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Group-Derived and Non-Group-Derived Dual BG-Algebra

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Abstract. This study introduces the notion of the dual BG-algebra. The axioms are presented and are shown to be independent. Fundamental properties of the dual BG-algebra are also provided. The concept of a group-derived dual BG-algebra and its characterization were established. Moreover, a non-group-derived dual BG-algebra can be constructed from a set containing at least 3 elements. Lastly, this paper also presented a Python script used to verify whether a given Cayley table is a dual BG-algebra. This was utilized throughout the process of this study.

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Key Words and Phrases: BG-algebra, dual BG-algebra, dual algebra, group-derived algebra

1. Introduction

Since the 1960s, numerous classes of algebras have been introduced, beginning with BCK/BCI-algebras [1], later extended to BCH-algebras [2, 3], and further generalized to BH-algebras [4]. Neggers and Kim subsequently developed d-algebras [5] and B-algebras [6], while Kim and Kim introduced BG-algebras as a generalization of B-algebras [7].

In parallel, dual algebras were also studied. Kim and Yon investigated dual BCK-algebras and their relation to MV-algebras [8], Kim and Kim proposed BE-algebras [9], Walendziak showed commutative BE-algebras coincide with dual BCK-algebras [10], Meng defined dual BCI-algebras and CI-algebras [11], with Saeid proved the equivalence of CI-algebras and dual Q-algebras [12], and Belleza and Vilela introduced the dual B-algebra and established its relationship to BCK-algebra, CI-algebra, and the dual BCI-algebra [13].

While many algebras and dual algebras have been established and interconnected, no work has addressed the dual BG-algebra. This study introduces its definition, examines its properties, and establishes key characterizations, with particular emphasis on group-derived and non-group-derived dual BG-algebras.

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2. Preliminaries

Definition 1. *[14] A **binary operation** on a nonempty set B (or simply an **operation** on B) is a function $f : B \times B \to B$. Commonly, the symbol $*$ is used instead of f to denote the operation and write $a * b$ instead of $f(a, b)$.*

Definition 2. *[14] A **group** is a nonempty set G equipped with a binary operation "$*$" that satisfies the following axioms:*

- (i) ***associativity**: $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$;*
- (ii) ***identity element**: there is an element $e \in G$ such that $a * e = a = e * a$ for every $a \in G$; and*
- (iii) ***inverse element**: for each $a \in G$, there is an element $d \in G$ such that $a * d = e$ and $d * a = e$.*

*The number of elements in G is called the **order of** G and is denoted by $|G|$.*

Theorem 1. *[14] Let G be a group and let $a, b, c \in G$. Then*

- (i) *G has a unique identity element e;*
- (ii) *Each element $a \in G$ has a unique inverse denoted by a^{-1};*
- (iii) ***cancellation law**: for all $a, b, c \in G$, if either $a * b = a * c$ or $b * a = c * a$, then $b = c$;*
- (iv) *$(a * b)^{-1} = b^{-1} * a^{-1}$; and*
- (v) *$\left(a^{-1}\right)^{-1} = a$ for all $a \in G$.*

Definition 3. *[13] A **dual B-algebra** X is a triple $(X, \circ, 1)$ where X is a nonempty set with a binary operation $\circ$ and a constant 1 satisying the following axioms for all $x, y, z \in X$:*

$$(DB1)\ x \circ x = 1 \quad (DB2)\ 1 \circ x = x \quad (DB3)\ x \circ (y \circ z) = ((y \circ 1) \circ x) \circ z$$

Example 1. *[13] Let $X = \{e, a, b, c\}$ be a set with the following Cayley table:*

Table 1: Cayley table of the dual B-algebra $(X, \circ, e)$

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Then $(X, \circ, e)$ is a dual B-algebra.

Lemma 1. *[13] Let X be a dual B-algebra. For any $x, y \in X$, $(y \circ 1) \circ (y \circ x) = x$.*

Definition 4. *[7] A BG-**algebra** is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms for all $x, y \in X$:*

$$(BG1)\; x * x = 0 \quad (BG2)\; x * 0 = x \quad (BG3)\; (x * y) * (0 * y) = x$$

Example 2. *[7] Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:*

Table 2: Cayley table of the BG-algebra $(X, *, 0)$

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is a BG-algebra.

3. Dual BG-Algebra

In this section, all finite dual BG-algebra examples were verified using a Python script developed by the author found in the Appendix section.

Definition 5. *A **dual** BG-**algebra** X is a triple $(X, \circ, 1)$ where X is a nonempty set with a binary operation $\circ$ and a constant 1 satisying the following axioms for all $x, y \in X$:*

$$(DBG1)\; x \circ x = 1 \quad (DBG2)\; 1 \circ x = x \quad (DBG3)\; (y \circ 1) \circ (y \circ x) = x$$

Example 3. *Let $X = \{1, a, b, c, d, e\}$ be a set with the following Cayley table:*

Table 3: Cayley table of the dual BG-algebra $(X, \circ, 1)$

$\circ$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	b	1	a	d	e	c
b	a	b	1	e	c	d
c	c	d	e	1	a	b
d	d	e	c	b	1	a
e	e	c	d	a	b	1

Then $(X, \circ, 1)$ is a dual BG-algebra.

Example 4. *Let $X = \mathbb{R}\backslash\{0\}$. Define the binary operation $\circ$ as $x \circ y = \dfrac{y}{x}$ for all $x, y \in X$. Note that X satisfies $(DBG1)$: $x \circ x = \dfrac{x}{x} = 1$, $(DBG2)$: $1 \circ x = \dfrac{x}{1} = x$, and $(DBG3)$: $(y \circ 1) \circ (y \circ x) = \dfrac{y \circ x}{y \circ 1} = \dfrac{\frac{x}{y}}{\frac{1}{y}} = \dfrac{x}{1} = x$. Therefore, $(X, \circ, 1)$ is a dual BG-algebra.*

Example 5 shows that the axioms are independent.

Example 5. *Let* $X_1 = (X, \circ_1, 1)$*,* $X_2 = (X, \circ_2, 1)$*, and* $X_3 = (Y, \circ_3, 1)$ *where* $X = \{1, a, b\}$ *and* $Y = \{1, a, b, c\}$*. The Cayley tables of the binary operations* $\circ_1$*,* $\circ_2$*, and* $\circ_3$ *are shown in Table 4.*

Table 4: Cayley tables of X_1, X_2, and X_3

$\circ_1$	1	a	b
1	1	a	b
a	a	1	b
b	1	a	b

$\circ_2$	1	a	b
1	1	b	a
a	b	1	a
b	a	b	1

$\circ_3$	1	a	b	c
1	1	a	b	c
a	b	1	a	c
b	c	a	1	b
c	a	b	c	1

The axioms $(DBG2)$ *and* $(DBG3)$ *hold for* X_1*. However,* $(DBG1)$ *does not hold since* $b \circ_1 b = b \neq 1$*. For* X_2*,* $(DBG1)$ *and* $(DBG3)$ *are satisfied but not* $(DBG2)$ *since* $1 \circ_2 a = b \neq a$*.* $(DBG1)$ *and* $(DBG2)$ *are satisfied in* X_3 *but fails on* $(DBG3)$ *since* $(b \circ_3 1) \circ_3 (b \circ_3 a) = c \circ_3 a = b \neq a$*.*

Example 6. *Consider the dual BG-algebra* $(X, \circ, 1)$ *in Example 3 where* $X = \{1, a, b, c, d, e\}$*. Define the binary operation "$*$" as* $x * y = y \circ x$ *for all* $x, y \in X$*. The Cayley table of* $(X, *, 1)$ *is shown below.*

Table 5: Cayley table of the BG-algebra $(X, *, 1)$

$*$	1	a	b	c	d	e
1	1	b	a	c	d	e
a	a	1	b	d	e	c
b	b	a	1	e	c	d
c	c	d	e	1	b	a
d	d	e	c	a	1	b
e	e	c	d	b	a	1

Note that $(X, *, 1)$ *satisfies* $(BG1)$*:* $x*x = x\circ x = 1$ *by* $(DBG1)$*,* $(BG2)$*:* $x*1 = 1\circ x = x$ *by* $(DBG2)$*, and* $(BG3)$*:* $(x*y)*(1*y) = (1*y)\circ(x*y) = (y\circ 1)\circ(y\circ x) = x$ *by* $(DBG3)$*. Thus,* $(X, *, 1)$ *is a BG-algebra where* 1 *acts as the constant element.*

Example 7. *Consider the BG-algebra* $(X, *, 1)$ *in Example 6 where* $X = \{1, a, b, c, d, e\}$*. Let "$\circ$" be a binary operation where* $x \circ y = y * x$ *for all* $x, y \in X$*. Then the Cayley table of* $(X, \circ, 1)$ *is the same as the Cayley table in 3 and so* $(X, \circ, 1)$ *is a dual BG-algebra.*

Every dual BG-algebra corresponds to a BG-algebra by commuting the operation. This is formalized in the next proposition.

Proposition 1. *Let* $(X, \circ, 1)$ *be a dual BG-algebra. Then* $(X, *, 1)$ *is a BG-algebra where* $x * y = y \circ x$ *for all* $x, y \in X$ *and* 1 *corresponds to the constant element.*

Proof. Suppose $(X, \circ, 1)$ is a dual BG-algebra and define "$*$" as follows: $x * y = y \circ x$ for all x, y in X. Then $(X, \circ, 1)$ satisfies (BG1): $x * x = x \circ x = 1$ by (DBG1), (BG2): $x * 1 = 1 \circ x = x$ by (DBG2), and (BG3): $(x*y)*(1*y) = (1*y)\circ(x*y) = (y\circ 1)\circ(y\circ x) = x$ by (DBG3). Thus, $(X, *, 1)$ is a BG-algebra. □

Example 8. *The dual BG-algebra* $(\mathbb{R}\backslash\{0\}, \circ, 1)$ *where* $x \circ y = \dfrac{y}{x}$ *for all* x, y *in* $\mathbb{R}\backslash\{0\}$ *from Example 4 is not a BG-algebra since* $x \circ 1 = \dfrac{1}{x} \neq x$*, failing to satisfy* $(BG2)$*.*

By Example 8, there exists a dual BG-algebra that is not a BG-algebra, which leads to the next remark.

Remark 1. *Not every dual BG-algebra is a BG-algebra.*

Example 9. *Let* $X = \{1, a, b, c\}$*. Consider the following Cayley table for the binary operation "*$\circ$*". Then* $(X, \circ, 1)$ *is a dual BG-algebra. Note that for any* $x, y \in X$*,* $x\circ y = y\circ x$*.*

Table 6: Cayley table of the dual BG-algebra $(X, \circ, 1)$

$\circ$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Now, let $x * y = y \circ x$ for a binary operation "$*$" where x, y in X. Then $(X, *, 1)$ is a BG-algebra by Proposition 1. Because $(X, \circ, 1)$ satisfies $x \circ y = y \circ x$ for all $x, y \in X$, it follows that it is also a BG-algebra. Thus, there exists a dual BG-algebra that is a BG-algebra at the same time. This is formalized in the next theorem.

Theorem 2. *Let* $(X, \circ, 1)$ *be a dual BG-algebra satisfying* $x \circ y = y \circ x$ *for all* x, y *in* X*. Then* $(X, \circ, 1)$ *is also a BG-algebra.*

Proof. Suppose $(X, \circ, 1)$ is a dual BG-algebra where $x \circ y = y \circ x$ for all x, y in X. Then $(X, \circ, 1)$ satisfies (BG1): $x \circ x = 1$ by (DBG1), (BG2): $x \circ 1 = 1 \circ x = x$ by (DBG2), and (BG3): $(x \circ y) \circ (1 \circ y) = (1 \circ y) \circ (x \circ y) = (y \circ 1) \circ (y \circ x) = x$ by (DBG3). Thus, $(X, \circ, 1)$ is also a BG-algebra. □

Example 10. *Consider the BG-algebra* $(X, *, 0)$ *in Example 2. This is not a dual BG-algebra because* $(2 * 0) * (2 * 1) = 2 * 2 = 0 \neq 1$*, failing to satisfy* $(DBG3)$*.*

By Example 10, there exists a BG-algebra that is not a dual BG-algebra, which leads to the next remark.

Remark 2. *Not every BG-algebra is a dual BG-algebra.*

Lemma 2 shows some properties of the dual BG-algebra.

Lemma 2. *Let* $(X, \circ, 1)$ *be a dual BG-algebra. Then for any* x, y, z *in* X*,* *(i)* $x = (x \circ 1) \circ 1$; *(iv)* *if* $x \circ y = 1$, *then* $x = y$;

(ii) $x = y \circ [(y \circ 1) \circ x]$; *(v)* *if* $x \circ 1 = y \circ 1$, *then* $x = y$; *and*

(iii) $x \circ y = x \circ z$ *implies* $y = z$; *(vi)* *if* $x \circ y = 1$, *then* $(x \circ z) \circ (y \circ z) = 1$.

Proof. Let $(X, \circ, 1)$ be a dual BG-algebra and x, y, z in X. By replacing y with x in (DBG3) and applying (DBG1), (i) is proved as shown: $x = (y \circ 1) \circ (y \circ x) = (x \circ 1) \circ (x \circ x) = (x \circ 1) \circ 1$. For (ii), replace y with $y \circ 1$ in (DBG3) and apply (i), that is, $x = [(y \circ 1) \circ 1] \circ [(y \circ 1) \circ x] = y \circ [(y \circ 1) \circ x]$. To show (iii), replace y with x in (ii) and so $x = x \circ [(x \circ 1) \circ x]$. Using (i), (iv) immediately follows. By hypothesis and (DBG3), $y = (x \circ 1) \circ (x \circ y) = (x \circ 1) \circ (x \circ z) = z$, proving (v). For (vi), if $x \circ y = 1$, then $x \circ y = x \circ x$ by (DBG1). This implies $x = y$ using (v). By (DBG3), $y = (y \circ 1) \circ (y \circ y)$. This implies $y = (x \circ 1) \circ 1$ after applying the hypothesis and (DBG1). Using (i), $x = y$ as needed in (vii). Finally, if $x \circ y = 1$, then $x = y$ by (vi). So, $(x \circ z) \circ (y \circ z) = (x \circ z) \circ (x \circ z) = 1$ by (DBG1), which proves (viii). □

For any dual BG-algebra, replacing y with x on the right-hand side in Lemma 2(ii) yields $x = x \circ [(x \circ 1) \circ x]$. Also, using Lemma 2(i), $x \circ y = [(x \circ 1) \circ 1] \circ y$ follows. This is formalized in the next remark.

Remark 3. *Let* $(X, \circ, 1)$ *be a dual BG-algebra. Then for any* x, y, z *in* X,

$$(i)\; x = x \circ [(x \circ 1) \circ x]\,; \quad (ii)\; x \circ y = [(x \circ 1) \circ 1] \circ y.$$

The next theorem characterizes the dual BG-algebra given any algebra with a binary operation and a constant element, which will be referred to as an algebra of type $(2, 0)$.

Theorem 3. *Let* $(X, \circ, 1)$ *be an algebra of type* $(2, 0)$. *Then* $(X, \circ, 1)$ *is a dual BG-algebra if and only if for any* $x, y \in X$,

$$(i)\; 1 \circ x = x; \quad (ii)\; (y \circ 1) \circ (y \circ x) = x; \quad (iii)\; x \circ y = 1 \text{ if and only if } x = y.$$

Proof. Let $(X, \circ, 1)$ be a dual BG-algebra. Then (i) and (ii) immediately follow from (DBG2) and (DBG3). If $x \circ y = 1$, then $x = y$ by Lemma 2(iv). If $x = y$, then $x \circ y = y \circ y = 1$ by (DBG1). This proves (iii). So, X satisfies (i), (ii), and (iii). Conversely, (DBG1) is implied from (iii), (DBG2) is equivalent to (i), and (DBG3) is equivalent to (ii). Hence, $(X, \circ, 1)$ is a dual BG-algebra. □

Every group can generate a dual BG-algebra given a condition. This is formalized in the next proposition.

Proposition 2. *Let* $(X, *, 1)$ *be a group where* 1 *is the identity element. Then* $(X, \circ, 1)$ *is a dual BG-algebra assuming* $x \circ y = x^{-1} * y$ *for any* $x, y \in X$. *The dual BG-algebra* $(X, \circ, 1)$ *is said to be* ***group-derived***.

Proof. Let $(X, *, 1)$ be a group where x, y in X. Then $(X, \circ, 1)$ satisfies (DBG1): $x \circ x = x^{-1} * x = 1$, (DBG2): $1 \circ x = 1^{-1} * x = 1 * x = x$, and (DBG3): $(y \circ 1) \circ (y \circ x) =$

$(y \circ 1)^{-1} * (y \circ x) = \left(y^{-1} * 1\right)^{-1} * (y^{-1} * x) = y * \left(y^{-1} * x\right) = \left(y * y^{-1}\right) * x = 1 * x = x$. Thus, $(X, \circ, 1)$ is a dual BG-algebra. □

Example 11 shows a dual BG-algebra that is *non-group-derived.*

Example 11. *Let $(X, \circ, 1)$ with the following Cayley table:*

Table 7: Cayley table of the non-group-derived dual BG-algebra $(X, \circ, 1)$

$\circ$	1	a	b
1	1	a	b
a	a	1	b
b	b	a	1

Then $(X, \circ, 1)$ is a dual BG-algebra. To show that it is non-group-derived, assume first that $(X, \circ, 1)$ is group-derived. So, $x \circ y = x^{-1} * y$ for any x, y from a group $(X, *, 1)$ where 1 is the identity element. Since X has only 3 elements, the inverse of a has to be b, that is, $a^{-1} = b$. Hence, $b = a \circ b = a^{-1} * b = b * b = a$, which is a contradiction.

By Example 11, there exists a dual BG-algebra that is non-group-derived, which leads to the next remark.

Remark 4. *Not all dual BG-algebras are group-derived.*

The next theorem shows that a dual BG-algebra is group-derived when it satisfies a certain identity. To establish this, define a binary operation, show that it is a group, and then use Proposition 2.

Theorem 4. *Let $(X, \circ, 1)$ be a dual BG-algebra with the identity $x \circ (y \circ z) = ((x \circ (y \circ 1)) \circ 1) \circ z$ for all $x, y, z \in X$. Then $(X, \circ, 1)$ is group-derived.*

Proof. Define a binary operation "$*$" on X as $x * y = (x \circ 1) \circ y$. Note that $x * 1 = (x \circ 1) \circ 1 = x$ by Lemma 2(i) and $1 * x = (1 \circ 1) \circ x = 1 \circ x = x$ by (DBG1) and (DBG2). So 1 is the identity element with respect to the binary operation "$*$". Also $x * (x \circ 1) = (x \circ 1) \circ (x \circ 1) = 1$ by (DBG1) and $(x \circ 1) * x = ((x \circ 1) \circ 1) \circ x = x \circ x = 1$ by Lemma 2(i) and (DBG1). Thus, $x \circ 1$ is the inverse for x. Now, $x * (y * z) = (x \circ 1) \circ (y * z) = (x \circ 1) \circ [(y \circ 1) \circ z] = [[(x \circ 1) \circ [(y \circ 1) \circ 1]] \circ 1] \circ z$ by replacing x with $x \circ 1$ and y with $y \circ 1$ in the given identity. By Lemma 2(i), $x * (y * z) = [[(x \circ 1) \circ y] \circ 1] \circ z$. The continuation is as follows: $x * (y * z) = [(x * y) \circ 1] \circ z = (x * y) * z$. Hence, $(X, *, 1)$ is a group by Definition 2. Observe that $x^{-1} * y = \left(x^{-1} \circ 1\right) \circ y = [(x \circ 1) \circ 1] \circ y = x \circ y$. Therefore, $(X, \circ, 1)$ is a group-derived dual BG-algebra by Proposition 2. □

Consider Example 11. Note that $b \circ (a \circ b) = b \circ b = 1$ while $((b \circ (a \circ 1)) \circ 1) \circ b = b$. So, the condition $x \circ (y \circ z) = ((x \circ (y \circ 1)) \circ 1) \circ z$ in Theorem 4 is not necessarily true in general.

Example 12. *Consider the dual BG-algebra $(X, \circ, 1)$ from Example 9. It satisfies the identity $x \circ (y \circ z) = ((x \circ (y \circ 1)) \circ 1) \circ z$. Hence, $(X, \circ, 1)$ is a group-derived dual BG-algebra by Theorem 4.*

Theorem 5 characterizes a group-derived dual BG-algebra through the dual B-algebra.

Theorem 5. *Let* $(X, \circ, 1)$ *be an algebra of type* $(2, 0)$*. Then* $(X, \circ, 1)$ *is a dual* B*-algebra if and only if it is a group-derived dual* BG*-algebra.*

Proof. Let $(X, \circ, 1)$ be a dual B-algebra. (DBG1), (DBG2), and (DBG3) immediately follow from (DB1), (DB2), and Lemma 1, respectively. Thus, $(X, \circ, 1)$ is a dual BG-algebra.

Define a binary operation "$*$" on X as $x*y = (x\circ 1)\circ y$. Note that $x*1 = (x\circ 1)\circ 1 = x$ by Lemma 2(i) and $1 * x = (1 \circ 1) \circ x = 1 \circ x = x$ by (DBG1) and (DBG2). So 1 is the identity element with respect to the binary operation "$*$". Also $x * (x \circ 1) = (x \circ 1) \circ (x \circ 1) = 1$ by (DBG1) and $(x \circ 1) * x = ((x \circ 1) \circ 1) \circ x = x \circ x = 1$ by Lemma 2(i) and (DBG1). Thus, $x \circ 1$ is the inverse for x. Now, $x * (y * z) = (x \circ 1) \circ (y * z) = (x \circ 1) \circ [(y \circ 1) \circ z] = [[(y \circ 1) \circ 1] \circ (x \circ 1)] \circ z$ by replacing x with $x \circ 1$ and y with $y \circ 1$ in (DB3). By Lemma 2(i), $x * (y * z) = [y \circ (x \circ 1)] \circ z$. Using (DB3), $y \circ (x \circ 1) = [(x \circ 1) \circ y] \circ 1$ and so $x * (y * z) = [[(x \circ 1) \circ y] \circ 1] \circ z$. The continuation is as follows: $x * (y * z) = [(x * y) \circ 1] \circ z = (x * y) * z$. Thus, $(X, *, 1)$ is a group by Definition 2. Note that $x^{-1} * y = \left(x^{-1} \circ 1\right) \circ y = [(x \circ 1) \circ 1] \circ y = x \circ y$. Therefore, $(X, \circ, 1)$ is a group-derived dual BG-algebra by Proposition 2.

Conversely, let $X = (X, \circ, 1)$ be a group-derived dual BG-algebra. Then (DB1) and (DB2) follow from (DBG1) and (DBG2), respectively. Now, since X is group-derived, then it was generated from a group $(X, *, 1)$ where $x^{-1} * y = x \circ y$ where $x, y \in X$. Now, $x \circ (y \circ z) = x^{-1} * (y \circ z) = x^{-1} * \left(y^{-1} * z\right) = \left(x^{-1} * y^{-1}\right) * z$. Associative property can be applied since $(X, *, 1)$ is a group. Using Theorem 1, the continuation is as follows: $x\circ(y\circ z) = (y * x)^{-1} * z = \left[\left(y^{-1}\right)^{-1} * x\right]^{-1} * z = \left[\left(y^{-1} * 1\right)^{-1} * x\right]^{-1} * z = \left[(y \circ 1)^{-1} * x\right]^{-1} * z = [(y \circ 1) \circ x]^{-1} * z = ((y \circ 1) \circ x) \circ z$. Hence, (DB3) is satisfied and so $(X, \circ, 1)$ is a dual B-algebra. This proves the theorem. □

Example 13. *Consider the dual* BG*-algebra* $(X, \circ, 1)$ *from Example 9. It was shown that it is a group-derived dual* BG*-algebra in Example 12. Thus,* $(X, \circ, 1)$ *is also a dual* B*-algebra by Theorem 5. Indeed, treating* 1 *as* e*,* $(X, \circ, 1)$ *is a dual* B*-algebra by Example 1.*

It is evident from the proof of Theorem 5 that every dual B-algebra is a dual BG-algebra. Consequently, non-group-derived dual BG-algebras are not dual B-algebras.

Theorem 6 can be used to construct infinitely many dual BG-algebras. These dual BG-algebras are also shown to be non-group-derived.

Theorem 6. *Define a binary operation "*$\circ$*" on a set* X *where* $1 \in X$ *by*

$$x \circ y = \begin{cases} x & \text{if } y = 1 \\ 1 & \text{if } x = y \\ y & \text{otherwise} \end{cases},$$

for any $x, y \in X$*, then* $(X, \circ, 1)$ *is a dual* BG*-algebra. Moreover, if* X *has at least 3 elements, then* $(X, \circ, 1)$ *is non-group-derived.*

Proof. Let $x, y \in X$. Note that $(X, \circ, 1)$ satisfies (DBG1): $x \circ x = 1$ and (DBG2): $1 \circ x = x$. Now, if $y = 1$, then $(y \circ 1) \circ (y \circ x) = (1 \circ 1) \circ (1 \circ x) = 1 \circ x = x$. Assume $y \neq 1$. If $x = y$, then $(y \circ 1) \circ (y \circ x) = (x \circ 1) \circ (x \circ x) = x \circ 1 = x$. If $x \neq y$, then $(y \circ 1) \circ (y \circ x) = y \circ x = x$. Hence, $(X, \circ, 1)$ satisfies (DBG3). Therefore, $(X, \circ, 1)$ is a dual BG-algebra.

Now, assume X has at least 3 elements. Let $x, y, z \in X$ be unique elements and $z \neq 1$. Then $x \circ z = y \circ z = z$. Assume $(X, \circ, 1)$ is a group-derived dual BG-algebra obtained from the group $(X, *)$, then $z = x \circ z = x^{-1} * z$ and $z = y \circ z = y^{-1} * z$. This implies $x = y$, which is a contradiction since they are unique by hypothesis. So, the dual BG-algebra $(X, \circ, 1)$ has to be non-group-derived. □

The condition for X in Theorem 6 to have at least 3 elements is necessary. To see this, if X has only one element, then it has to be 1 and so $X = \{1\}$. If X has two elements, say $X = \{1, a\}$, then $1 \circ 1 = 1$, $1 \circ a = a$, $a \circ 1 = a$, and $a \circ a = 1$. In any case, $(X, \circ, 1)$ satisfies $x \circ (y \circ z) = ((x \circ (y \circ 1)) \circ 1) \circ z$. Therefore, $(X, \circ, 1)$ is a group-derived dual BG-algebra by Theorem 4 when X has only 1 or 2 elements.

Example 14. *Let* $X_1 = (X, \circ_1, 1)$, $Y_1 = (Y, \circ_2, 1)$, *and* $Z_1 = (Z, \circ_3, 1)$ *where* $X = \{1, a, b, c\}$, $Y = \{1, a, b, c, d\}$, *and* $Z = \{1, a, b, c, d, e\}$. *The Cayley tables of the binary operations* $\circ_1$, $\circ_2$, *and* $\circ_3$ *are shown in Table 8.*

Table 8: Cayley tables of X_1, Y_1, and Z_1

$\circ_1$	1	a	b	c
1	1	a	b	c
a	a	1	b	c
b	b	a	1	c
c	c	a	b	1

$\circ_2$	1	a	b	c	d
1	1	a	b	c	d
a	a	1	b	c	d
b	b	a	1	c	d
c	c	a	b	1	d
d	d	a	b	c	1

$\circ_3$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	b	c	d	e
b	b	a	1	c	d	e
c	c	a	b	1	d	e
d	d	a	b	c	1	e
e	e	a	b	c	d	1

Then X_1, Y_1, *and* Z_1 *are all non-group-derived dual BG-algebras by Theorem 6.*

Example 14 shows some of the dual BG-algebras that can be generated using Theorem 6.

4. Conclusion

It was shown that the axioms of the dual BG-algebra are independent. Not every dual BG-algebra is a BG-algebra and not every BG-algebra is dual BG-algebra. But it is possible that a dual BG-algebra is also a BG-algebra. A characterization for the dual BG-algebra was established. The notion of a group-derived and non-group-derived dual BG-algebras were also introduced. It was shown that a group-derived dual BG-algebra is characterized by a dual B-algebra and infinitely many non-group-derived dual BG-algebras can be constructed.

5. Recommendations

In one of the theorems, it was shown that every dual B-algebra is a dual BG-algebra. The authors recommend exploring the relationship of the dual BG-algebra to other algebras and dual algebras, and not just the dual B-algebra. Other structural properties may also be considered such as ideals, filters, homomorphisms, among others.

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Appendix

The following Python script developed by the author was used to verify if a given Cayley table is a dual BG-algebra. The verification result of Example 3 using the script is shown.

```
def dbg1(X, tbl):
  # x o x = 1 for all x in X

  shp = len(tbl)
  flag = True

  print(f"X = {X}")
  print("\nCayley table")
  print(np.array(tbl))
  print("")

  constant = X[0]
  ctr = 0

  for i in range(shp):
    ctr += 1

    left = tbl[i][i]
    right = constant

    if (left != right):
      print(f"{ctr}.\tx = {X[i]}:\t{X[i]} o {X[i]} != {constant}\t->
      \t {left} != {right}")
      flag = False
      break
    print(f"{ctr}.\tx = {X[i]}:\t{X[i]} o {X[i]} = {constant}\t->
    \t {left} = {right}")

  if (flag == True):
    print(f"\nx o x = {constant} for all x in X")
  else:
    print("\nX does not satisfy axiom dbg1.")

  return flag

def dbg2(X, tbl):
  # 1 o x = x for all x in X
```

```
  shp = len(tbl)
  flag = True

  print(f"X = {X}")
  print("\nCayley table")
  print(np.array(tbl))
  print("")

  constant = X[0]
  ctr = 0

  for i in range(shp):
    ctr += 1

    left = tbl[0][i]
    right = X[i]

    if (left != right):
      print(f"{ctr}.\tx = {X[i]}:\t{constant} o {X[i]} != {X[i]}\t->
      \t {left} != {right}")
      flag = False
      break

    print(f"{ctr}.\tx = {X[i]}:\t{constant} o {X[i]} = {X[i]}\t->
    \t {left} = {right}")

  if (flag == True):
    print(f"\n{constant} o x = x for all x in X")
  else:
    print("\nX does not satisfy axiom dbg2.")

  return flag

def dbg3(X, tbl):
  # (y o 1) o (y o x) = x for all x, y in X

  shp = len(tbl)
  flag = True

  print(f"X = {X}")
  print("\nCayley table")
  print(np.array(tbl))
  print("")
```

```
  constant = X[0]
  ctr = 0

  for i in range(shp):
    for j in range(shp):
      ctr += 1

      y_0 = tbl[j][0]
      y_x = tbl[j][i]

      left = tbl[X.index(y_0)][X.index(y_x)]
      right = X[i]

      if (left != right):
        print(f"{ctr}.\tx = {X[i]}, y = {X[j]}:
        \t({X[j]} o {constant}) o ({X[j]} o {X[i]}) != {X[i]}\t->
        \t{y_0} o {y_x} != {X[i]}\t->\t {left} != {right}")
        flag = False
        break

      if (left == right):
        print(f"{ctr}.\tx = {X[i]}, y = {X[j]}:
        \t({X[j]} o {constant}) o ({X[j]} o {X[i]}) = {X[i]}\t->
        \t{y_0} o {y_x} = {X[i]}\t->\t {left} = {right}")

    if (flag == False):
      break

  if (flag == True):
    print(f"\n(y o {constant}) o (y o x) = x for all x, y in X")
  else:
    print("\nX does not satisfy axiom dbg3.")

  return flag

def dbg(X, tbl):
  dbg_flag = False
  constant = X[0]

  print(f"checking dbg1: x o x = {constant} for all x in X ...")
  dbg1_flag = dbg1(X, tbl)
```

```
print("------------------------------------------------------------\n")

print(f"checking dbg2: {constant} o x = x for all x in X ...")
dbg2_flag = dbg2(X, tbl)

print("------------------------------------------------------------\n")

print("checking dbg3: \n")
print(f"(y o {constant}) o (y o x) = x for all x, y in X ...")
dbg3_flag = dbg3(X, tbl)

print("------------------------------------------------------------\n")

if (dbg1_flag, dbg2_flag, dbg3_flag) == (True, True, True):
  dbg_flag = True

if (dbg_flag == True):
  print("X satisfies dbg1, dbg2, and dbg3.")
  print("\nTherefore, X is a Dual BG-algebra.")
else:
  print("\nX is NOT a Dual BG-algebra.")

return dbg_flag
```

```
checking dbg1: x ∘ x = 1 for all x in X ...
X = [1, 'a', 'b', 'c', 'd', 'e']

Cayley table
[['1' 'a' 'b' 'c' 'd' 'e']
 ['b' '1' 'a' 'd' 'e' 'c']
 ['a' 'b' '1' 'e' 'c' 'd']
 ['c' 'd' 'e' '1' 'a' 'b']
 ['d' 'e' 'c' 'b' '1' 'a']
 ['e' 'c' 'd' 'a' 'b' '1']]

1.      x = 1:  1 ∘ 1 = 1       ⇒       1 = 1
2.      x = a:  a ∘ a = 1       ⇒       1 = 1
3.      x = b:  b ∘ b = 1       ⇒       1 = 1
4.      x = c:  c ∘ c = 1       ⇒       1 = 1
5.      x = d:  d ∘ d = 1       ⇒       1 = 1
6.      x = e:  e ∘ e = 1       ⇒       1 = 1

x ∘ x = 1 for all x in X
```

DBG1 Verification

```
checking dbg2: 1 ∘ x = x for all x in X ...
X = [1, 'a', 'b', 'c', 'd', 'e']

Cayley table
[['1' 'a' 'b' 'c' 'd' 'e']
 ['b' '1' 'a' 'd' 'e' 'c']
 ['a' 'b' '1' 'e' 'c' 'd']
 ['c' 'd' 'e' '1' 'a' 'b']
 ['d' 'e' 'c' 'b' '1' 'a']
 ['e' 'c' 'd' 'a' 'b' '1']]

1.      x = 1:  1 ∘ 1 = 1       ⇒       1 = 1
2.      x = a:  1 ∘ a = a       ⇒       a = a
3.      x = b:  1 ∘ b = b       ⇒       b = b
4.      x = c:  1 ∘ c = c       ⇒       c = c
5.      x = d:  1 ∘ d = d       ⇒       d = d
6.      x = e:  1 ∘ e = e       ⇒       e = e

1 ∘ x = x for all x in X
```

DBG2 Verification

```
checking dbg3: (y ∘ 1) ∘ (y ∘ x) = x for all x, y in X ...
X = [1, 'a', 'b', 'c', 'd', 'e']

Cayley table
[['1' 'a' 'b' 'c' 'd' 'e']
 ['b' '1' 'a' 'd' 'e' 'c']
 ['a' 'b' '1' 'e' 'c' 'd']
 ['c' 'd' 'e' '1' 'a' 'b']
 ['d' 'e' 'c' 'b' '1' 'a']
 ['e' 'c' 'd' 'a' 'b' '1']]

1.    x = 1, y = 1:   (1 ∘ 1) ∘ (1 ∘ 1) = 1   ⇒   1 ∘ 1 = 1   ⇒   1 = 1
2.    x = 1, y = a:   (a ∘ 1) ∘ (a ∘ 1) = 1   ⇒   b ∘ b = 1   ⇒   1 = 1
3.    x = 1, y = b:   (b ∘ 1) ∘ (b ∘ 1) = 1   ⇒   a ∘ a = 1   ⇒   1 = 1
4.    x = 1, y = c:   (c ∘ 1) ∘ (c ∘ 1) = 1   ⇒   c ∘ c = 1   ⇒   1 = 1
5.    x = 1, y = d:   (d ∘ 1) ∘ (d ∘ 1) = 1   ⇒   d ∘ d = 1   ⇒   1 = 1
6.    x = 1, y = e:   (e ∘ 1) ∘ (e ∘ 1) = 1   ⇒   e ∘ e = 1   ⇒   1 = 1
7.    x = a, y = 1:   (1 ∘ 1) ∘ (1 ∘ a) = a   ⇒   1 ∘ a = a   ⇒   a = a
8.    x = a, y = a:   (a ∘ 1) ∘ (a ∘ a) = a   ⇒   b ∘ 1 = a   ⇒   a = a
9.    x = a, y = b:   (b ∘ 1) ∘ (b ∘ a) = a   ⇒   a ∘ b = a   ⇒   a = a
10.   x = a, y = c:   (c ∘ 1) ∘ (c ∘ a) = a   ⇒   c ∘ d = a   ⇒   a = a
11.   x = a, y = d:   (d ∘ 1) ∘ (d ∘ a) = a   ⇒   d ∘ e = a   ⇒   a = a
12.   x = a, y = e:   (e ∘ 1) ∘ (e ∘ a) = a   ⇒   e ∘ c = a   ⇒   a = a
13.   x = b, y = 1:   (1 ∘ 1) ∘ (1 ∘ b) = b   ⇒   1 ∘ b = b   ⇒   b = b
14.   x = b, y = a:   (a ∘ 1) ∘ (a ∘ b) = b   ⇒   b ∘ a = b   ⇒   b = b
15.   x = b, y = b:   (b ∘ 1) ∘ (b ∘ b) = b   ⇒   a ∘ 1 = b   ⇒   b = b
16.   x = b, y = c:   (c ∘ 1) ∘ (c ∘ b) = b   ⇒   c ∘ e = b   ⇒   b = b
17.   x = b, y = d:   (d ∘ 1) ∘ (d ∘ b) = b   ⇒   d ∘ c = b   ⇒   b = b
18.   x = b, y = e:   (e ∘ 1) ∘ (e ∘ b) = b   ⇒   e ∘ d = b   ⇒   b = b
19.   x = c, y = 1:   (1 ∘ 1) ∘ (1 ∘ c) = c   ⇒   1 ∘ c = c   ⇒   c = c
20.   x = c, y = a:   (a ∘ 1) ∘ (a ∘ c) = c   ⇒   b ∘ d = c   ⇒   c = c
21.   x = c, y = b:   (b ∘ 1) ∘ (b ∘ c) = c   ⇒   a ∘ e = c   ⇒   c = c
22.   x = c, y = c:   (c ∘ 1) ∘ (c ∘ c) = c   ⇒   c ∘ 1 = c   ⇒   c = c
23.   x = c, y = d:   (d ∘ 1) ∘ (d ∘ c) = c   ⇒   d ∘ b = c   ⇒   c = c
24.   x = c, y = e:   (e ∘ 1) ∘ (e ∘ c) = c   ⇒   e ∘ a = c   ⇒   c = c
25.   x = d, y = 1:   (1 ∘ 1) ∘ (1 ∘ d) = d   ⇒   1 ∘ d = d   ⇒   d = d
26.   x = d, y = a:   (a ∘ 1) ∘ (a ∘ d) = d   ⇒   b ∘ e = d   ⇒   d = d
27.   x = d, y = b:   (b ∘ 1) ∘ (b ∘ d) = d   ⇒   a ∘ c = d   ⇒   d = d
28.   x = d, y = c:   (c ∘ 1) ∘ (c ∘ d) = d   ⇒   c ∘ a = d   ⇒   d = d
29.   x = d, y = d:   (d ∘ 1) ∘ (d ∘ d) = d   ⇒   d ∘ 1 = d   ⇒   d = d
30.   x = d, y = e:   (e ∘ 1) ∘ (e ∘ d) = d   ⇒   e ∘ b = d   ⇒   d = d
31.   x = e, y = 1:   (1 ∘ 1) ∘ (1 ∘ e) = e   ⇒   1 ∘ e = e   ⇒   e = e
32.   x = e, y = a:   (a ∘ 1) ∘ (a ∘ e) = e   ⇒   b ∘ c = e   ⇒   e = e
33.   x = e, y = b:   (b ∘ 1) ∘ (b ∘ e) = e   ⇒   a ∘ d = e   ⇒   e = e
34.   x = e, y = c:   (c ∘ 1) ∘ (c ∘ e) = e   ⇒   c ∘ b = e   ⇒   e = e
35.   x = e, y = d:   (d ∘ 1) ∘ (d ∘ e) = e   ⇒   d ∘ a = e   ⇒   e = e
36.   x = e, y = e:   (e ∘ 1) ∘ (e ∘ e) = e   ⇒   e ∘ 1 = e   ⇒   e = e

(y ∘ 1) ∘ (y ∘ x) = x for all x, y in X
```

DBG3 Verification

```
X satisfies dbg1, dbg2, and dbg3. Therefore, X is a Dual BG-algebra.

True
```

Example 3 is a Dual *BG*-Algebra