However, it was Charles Hermite who later defined a more generalized class of Hermite polynomials, which remain less known compared to the standard form, despite their significance. These polynomials have emerged as a fundamental tool in both pure and applied mathematics. Their relevance has recently increased due to their applications in quantum mechanics, engineering, physics, and other scientific fields.

Hermite polynomials form a set of mutually orthogonal functions with respect to the weight function $e^{-\chi^2}$ over the interval $(-\infty, \infty)$ [23, 25, 26]. They can be generated using the Rodrigues formula:

$$H_n(\chi) = (-1)^n e^{\chi^2} \frac{d^n}{d\chi^n}\left(e^{-\chi^2}\right), \quad n = 0, 1, 2, \ldots$$

The first few Hermite polynomials are given by

$$\begin{aligned}
H_0(\chi) &= 1, \\
H_1(\chi) &= 2\chi, \\
H_2(\chi) &= 4\chi^2 - 2, \\
H_3(\chi) &= 8\chi^3 - 12\chi, \\
H_4(\chi) &= 16\chi^4 - 48\chi^2 + 12, \\
H_5(\chi) &= 32\chi^5 - 160\chi^3 + 120\chi.
\end{aligned}$$

These polynomials satisfy several important identities. The derivative of $H_n(\chi)$ is given by

$$H_n'(\chi) = 2nH_{n-1}(\chi), \quad n \geq 1.$$

They also satisfy a recurrence relation

$$H_{n+1}(\chi) - 2\chi H_n(\chi) = -2nH_{n-1}(\chi), \quad n = 1, 2, \dots$$

Moreover, they exhibit symmetry properties based on the parity of n

$$H_n(-\chi) = (-1)^n H_n(\chi).$$

With respect to orthogonality, Hermite polynomials satisfy the following inner product identity over the entire real line

$$\int_{-\infty}^{\infty} e^{-\chi^2} H_m(\chi) H_n(\chi)\, d\chi = \begin{cases} 0, & m \neq n, \\ 2^n \sqrt{\pi}\, n!, & m = n. \end{cases}$$

3. Existence and uniqueness of the solution of the integral equation

First, we need to prove the existence and uniqueness of the solution of (1). To this end, we employ the Banach fixed point theorem.

Theorem 1. *Let $(C(D), \|\cdot\|)$ as the Banach space for continuous real-valued functions on D, with norm*

$$\|u\| = \max_{(\chi,t)\in D} |u(\chi, t)|.$$

Assume that the function $f(\chi, t)$ is continuous on D, and the kernel $k(\chi, t, s, y, u)$ is continuous on $D \times D \times \mathbb{R}$, and satisfies the Lipschitz condition in the fifth argument

$$|K(\chi, t, s, y, u) - K(\chi, t, s, y, v)| \leq L|u - v|$$

for all $(\chi, t, s, y) \in D \times D$ and all $u, v \in \mathbb{R}$, with a constant $L > 0$. If $\alpha = |\lambda|L < 1$, and $0 < \alpha < 1$ then the integral equation (1) has a unique solution $u \in C(D)$.

Proof. Define the operator T on $C(D)$ by

$$(Tu)(\chi, t) = f(\chi, t) + \lambda \int_0^t \int_0^\chi K(\chi, t, s, y, u(s, y))\, ds\, dy.$$

We aim to show that T is a contraction on $C(D)$. For any $u, v \in C(D)$, we have

$$\begin{aligned}
\|Tu - Tv\| &= \max_{(\chi,t)\in D} |(Tu)(\chi, t) - (Tv)(\chi, t)| \\
&= \max_{(\chi,t)\in D} \left| \lambda \int_0^t \int_0^\chi [K(\chi, t, s, y, u(s, y)) - K(\chi, t, s, y, v(s, y))]\, ds\, dy \right| \\
&\leq |\lambda| \max_{(\chi,t)\in D} \int_0^t \int_0^\chi |K(\chi, t, s, y, u(s, y)) - K(\chi, t, s, y, v(s, y))|\, ds\, dy \\
&\leq |\lambda| L \max_{(\chi,t)\in D} \int_0^t \int_0^\chi |u(s, y) - v(s, y)|\, ds\, dy \\
&\leq |\lambda| L \|u - v\| \max_{(\chi,t)\in D} \int_0^t \int_0^\chi ds\, dy \\
&= |\lambda| L \|u - v\| \max_{(\chi,t)\in D} \chi\, t = \alpha \|u - v\|.
\end{aligned}$$

Since $\alpha = |\lambda|L < 1$, the operator T is a contraction. Therefore, by the Banach Fixed Point Theorem, T has a unique fixed point $u \in C(D)$, which is the unique solution of the integral equation.

To illustrate the convergence, S.Bazm studied the Bernoulli polynomials for integral equations [27], while Mao and Shen analyzed the Hermite–Galerkin method for fractional PDEs [25].

4. Description of the methods

In this section, we solve Eq.(1) using Bernoulli collocation method and Hermite Galerkin method.

4.1. Bernoulli collocation method

In this method, we approximate the unknown function $\tilde{u}(\chi, t)$ in equation (1) using a double series expansion of the form

$$\tilde{u}(\chi, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} B_i(\chi) B_j(t), \tag{8}$$

where $B_i(\chi)$ and $B_j(t)$ are Bernoulli polynomials and the unknown coefficients a_{ij} are be determined. This representation is then used to construct the approximate solution as follows.

By truncating the infinite series in equation (8), we obtain the following finite approximation

$$\tilde{u}(\chi, t) \approx \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_{ij} B_i(\chi) B_j(t), \tag{9}$$

where $N - 1$ is the chosen degree of approximation in both variables.

Substituting from (9) into (1) we get

$$\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_{ij} B_i(\chi) B_j(t) = f(\chi, t) + \lambda \int_0^t \int_0^\chi K\left(\chi, t, s, y, \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_{ij} B_i(s) B_j(y)\right) ds\, dy. \tag{10}$$

The unknown function $\tilde{u}(\chi, t)$ is approximated using Bernoulli polynomials as basis functions, while the collocation points χ_l, t_f are chosen as shifted Chebyshev points on the interval $[0, 1]$, given by

$$\chi_l = \frac{1}{2}\left(1 + \cos\left(\frac{(2l-1)\pi}{2N}\right)\right), \quad t_f = \frac{1}{2}\left(1 + \cos\left(\frac{(2f-1)\pi}{2N}\right)\right), \quad l, f = 1, 2, \ldots, N. \tag{11}$$

Equation (10) can be represented as

$$\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_{ij}B_i(\chi_l)B_j(t_f) = f(\chi_l, t_f) + \lambda \int_0^{t_f}\int_0^{\chi_l} K\left(\chi_l, t_f, s, y, \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_{ij}B_i(s)B_j(y)\right) ds\, dy. \tag{12}$$

By substituting the collocation points defined in (11) into equation (10), we obtain a system of nonlinear algebraic equations containing N^2 unknown coefficients a_{ij}, with indices $i, j = 0, 1, ..., N-1$. In this study, $N = 6$ was chosen as it provides sufficient accuracy while keeping the computational cost reasonable. All double integrals, including the nonlinear terms containing the kernel K, are numerically evaluated using MATLAB's integral2 function (adaptive quadrature method). The resulting nonlinear system is solved iteratively using the Picard iteration method with a convergence tolerance of 1×10^{-6}, with a maximum limit of 10 iterations to ensure numerical stability, leading to the approximate solution $\tilde{u}(\chi, t)$.

4.2. Hermite Galerkin method

Assume that $\tilde{u}(\chi, t)$ is an approximate solution of the two-dimensional Volterra integral equation (1). The Galerkin method with Hermite polynomials is applied, yielding the following approximation

$$\tilde{u}(\chi, t) \approx \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} c_{ij}H_i(\chi)H_j(t), \tag{13}$$

where the Hermite polynomials are $H_i(\chi)$ and $H_j(t)$ and the unknown Hermite coefficients c_{ij} are to be determined.

From the right-hand side of equation (13) and substituting into equation (1), we obtain

$$\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} c_{ij}H_i(\chi)H_j(t) = f(\chi, t) + \eta(\chi, t), \tag{14}$$

where

$$\eta(\chi, t) = \lambda \int_0^{t}\int_0^{\chi} K\left(\chi, t, s, y, \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} c_{ij}H_i(s)H_j(y)\right) ds\, dy.$$

Multiplying equation (14) by $H_b(\chi)H_r(t)$, also integrating both sides of equation (14) with respect to χ and t over $[0, 1]$, for $b, r = 0, 1, \ldots, N-1$, we have the following.

$$\begin{aligned}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} c_{ij}\int_0^1\int_0^1 & H_i(\chi)H_j(t)H_b(\chi)H_r(t)\, d\chi\, dt \\ &= \int_0^1\int_0^1 f(\chi, t)H_b(\chi)H_r(t)\, d\chi\, dt \\ &\quad + \int_0^1\int_0^1 \eta(\chi, t)H_b(\chi)H_r(t)\, d\chi\, dt.\end{aligned} \tag{15}$$

By substituting all combinations of $b, r = 0, 1, \ldots, N-1$ into equation (15), we obtain a system of N^2 nonlinear algebraic equations involving the unknown Hermite coefficients c_{ij}. In this work, $N = 6$ is chosen to balance computational efficiency and accuracy. All double integrals appearing in the formulation are numerically evaluated using MATLAB's integral2 function. The resulting nonlinear system is solved iteratively using the Picard iteration method with a convergence tolerance of 1×10^{-6}, leading to the approximate solution $\tilde{u}(\chi, t)$.

5. Numerical examples

To illustrate the previously described methods, several numerical examples of the two-dimensional nonlinear Volterra integral equation (2D-NVIE) are provided. The results were obtained using MATLAB R2025a.

Example 1. *Consider the following 2D-NVIE [8, 28]:*

$$u(\chi, t) = f(\chi, t) + \int_0^t \int_0^\chi (\chi s^2 + \cos y) u^2(s, y)\, ds\, dy \quad \chi, t \in [0, 1], \tag{16}$$

where

$$f(\chi, t) = \chi \sin t (1 - \frac{1}{9}\chi^2 \sin^2 t) + \frac{1}{10}\chi^6 (\frac{1}{2}\sin 2t - t),$$

and exact solution is $u(\chi, t) = \chi \sin t$.

The comparison of absolute errors of equation (16) for various values of χ and t, with using Bernoulli collocation (BC), Hermite–Galerkin (HG) methods with $N = 6$, the reproducing kernel space [8] with $N = 30$ and the Taylor collocation method [28] with $N = 64$, is presented in Table (1). Figures (1) and (2) show the absolute error distributions obtained by these methods. Table 2 shows the absolute errors of equation (16) for $(\chi, t) = (0.5, 0.5)$ using BC and HG methods at different N. The errors decrease as N increases, with BC converging much faster than HG due to the latter's higher computational cost from nested integrals. Figure 3 illustrates the same trend visually. Therefore, $N = 6$ was chosen as a suitable compromise between accuracy and efficiency.

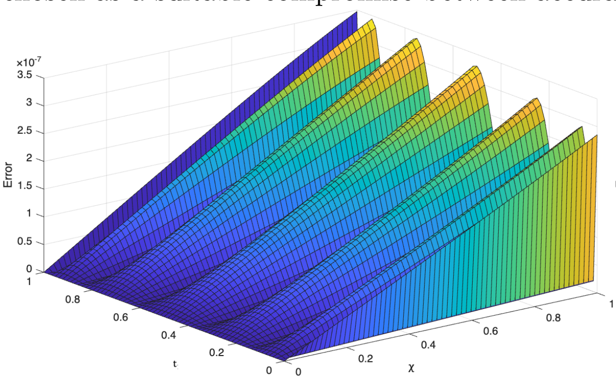

Figure 1: Absolute error of Example 1 by the BC method, $N = 6$.

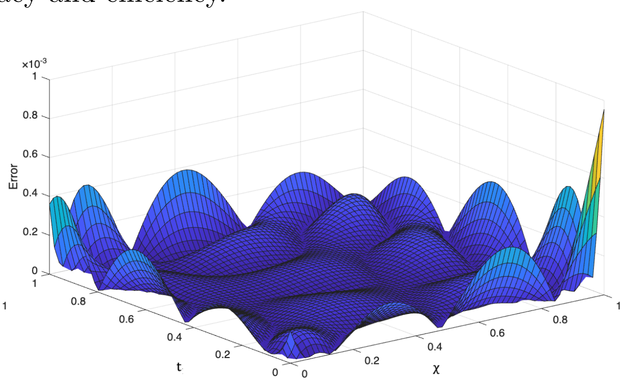

Figure 2: Absolute error for Example 1 by HG method, $N = 6$.

$(\chi,t)=\left(\frac{1}{2^p},\frac{1}{2^p}\right)$	**Bernoulli collocation method** $N=6$	**Hermite-Galerkin method** $N=6$	**Method of [8] with** $N=30$	**Method of [28] with** $N=M=64$
$p=1$	1.61×10^{-7}	3.70×10^{-5}	6.0×10^{-5}	3.45×10^{-7}
$p=2$	7.54×10^{-8}	5.56×10^{-5}	1.29×10^{-4}	1.28×10^{-10}
$p=3$	1.33×10^{-8}	8.06×10^{-5}	7.0×10^{-5}	2.22×10^{-11}
$p=4$	1.77×10^{-8}	1.0×10^{-4}	5.33×10^{-5}	1.67×10^{-13}
$p=5$	4.69×10^{-9}	3.54×10^{-5}	5.959×10^{-5}	3.23×10^{-13}
$p=6$	2.93×10^{-10}	4.33×10^{-5}	7.4588×10^{-4}	9.42×10^{-15}

Table 1: Numerical results for Example 1.

N	**BC method**	**HG method**
2	1.5726×10^{-2}	1.0118×10^{-2}
4	7.7782×10^{-5}	5.3089×10^{-5}
6	1.6191×10^{-7}	3.7095×10^{-5}
7	6.0440×10^{-11}	6.9195×10^{-5}

Table 2: Effect of N on the absolute error for $(\chi,t)=(0.5,0.5)$.

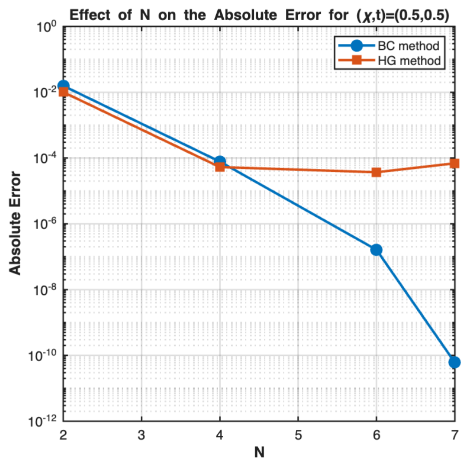

Figure 3: Effect of N on the absolute error for $(\chi,t)=(0.5,0.5)$.

Example 2. *Let us present the following 2D-NVIE [10]:*

$$u(\chi,t)=f(\chi,t)+\int_0^t\int_0^\chi u^2(s,y)\,ds\,dy \quad \chi,t\in[0,1], \tag{17}$$

where

$$f(\chi,t) = \chi^2 + t^2 - \frac{1}{45}\chi t(9\chi^4 + 10\chi^2 t^2 + 9t^4),$$

and the exact solution is $u(\chi,t) = \chi^2 + t^2$.

Table (3) presents the absolute errors of equation (17) calculated using the Bernoulli collocation (BC) and Hermite–Galerkin (HG) methods for various values of χ and t. Figures (4)–(5) illustrate the absolute error distributions obtained by these methods. Furthermore, a comparison with the rationalized Haar functions method [10] with N = 32 is also included.

$(\chi,t) = \left(\frac{1}{2^p}, \frac{1}{2^p}\right)$	**Bernoulli collocation method** $N = 6$	**Hermite-Galerkin method** $N = 6$	**Method of [10] with** $N = 32$
$p = 2$	5.37×10^{-12}	6.92×10^{-5}	5.90×10^{-5}
$p = 3$	1.34×10^{-12}	5.06×10^{-5}	9.06×10^{-7}
$p = 4$	1.15×10^{-12}	9.52×10^{-6}	1.29×10^{-8}
$p = 5$	5.70×10^{-13}	2.49×10^{-4}	1.43×10^{-10}
$p = 6$	1.62×10^{-13}	4.67×10^{-4}	2.00×10^{-12}

Table 3: Numerical results for Example 2.

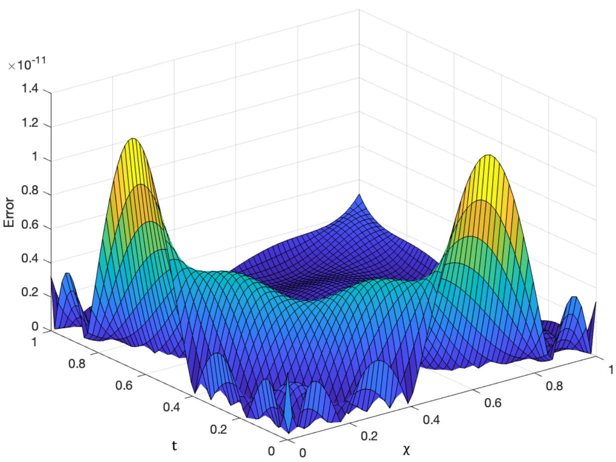

Figure 4: Absolute error of Example 2 by the BC method with $N = 6$.

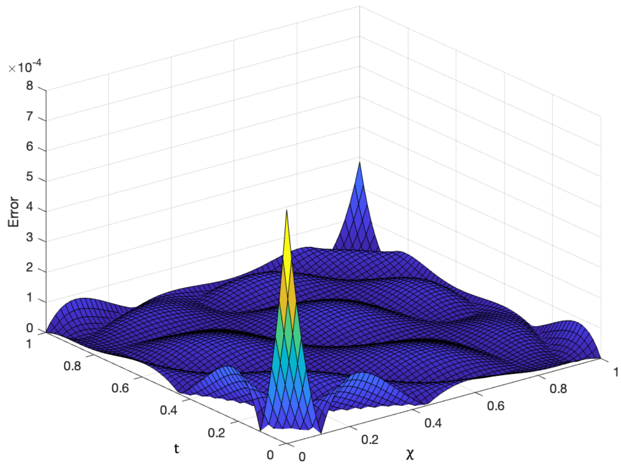

Figure 5: Absolute error for Example 2 by the HG method with $N = 6$.

Example 3. *Let us present the following 2D-NVIE [29]:*

$$u(\chi,t) = f(\chi,t) + \int_0^t \int_0^\chi (\chi + t)e^{u(s,y)}\, ds\, dy \quad \chi, t \in [0,1], \tag{18}$$

where

$$f(\chi, t) = (\chi + t)(e^{\chi} + e^{t} - e^{(\chi+t)}).$$

The exact solution is given by $u(\chi, t) = \chi + t$.

Table (4) shows the comparison of the absolute errors of equation (18) calculated using the Bernoulli collocation (BC) and Hermite–Galerkin (HG) methods for various values of χ and t. Figures (6)–(7) illustrate the absolute error distributions obtained by these methods. A comparison with the extrapolation method [29] for $m = n = 2^6$ is also provided.

(χ, t)	**Bernoulli collocation method** $N = 6$	**Hermite-Galerkin method** $N = 6$	**Method of [29] with** $m = n = 2^6$
$(0.1, 0.1)$	1.87×10^{-11}	5.24×10^{-6}	1.5×10^{-7}
$(0.2, 0.2)$	4.52×10^{-12}	4.36×10^{-5}	1.5×10^{-6}
$(0.3, 0.3)$	2.58×10^{-11}	8.14×10^{-6}	6.4×10^{-6}
$(0.4, 0.4)$	3.73×10^{-12}	1.91×10^{-5}	1.9×10^{-5}
$(0.5, 0.5)$	3.98×10^{-11}	1.67×10^{-5}	4.7×10^{-5}
$(0.6, 0.6)$	8.77×10^{-11}	3.09×10^{-5}	1.0×10^{-4}
$(0.7, 0.7)$	2.83×10^{-11}	5.62×10^{-5}	2.0×10^{-4}
$(0.8, 0.8)$	6.36×10^{-12}	5.79×10^{-5}	3.8×10^{-4}
$(0.9, 0.9)$	2.01×10^{-9}	1.5×10^{-4}	6.8×10^{-4}

Table 4: Numerical results for Example 3.

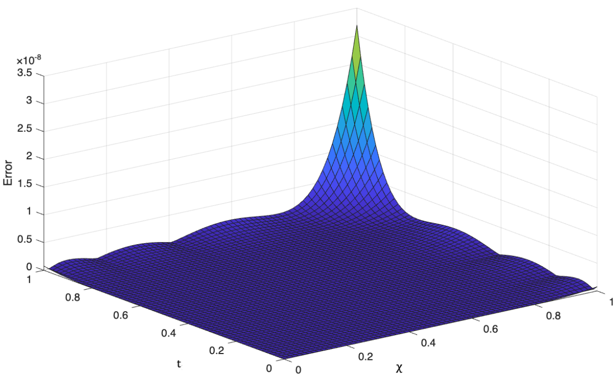

Figure 6: Absolute error for Example 3 by the BC method with $N = 6$.

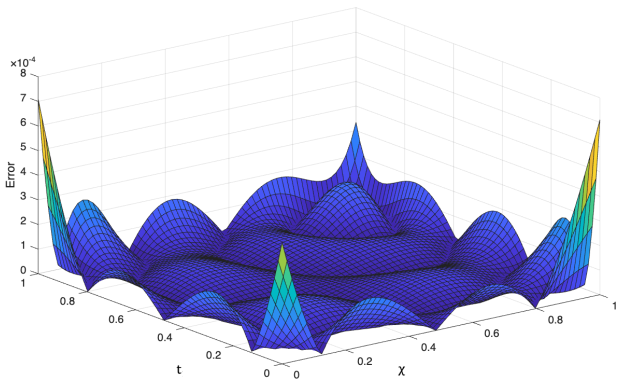

Figure 7: Absolute error for Example 3 by HG method with $N = 6$.

6. Results and Discussion

As shown in the numerical results for the three test examples (Tables (1)–(4) and Figures (1)–(7)), the Bernoulli collocation (BC) method provides smaller absolute errors compared to the Hermite–Galerkin (HG) method. Although $N = 6$ was generally selected as a suitable compromise between accuracy and computational cost, a higher value of N (e.g., $N = 7$ in Example (1)) was occasionally used to investigate the error behavior. The computational cost of the HG method increases significantly with N due to the evaluation of multiple nested integrals. The numerical results indicate that the BC method consistently outperforms the HG method. This superiority is mainly due to the better conditioning of the BC formulation and the suitability of the Bernoulli basis functions for smooth nonlinear kernels, which leads to faster convergence and reduced numerical oscillations.

7. Conclusions

In this article, we employed Bernoulli collocation and Hermite–Galerkin methods to approximate the solution of two-dimensional nonlinear Volterra integral equations. Numerical results show that the Bernoulli collocation method consistently provides more accurate and efficient results than the Hermite–Galerkin method for the same number of points N. In our computations, we used $N = 6$, as increasing N for the Hermite–Galerkin method is difficult due to the complexity of its integrals, while the Bernoulli collocation method handles larger N more efficiently. Comparisons with previously published methods, such as the reproducing kernel space [8], Taylor collocation [28], rationalized Haar functions [10], and extrapolation methods [29], further demonstrate the superiority of our approaches. The better performance of the Bernoulli collocation method is mainly due to the suitability of the Bernoulli basis functions for smooth nonlinear kernels and improved conditioning, leading to faster convergence and reduced numerical oscillations. Furthermore, the present work can be extended to fractional-order versions of two-dimensional Volterra integral equations to capture memory and nonlocal effects, as illustrated in related studies [19, 30].

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Appendix

A. Implementation of the Bernoulli Collocation Method

Algorithm 1 Bernoulli Collocation

Maximum degree N; Tolerance tol; Maximum iterations $I_{\max}$; Nonlinear function $g(u)$.
1. Initialize $C^{(0)}$ (coefficient vector) to zeros (or a suitable initial guess).
2. Define Collocation Points x_i, t_j using Chebyshev roots: $x_{\text{points}} = \frac{1}{2}\left[\cos\left(\frac{(2i-1)\pi}{2N}\right) + 1\right]$.
3. Generate Bernoulli basis functions $B_k(x)$ for $k = 0, \ldots, N-1$.
for $k = 1$ **to** $I_{\max}$ **do**
4. Compute the approximate solution $u^{(k-1)}(s, y)$ from the previous iteration using $C^{(k-1)}$:
$u^{(k-1)}(s, y) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} C_{m,n}^{(k-1)} B_m(s) B_n(y)$.
5. Initialize the system matrices for the current iteration:
$A^{(k)} \leftarrow \text{zeros}(N^2, N^2)$, $F^{(k)} \leftarrow \text{zeros}(N^2, 1)$.
6. Assemble the system at each collocation point (x_i, t_j) for $i, j = 1, \ldots, N$ (Total N^2 points): **for** $i = 1$ **to** N **do**
  for $j = 1$ **to** N **do**
    a. Compute the integral of the known non-linear term (from the previous approximation $u^{(k-1)}$):

$$J_{i,j}^{(k-1)} = \lambda \int_0^{t_j} \int_0^{x_i} K(x_i, t_j, s, y, u^{(k-1)}(s, y))\, ds\, dy$$

    b. Define the right-hand side vector F:

$$F_{(i-1)N+j}^{(k)} = f(x_i, t_j) + J_{i,j}^{(k-1)}$$

    c. Define the system matrix A (which remains constant for the Bernoulli basis):

$$A_{(i-1)N+j, mN+n}^{(k)} = B_m(x_i) B_n(t_j)$$

7. Solve the linear system for the new coefficients:
$C^{(k)} \leftarrow (A^{(k)})^{-1} F^{(k)}$.
8. Check for convergence:
if $\|C^{(k)} - C^{(k-1)}\|_\infty < tol$ **then**
  return $C^{(k)}$ and **break**.
9. Update the coefficients:
$C^{(k-1)} \leftarrow C^{(k)}$.