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Accelerated Tseng's Method for Finding Common Solutions of Fixed Point, Variational Inequality, and Zeros Problems in Reflexive Banach Spaces

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Abstract. In this paper, we propose a novel accelerated extrapolation version of Tseng's algorithm with a self-adaptive step size for approximating a common solution to pseudomonotone variational inequality problems, zeros of maximal and Bregman inverse strongly monotone operators, and common fixed points of a finite family of Bregman demigeneralized mappings in a smooth, strictly convex and real reflexive Banach space. Using the Bregman distance technique, we establish a strong convergence result under mild assumptions, without requiring prior knowledge of the Lipschitz constant of the operator with application to a convex minimization problem (CMP). Our findings generalize and improve several results in the existing literature. Finally, we provide numerical example to demonstrate the effectiveness of our algorithm over recently announced results in the literature. Our results generalize and improve upon many existing findings in the literature.

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1. Introduction

Let E be a real reflexive Banach space with its dual space E^*, C a nonempty, closed, and convex subset of E and $B : E \to 2^{E^*}$ be a multi-valued mapping with domain

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$D(B) = \{x^* \in E : Bx^* \neq \emptyset\}$. Then B is monotone if $\forall\ x, y \in E$, with

$$u^* \in Bx \quad and \quad v^* \in By, \quad then \quad \langle x - y, u^* - v^* \rangle \geq 0 \quad hold.$$

The study of monotone maps was first introduced by Minty in 1960 in order to ease the abstract study of electrical networks (see, [1] for more details). Interest in monotone operators stems mainly from the fact that, they are applicable in various areas of science and engineering (see, for example [2, 3] and references therein). Thus, a fundamental problem of interest in studying monotone operators in Banach space is as follows:

$$find \quad x^* \in E \quad such\ that \qquad 0 \in Bx^*. \tag{1}$$

Numerous problems in applications can be transformed into the form of the inclusion problem (1). For example, problems arising from convex minimization, variational inequality, Hammerstein equations and evolution equations can be transformed into the form of the inclusion problem (1) (see, for example [4] and reference therein). Iterative methods for approximating solution of inclusion problem (1) have been studied extensively by various authors in Hilbert space and in more general Banach space (see, for example [4–6] and references therein). One of the methods for approximating solution(s) of (1) in Hilbert space is the proximal point algorithm (PPA) introduced by Martinet [7]. Let $x_0 \in E$, then

$$x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, 3, ..., \tag{2}$$

where $\{r_n\} \in (0, \infty)$ and J_{r_n} is the resolvent of B.
Let $F : C \to E^*$ be a mapping. The problem of finding a point $x^* \in C$ such that

$$\langle Fx^*, y - x^* \rangle \geq 0, \quad \forall\ y \in C, \tag{3}$$

is called a variational inequality problem, denoted by $VI(C, F)$. Variational inequality problems (VIPs) originated from efforts to solve optimization problems involving infinite-dimensional functions and calculus of variations, as developed by Hartman and Stampacchia [8]. Since then, VIPs have found applications in numerous scientific and mathematical fields, including networking, image recovery, resource allocation, and optimal control [9–13]. These problems can be expressed as either variational inequalities or fixed point problems, making the study of their common solutions significant [14, 15].

To solve VIPs, various iterative methods, primarily based on projection methods, have been proposed. Goldstein [16] introduced the earliest projection method, an extension of the gradient projection technique. Korpelevich [17] proposed the extragradient method (EGM) to reduce the stringent condition of strong monotonicity on the operator. The sequence generated by EGM converges weakly to a solution of VIP, but the method requires two projections per iteration, which can be computationally expensive.

To address this, several modifications have been suggested. Tseng [18] developed Tseng's extragradient method (TEGM), which also converges weakly to a solution in a real Hilbert space. Censor et al. [19] introduced the subgradient extragradient method (SEGM), which replaces one projection with a projection onto a half-space, simplifying implementation [18–20]. Kraikaew and Saejung [21] combined SEGM with the Halpern method to achieve strong convergence. In 1964, Polyak [22] introduced the inertial extrapolation process to accelerate convergence of iterative methods. This technique has inspired numerous inertial-type iterative schemes (see [13, 23–25] and references therein). Recently, Uzor et al. [26] proposed a viscosity-type inertial Tseng's extragradient algorithm for solving VIPs in real Hilbert spaces.

Bregman distances, introduced by Bregman [27], offer an efficient technique for designing and analyzing optimization algorithms. Ali et al. [28] proposed a modified inertial subgradient extragradient method for approximating solutions without prior knowledge of the Lipschitz constant of the operator.
Reich and Sabach [29] introduced the concept of Bregman strongly nonexpansive mappings using Bregman distance function. They also studied the convergence of two iterative algorithms for finding common fixed point of finitely Bregman strongly nonexpansive mappings in reflexive Banach spaces.

Recently, Orouji et al. [30] introduced the following shrinking projection method for approximating a common element in the set of zeros of maximal and Bregman inverse strongly monotone mappings and the set of common fixed points of a finite family of Bregman k-demimetric mappings in a reflexive Banach space. For $x_1 \in C$ and $C_1 = Q_1 = C$, let $\{x_n\}$ be a sequence defined iteratively as:

$$\begin{cases} y_n = \nabla f^*(\sum_{j=1}^{M} \xi_j((1-\lambda_n)\nabla f(x_n) + \lambda_n \nabla f(T_j x_n))), \\ z_n = \nabla f^* \sum_{i=1}^{N} \sigma_i \nabla f Q_{\eta_n} B^f_{i,\eta_n}(y_n), \\ u_n = J_{r_n} z_n, \\ C_{n+1} = \{z \in C_n : D_f(z, y_n) \leq D_f(z, x_n), D_f(z, z_n) \leq D_f(z, y_n), \\ \langle \nabla f(z_n) - \nabla f(u_n), z_n - z\rangle \geq D_f(z_n, u_n)\}, \\ Q_{n+1} = \{z \in Q_n : \langle \nabla f(x_1) - \nabla f(x_{n+1}), x_{n+1} - z\rangle \geq 0\}, \\ x_{n+1} = Proj^f_{C_{n+1}\cap Q_n}(x_1), \ \forall\, n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\lambda_n\} \subseteq (0, 1), \{\eta_n\}, \{r_n\} \subseteq (0, +\infty), \{\xi_1, \xi_2, ..., \xi_M\}, \{\sigma_1, \sigma_2, ..., \sigma_N\} \subseteq (0, 1)$ and $a, b, c \in \mathbb{R}$. They showed that the sequence $\{x_n\}$ generated by algorithm (4) converges strongly to an element $w_0 \in \Omega$ where $w_0 = Proj^f_{\Omega} x_1$ and
$\Omega = A^{-1}(0) \bigcap(\bigcap_{j=1}^{M} F(T_j)) \bigcap(\bigcap_{i=1}^{N}(B_i + G)^{-1}0^*) \neq \emptyset$.
Thus, the following questions arise:

(i) Can we dispense with the sets C_n and Q_n in the algorithm of Orouji et al. [30] and still obtain strong convergence?

(ii) Can we provide a new inertial - type Tseng's extragradient algorithm with self - adaptive step size using Bregman distance technique, for approximating a common element in the set of solutions of pseudomonotone variational inequality problem and zeros of Bregman inverse strongly monotone mappings in a real reflexive Banach space such that its implementation does not require a prior knowledge of Lipschitz constant of the operator?

(iii) Can we approximate such solution as mentioned above which happen to be a common fixed point of a finite family of Bregman demigeneralized mappings in a real reflexive Banach space?

Motivated by these developments, we introduce a new accelerated extrapolation Tseng's algorithm with a self-adaptive step size for approximating a common element of the solution set of pseudomonotone VIPs, zeros of maximal and Bregman inverse strongly monotone mappings and the set of common fixed points of a finite family of Bregman demigeneralized mappings in "a smooth, strictly convex and real reflexive Banach space. Using the Bregman distance technique, we prove a strong convergence theorem for our algorithm without prior knowledge of the Lipschitz constant of the operator under mild assumptions. Our results generalize and improve upon many existing findings in the literature.

2. Preliminaries

In this section, we introduce essential definitions and lemmas required for this paper. Let E be a real reflexive Banach space with its dual space E^*, and let C be a nonempty, closed, and convex subset of E. We denote the duality pairing between E and E^* by $\langle \cdot, \cdot \rangle$, and the domain of a function $f : E \to (-\infty, +\infty]$ is denoted by $\operatorname{dom} f := \{x \in E : f(x) < +\infty\}$.

Let $x \in \operatorname{int}(\operatorname{dom} f)$:

(T1) The subdifferential of f is a function $\partial f : E \to E^*$ defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \forall y \in E\}.$$

(T2) The Fenchel conjugate of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}.$$

(T3) For any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, the right-hand derivative of f at x in the direction of y is

$$f^0(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be Gateaux differentiable at x if the limit as $t \to 0$ in $(T3)$ exists for each y. In this case, the gradient of f at x is the linear function $\nabla f : E \to (-\infty, +\infty]$ defined by $\langle \nabla f(x), y \rangle = f^0(x, y)$ for all $y \in E$. If f is Frechet differentiable at x, the limit as $t \to 0$ in $(T3)$ is attained uniformly in y with $\|y\| = 1$.

A function f on E is strongly coercive if

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

Definition 1. *A function f is:*

(i) Essentially smooth if ∂f is locally bounded and single-valued on its domain;

(ii) Essentially strictly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $dom\,\partial f$;

(iii) A Legendre function if it is both essentially smooth and essentially strictly convex.

Remark 1. *If E is a reflexive Banach space and f is a Legendre function, then:*

(i) f is essentially smooth if and only if f^* is essentially strictly convex;

(ii) $(\partial f)^{-1} = \partial f^*$;

(iii) f is Legendre if and only if f^* is a Legendre function;

(iv) If f is a Legendre function, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $ran\,\nabla f = dom\,\nabla f^* = int(dom\,f^*)$ and $ran\,\nabla f^* = dom\,f = int(dom\,f)$.

Definition 2. *Let $f : E \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : domf \times int(domf) \longrightarrow (-\infty, +\infty]$ defined by*

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \tag{5}$$

for all $x \in domf$ and $y \in int(domf)$ is called the Bregman distance with respect to f (see, for more details [27, 31]). It is well known that Bregman distance satisfies the following properties for any $x, w \in domf$ and $y, z \in int(domf)$:

(1) three point identity

$$D_f(z, x) := D_f(z, y) + D_f(y, x) + \langle \nabla f(y) - \nabla f(x), z - y\rangle \tag{6}$$

(2) four point identity

$$\begin{aligned} D_f(x, y) + D_f(w, z) - D_f(x, z) - D_f(w, y) \\ = \langle \nabla f(z) - \nabla f(y), x - w\rangle \end{aligned} \tag{7}$$

Definition 3. *A Gâteaux differentiable function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined on a reflexive real Banach space E is said to be strongly convex if there exists a constant $\beta > 0$ such that*

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \beta||x - y||^2,\ \forall\, x, y \in domf,$$

equivalently

$$f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle + \frac{\beta}{2}||x - y||^2,\ \forall\, x, y \in domf.$$

If E is a smooth and strictly convex Banach space, then $f(x) = \frac{1}{2}||x||^2$ is a strongly coercive, bounded, uniformly Fréchet differentiable and strongly convex function with strong *convexity constant* $\beta \in (0,1]$ *and Fenchel conjugate* $f^*(x^*) = \frac{1}{2}||x^*||^2$. *It can be easily shown that if* f *is a strongly convex function with constant* $\beta > 0$, *then, for all* $x \in domf$, *and* $y \in int(domf)$, *(see, [32] for more details),*

$$D_f(x,y) \geq \frac{\beta}{2}||x-y||^2. \tag{8}$$

Definition 4. *Let* B *and* S *be the closed unit ball and the unit sphere of a Banach space* E *defined by* $B_r = \{w \in E : ||w|| \leq r\}$ *for all* $r > 0$ *and* $S_E = \{x \in E : ||x|| = 1\}$ *respectively. Then, the function* $f : E \longrightarrow \mathbb{R}$ *is said to be uniformly convex on bounded subsets of* E *(see, for example [33] and reference therein) if* $\rho_r : [0,+\infty) \longrightarrow [0,+\infty)$ *defined by*

$$\rho_r(t) \quad = \inf_{x,y\in B_r, ||x-y||=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$

which satisfies $\rho_r(t) > 0$ *for all* $r, t > 0$. *The function* ρ_r *is called the gauge of uniform convexity of* f.

Definition 5. *Let* $T : C \to C$ *be a mapping.*

(i) A point $x \in C$ *is called a fixed point of* T *if* $Tx = x$, *where* $F(T) := \{x \in C : Tx = x\}$ *is the set of fixed point of* T.

(ii) A point $x \in C$ *is said to be asymptotic fixed point of* T, *if there exists a sequence* $\{x_n\} \subseteq C$ *such that* $x_n \rightharpoonup x$ *and* $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. *We denote the set of all asymptotic fixed point of* T *by* $\hat{F}(T)$.

A map $T : C \to C$ is called Bregman quasi nonexpansive if $F(T) \neq \emptyset$ and $D_f(p,Tx) \leq D_f(p,x)$ for all $x \in C$ and $p \in F(T)$. T is said to be Bregman quasi strictly pseudocontractive [34] if there exists a constant $\lambda \in [0,1)$ and $F(T) \neq \emptyset$ such that $D_f(p,Tx) \leq D_f(p,x) + \lambda D_f(x,Tx)$ for all $x \in C$ and $p \in F(T)$.

Definition 6. *Let* E *be a reflexive Banach space,* C *a nonempty closed and convex subset of* E, *let* η *be a real number with* $\eta \in (-\infty,1)$. *Then the mapping* $T : C \to E$ *with* $F(T) \neq \emptyset$ *is called* $(\eta,0)$*-Bregman demigeneralized, if for any* $x \in C$ *and* $q \in F(T)$,

$$\langle x-q, \nabla f(x) - \nabla f(Tx)\rangle \geq (1-\eta)D_f(x,Tx), \tag{9}$$

where $F(T)$ *is the set of fixed points of* T.

The modulus of total convexity at $x \in int(domf)$ is the function $v_f(x,.) : [0,+\infty) \longrightarrow [0,+\infty)$ defined by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in domf, ||y-x|| = t\}.$$

The function f is called totally convex at $x \in int(domf)$ if $v_f(x,t)$ is positive for any $t > 0$. This concept was first introduced by [35].

Definition 7. *Let $C \subseteq int(domf)$ be a nonempty, closed and convex subset of a real Banach space E, where $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and Gâteaux differentiable function. The Bregman projection with respect to f of $x \in int(domf)$ onto C is defined as the unique vector $Proj_C^f(x) \in C$, which satisfies*

$$D_f(Proj_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Definition 8. *Let $F : C \to E^*$ be a mapping. Then F is said to be*

(i) monotone if the following inequality hold

$$\langle Fx - Fy, x - y\rangle \geq 0, \quad \forall\, x, y \in C.$$

(ii) pseudomonotone if

$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0, \quad \forall\, x, y \in C.$$

(iii) Lipschitz continuous if there exists a constant $L > 0$ such that

$$||Fx - Fy|| \leq L||x - y||, \quad \forall\, x, y \in C.$$

(iv) weakly sequentially continuous if for any $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ implies $Fx_n \rightharpoonup Fx$.

Definition 9. *A map $B : E \longrightarrow 2^{E^*}$ is called Bregman inverse strongly monotone on C, if $C \cap (int(domf)) \neq \emptyset$, and for any $x, y \in C \cap (int(domf)) \neq \emptyset$, we have*

$$\langle Bx - By, \nabla f^*(\nabla f(x) - Bx) - \nabla f^*(\nabla f(y) - By)\rangle \geq 0. \tag{10}$$

Let A be a maximal monotone mapping with $A^{-1}(0) \neq \emptyset$ and $f : E \to (-\infty, +\infty)$ be uniformly Fréchet differentiable and bounded on bounded subsets of E, then the *resolvent of A with respect to f* and $\lambda > 0$ defined by

$$Res_A^f(x) = (\nabla f + \lambda A)^{-1} \circ \nabla f(x),$$

is single-valued, Bregman quasi-nonexpansive mapping from E onto $D(A)$ and $F(Res_A^f) = A^{-1}(0)$ (for more details, see [29]).
Let $B : E \longrightarrow 2^{E^*}$ be a mapping, then the map defined by

$$B_\lambda^f := \nabla f^* \circ (\nabla f - \lambda B) : E \longrightarrow E \tag{11}$$

is called an antiresolvent associated with B for any $\lambda > 0$.

Definition 10. *Let E be a real reflexive Banach space, $f : E \longrightarrow (-\infty, +\infty]$ be a uniformly Fréchet differentiable function and bounded on bounded subsets of E and A be a maximal monotone mapping. Then, for any $\lambda > 0$, the resolvent of A defined by*

$$Res_A^f(x) = (\nabla f + \lambda A)^{-1} \circ \nabla f(x), \tag{12}$$

is a single valued Bregman quasi nonexpansive mapping from E onto $dom(A)$ and $F(Res_A^f) = A^{-1}0$. We denote by $A_\lambda = (\frac{1}{\lambda})(\nabla f - \nabla f(Res_A^f))$ the Yosida approximation of A for any $\lambda > 0$. We have from [36] that $A_\lambda(x) \in A(Res_A^f(x))$, for all $x \in E$, $\lambda > 0$, (see for example [29]).

Lemma 1. *[37] Let $G : E \longrightarrow 2^{E^*}$ be a maximal monotone mapping and $B : E \longrightarrow E^*$ be Bregman inverse strongly monotone mapping such that $(G + B)^{-1}(0) \neq \emptyset$. Also, let $f : E \longrightarrow \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of E. Then,*

(i) $(G + B)^{-1}(0) = F(Res_{\lambda G}^f \circ B_\lambda^f)$

(ii) $Res_{\lambda G}^f \circ B_\lambda^f$ *is a Bregman strongly nonexpansive mapping such that*

$$F(Res_{\lambda G}^f \circ B_\lambda^f) = \hat{F}(Res_{\lambda G}^f \circ B_\lambda^f) \tag{13}$$

(iii)

$$D_f(u, Res_{\lambda G}^f \circ B_\lambda^f(x)) + D_f(Res_{\lambda G}^f \circ B_\lambda^f(x), x) \leq D_f(u, x), \tag{14}$$

for all $u \in (G + B)^{-1}(0)$, $x \in E$ and $\lambda > 0$.

Lemma 2. *[36] Let $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and $A : E \longrightarrow 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0) \neq \emptyset$. Then,*

$$D_f(p, Res_{rA}^f(x)) + D_f(Res_{rA}^f(x), x) \leq D_f(p, x), \tag{15}$$

for all $r > 0$, $p \in A^{-1}(0)$ and $x \in E$.

Lemma 3. *[38, 39] Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x \in E$. Then the Bregman projection $Proj_C^f : E \longrightarrow C$ satisfies the following properties:*

(i) $z = Proj_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \ \forall\, y \in C$,

(ii) $D_f(y, proj_C^f(x)) + D_f(proj_C^f(x), x) \leq D_f(y, x), \ \forall\, y \in C$ *and* $x \in E$.

Let $f : E \longrightarrow (-\infty, +\infty]$ be convex, Legendre function Gâteaux differentiable function. Following [31, 40] we make use of the function $V_f : E \times E^* \longrightarrow [0, +\infty)$ defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \ \forall \, x \in E \text{ and } x^* \in E^*. \tag{16}$$

Then, the following assertions hold:

(i) V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \ \forall \, x \in E \text{ and } x^* \in E^*. \tag{17}$$

Thus, from (16) *it is obvious that $D_f(x, y) = V_f(x, \nabla f(y))$ and V_f is convex in the second variable. Therefore for $\lambda \in (0, 1)$ and $x, y \in E$, we have*

$$D_f(z, \nabla f^*(\lambda \nabla f(x) + (1 - \lambda)\nabla f(y))) \ \ \leq \ \ \lambda D_f(z, x) + (1 - \lambda)D_f(z, y) \tag{18}$$

Moreover by subdifferential inequality (see, for example [41] and reference therein), we have

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \ \ \leq \ \ V_f(x, x^* + y^*), \ \forall \, x \in E \text{ and } x^*, y^* \in E^*. \tag{19}$$

Lemma 4. *[42] If $f : E \longrightarrow (-\infty, +\infty]$ is a proper lower semi-continuous and convex function, then $f^* : E^* \longrightarrow (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function. Thus, V_f is convex in the second variable. Hence, for all $u \in E$, we have*

$$D_f(u, \nabla f^*(\sum_{i=1}^{M} \tau_i \nabla f(x_i))) \ \ \leq \ \ \sum_{i=1}^{M} \tau_i D_f(u, x_i), \tag{20}$$

where $\{x_i\} \subset E$ and $\{\tau_i\}_{i=1}^{M} \subset (0, 1)$ satisfying $\sum_{i=1}^{M} \tau_i = 1$.

Lemma 5. *[43] Let E be a Banach space, $r > 0$ be a constant and $f : E \longrightarrow \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then*

$$f(\sum_{k=0}^{n} \alpha_k x_k) \leq \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(||x_i - x_j||), \ \forall \, i, j \in \{1, 2, ...n\} \tag{21}$$

$x_k \in B_r, \alpha_k \in (0, 1)$ and $k = 0, 1, 2, ...n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 6. *[35] If the dom$f$ contains at least two points, then the function $f : E \longrightarrow (-\infty, +\infty]$ is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 7. *[44] Let $f : E \longrightarrow (-\infty, +\infty]$ be a uniformly Fréchet differentiable function and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to strong topology of E^*.*
Recalll that the function f is called sequentially consistent [38] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n \to \infty} D_f(x_n, y_n) = 0 \Longrightarrow \lim_{n \to \infty} ||x_n - y_n|| = 0. \tag{22}$$

Lemma 8. *[36] Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function. If $x \in E$ and the sequence $\{D_f(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 9. *[45] Consider the variational inequality problem $VI(C, F)$. Suppose the mapping $h : [0,1] \longrightarrow E^*$ defined by $h(t) = F(tx + (1-t)y)$ and $t \in [0,1]$ is continuous for all $x, y \in C$ (i.e, h is hemicontinuous), then $M(C,F) \subset VI(C,F)$. Thus, if F is pseudomonotone, then $VI(C,F)$ is closed, convex and $VI(C,F) = M(C,F)$*

Lemma 10. *[46] Let $\{b_n\}$ be a sequence of nonnegative real numbers such that*

$$b_{n+1} \leq (1-\psi_n)b_n + \psi_n\sigma_n, \ \ n \geq 1,$$

where $\{\psi_n\} \subset (0,1)$ with $\lim\limits_{n\to\infty} \psi_n = 0$, $\sum_{n=1}^{\infty} \psi_n = \infty$ and $\{\sigma_n\}$ is a sequence of real numbers. If $\limsup\limits_{k\to\infty} \sigma_{n_k} \leq 0$ for every subsequence $\{b_{n_k}\}$ of $\{b_n\}$ satisfying the condition

$$\liminf_{k\to\infty}(b_{n_k+1} - b_{n_k}) \geq 0,$$

then, $b_n \longrightarrow 0$ as $n \longrightarrow \infty$.

The following lemma plays an important role in the proof of our result.

Lemma 11. *[47] Let E be a reflexive Banach space and C a nonempty closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive, Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E. Let η be a real number with $\eta \in (-\infty, 0)$ and T an $(\eta, 0)$-Bregman demigeneralized mapping of C into E. Then $F(T)$ is closed and convex.*

3. Main Results

In order to obtain strong convergence of our algorithm, we make the following assumptions:

(A1) Let E be "a smooth, strictly convex and real reflexive Banach space and C be nonempty, closed and convex subset of E. Suppose that $\{B_i\}_{i=1}^N$ is a finite family of Bregman inverse strongly monotone mappings of C into E and $\{B_i^f, \tau_n\}_{i=1}^N$ the family of antiresolvent mappings of $\{B_i\}_{i=1}^N$. Let $A : E \longrightarrow 2^{E^*}$ and $G : E \longrightarrow 2^{E^*}$ be maximal monotone mappings on E and $Q_n = Res_{\tau_n G}^f = (\nabla f + \tau_n G)^{-1}\nabla f$ and $J_r = Res_{rA}^f = (\nabla f + rA)^{-1}\nabla f$ be the resolvents of G and A for $\tau_n > 0$ and $r > 0$ respectively.

(A2) The operator $F : E \longrightarrow E^*$ is pseudomonotone, L - Lipschitz continuous and weakly sequentially continuous on E.

(A3) For each $j \in \{1, 2, ..., M\}$, $\{T_j\}$ be a finite family of Bregman $(\nu_j, 0)$ - demigeneralized mapping of E into itself and $\nu_j \in (-\infty, 0)$ such that $F(T_j) \neq \emptyset$.
Assume $\Omega = A^{-1}(0) \bigcap (\bigcap_{j=1}^M F(T_j)) \bigcap (\bigcap_{i=1}^N (B_i + G)^{-1}0) \neq \emptyset$.

(A4) The solution set $\Gamma = VI(C,F)\bigcap \Omega \neq \emptyset$.

(A5) The function $f : E \longrightarrow \mathbb{R}$ satisfies the following:

(1) f is proper, convex and lower semi-continuous;

(2) f is uniformly Fréchet differentiable and totally convex on bounded subsets of E;

(3) f is strongly convex on E with strong convexity constant $\beta > 0$;

(4) f is a strongly coercive and Legendre function which is bounded on bounded subsets of E.

(A6) Assume that the control sequences satisfy:

(i) $\{\alpha_n\} \subset (0,1)$ satisfies $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) Choose a positive sequence $\{\psi_n\}$ such that $\psi_n \in (0, \frac{\beta}{2})$ for all $n \geq 0$, β satisfy condition (8), and $\lim\limits_{n\to\infty} \frac{\psi_n}{\alpha_n} = 0$.

Algorithm 1. *Initialization: Take* $\lambda_1 > 0, \mu \in (0,\beta)$, $\theta \in (0,1)$. *Select initial data* $u, x_0, x_1 \in E$ *and set* $n = 1$.
Step 1 : Given x_{n-1}, x_n *and* θ_n *for each* $n \geq 1$, *choose* θ_n *such that* $\theta_n \in [0, \bar{\theta}_n]$ *with* $\bar{\theta}_n$ *defined by*

$$\bar{\theta}_n = \begin{cases} \min\{\frac{\psi_n}{||\nabla f(x_n) - \nabla f(x_{n-1})||}, \theta\}, & if \quad x_n \neq x_{n-1}, \\ \theta, & otherwise \end{cases} \tag{23}$$

Step 2: Compute

$$\begin{cases} u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))), \\ y_n = Proj_C^f(\nabla f^*(\nabla f(u_n) - \lambda_n F(u_n))). \end{cases} \tag{24}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu||u_n - y_n||}{||F(u_n) - F(y_n)||}, \ \ \lambda_n\}, & if \quad F(u_n) \neq F(y_n), \\ \lambda_n, & otherwise. \end{cases} \tag{25}$$

If $y_n = u_n$, *then set* $z_n = u_n$ *for some* $n \geq 1$. *Else go to step 3.*
Step 3: Compute

$$\begin{cases} z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Fy_n - Fu_n)), \\ w_n = \nabla f^*(\sum_{j=1}^{M} \varphi_j((1-\beta_n)\nabla f(z_n) + \beta_n \nabla f(T_j z_n))), \\ t_n = \nabla f^*(\eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(J_{r_n} t_n)), \quad \forall \ \ n \geq 1. \end{cases} \tag{26}$$

Set $n := n + 1$ *and return to Step 1,*
where $\{\beta_n\} \subseteq (0,1)$, $\{\tau_n\}, \{r_n\} \subseteq (0,+\infty)$ $\{\varphi_1, \varphi_2, ..., \varphi_M\}, \{\sigma_1, \sigma_2, ..., \sigma_N\} \subseteq (0,1)$ *and* $a, b \in \mathbb{R}$ *satisfy the following:*

(i) $0 < a \leq \beta_n \leq \min\{1-\nu_1, 1-\nu_2, ..., 1-\nu_m\}$, *let* $\nu := \max\{\nu_j, 1 \leq j \leq M\}$;

(ii) $0 < b \leq r_n$, $\forall\, n \in \mathbb{N}$;

(iii) $\sum_{j=1}^{M} \varphi_j = 1$;

(iv) $0 < r \leq \eta_n, \delta_n, \xi_n, \sigma_{i,n} \leq \mu < 1$ *and* $\eta_n+\delta_n+\xi_n+\sum_{i=1}^{N} \sigma_{i,n} = 1$, $\forall\, i \in \mathbb{N}$ *and* $\forall\, n \in \mathbb{N}$.

In order to prove the strong convergence result of Algorithm 1 we first prove the following lemma which plays an important role in the proof of the main result.

Lemma 12. *Suppose that* $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{\lambda_n\}$ *are sequences generated by Algorithm* 1 *and assumptions* $(A1)$ - $(A6)$ *hold, then*

(i) If $u_n = y_n$ *for some* $n \geq 1$, *then* $u_n \in VI(C, F)$.

(ii) The sequence $\{\lambda_n\}$ *generated by* (25) *is a nonincreasing sequence and* $\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}$.

Proof. (1) Suppose that $u_n = y_n$ *for some* $n \geq 1$. *Then from Algorithm* 1, *we have*

$$u_n = Proj_C^f(\nabla f^*(\nabla f(u_n) - \lambda_n F(u_n))).$$

Thus, $u_n \in C$. *Using the definition of* $\{y_n\}$ *in Algorithm* 1 *and the property of Bregman projection* $Proj_C^f$ *onto* C *in Lemma* 3, *we have*

$$\langle \nabla f(u_n) - \lambda_n F(u_n) - \nabla f(u_n), u_n - y\rangle \geq 0, \quad \forall\, y \in C.$$

Thus,

$$\langle -\lambda_n F(u_n), u_n - y\rangle = \lambda_n \langle F(u_n), y - u_n\rangle \geq 0, \quad \forall\, y \in C.$$

Since $\lambda_n \geq 0$, *we obtain that* $\langle F(u_n), y - u_n\rangle \geq 0$. *Hence,* $u_n \in VI(C, F)$.
(2) *It follows from (*25*) that* $\lambda_{n+1} \leq \lambda_n$, *for all* $n \in \mathbb{N}$. *Furthermore, since* F *is a Lipschitz continuous mapping with positive constant* L, *in a case where* $F(u_n) - F(y_n) \neq 0$, *and the sequence* $\{\lambda_n\}$ *is nonincreasing, we obtain*

$$\frac{\mu||u_n - y_n||}{||F(u_n) - F(y_n)||} \quad \geq \quad \frac{\mu||u_n - y_n||}{L||u_n - y_n||} = \frac{\mu}{L}.$$

Thus $\{\lambda_n\}$ *is bounded below by* $\min\{\frac{\mu}{L}, \lambda_1\}$, *we conclude that*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}.$$

Remark 2. *We have from* (23) *of Algorithm* 1 *that* $\theta_n||x_n - x_{n-1}|| \leq \psi_n$ *for each* $n \geq 1$, *which together with* $\lim_{n\to\infty} \frac{\psi_n}{\alpha_n} = 0$ *implies*

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \leq \lim_{n\to\infty} \frac{\psi_n}{\alpha_n} = 0. \tag{27}$$

Lemma 13. *Suppose that assumptions (A1)-(A6) holds, let* $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by Algorithm* 1. *Let* $\{u_{n_k}\}$ *be a subsequence of* $\{u_n\}$ *which converges weakly to* $\bar{x} \in E$ *and* $\lim_{k\to\infty} ||u_{n_k} - y_{n_k}|| = 0$, *then* $\bar{x} \in VI(C, F)$.

Proof. Using the definition of $y_{n_k} = Proj_C^f(\nabla f^*(\nabla f(u_{n_k}) - \lambda_{n_k} F(u_{n_k})))$ *and Lemma* 3(*i*), *we have that for all* $z \in C$,

$$\langle \nabla f u_{n_k} - \lambda_{n_k} F(u_{n_k}) - \nabla f y_{n_k}, z - y_{n_k}\rangle \leq 0.$$

This implies that

$$\langle \nabla f u_{n_k} - \nabla f y_{n_k}, z - y_{n_k}\rangle \leq \lambda_{n_k} \langle F(u_{n_k}), z - y_{n_k}\rangle.$$

Then for all $z \in C$, *we have*

$$\frac{1}{\lambda_{n_k}} \langle \nabla f u_{n_k} - \nabla f y_{n_k}, z - y_{n_k}\rangle + \langle F(u_{n_k}), y_{n_k} - u_{n_k}\rangle \leq \langle F(u_{n_k}), z - u_{n_k}\rangle \tag{28}$$

Fixing $z \in C$ *and letting* $k \longrightarrow +\infty$ *in (*28*) also remembering that* $||y_{n_k} - u_{n_k}|| \longrightarrow 0$ *as* $k \longrightarrow \infty$ *together with the fact that* $\liminf_{k\to\infty} \lambda_{n_k} > 0$, *we have*

$$\liminf_{k\to\infty} \langle F(u_{n_k}), z - u_{n_k}\rangle \geq 0.$$

Let $\{\varepsilon_k\}$ *be a decreasing nonnegative sequence such that* $\lim_{n\to\infty} \varepsilon_k = 0$. *For each* ε_k, *we denote the smallest positive integer* N_k *such that for all* $k \geq N_k$,

$$\langle F(u_{n_k}), z - u_{n_k}\rangle + \varepsilon_k \geq 0. \tag{29}$$

Furthermore, as $\{\varepsilon_k\}$ *is decreasing,* $\{N_k\}$ *is increasing. Thus, if there exists a subsequence* $\{u_{n_{k_i}}\} \subset \{u_{n_k}\}$, *such that for each* $i \geq 1$, $F(u_{n_{k_i}}) \neq 0$, *and setting*

$$s_{n_{k_i}} = \frac{J^{-1} F(u_{n_{k_i}})}{||F(u_{n_{k_i}})||^2},$$

we have $\langle F(u_{n_{k_i}}), s_{n_{k_i}}\rangle = 1$ *for each* $i \geq 1$. *It follows from (*29*) that for each* $i \geq 1$

$$\langle F(u_{n_{k_i}}), z + \varepsilon_k s_{n_{k_i}} - u_{n_{k_i}}\rangle \geq 0. \tag{30}$$

Thus, since F *is pseudomonotone, we obtain from (*30*) that*

$$\langle F(z + \varepsilon_k s_{n_{k_i}}), z + \varepsilon_k s_{n_{k_i}} - u_{n_{k_i}}\rangle \geq 0. \tag{31}$$

Since $\{u_{n_k}\}$ *converges weakly to* $\bar{x} \in C$, *and* F *is weakly sequentially continuous, we have that* $F(u_{n_k})$ *converges weakly to* $F(\bar{x})$. *If* $F(\bar{x}) = 0$, *then* $\bar{x} \in VI(C, F)$. *Suppose that* $F(\bar{x}) \neq 0$. *Then, by sequentially weakly lower semicontinuity of the norm, we have the following*

$$0 < ||F(\bar{x})|| \leq \liminf_{k\to\infty} ||F(u_{n_k})||.$$

Since $\{u_{n_{k_i}}\} \subset \{u_{n_k}\}$ *and* $\varepsilon_k \longrightarrow 0$ *as* $k \longrightarrow \infty$, *we obtain*

$$0 \leq \limsup_{k\to\infty} ||\varepsilon_k s_{n_k}|| = \limsup_{k\to\infty} (\frac{\varepsilon_k}{||F(u_{n_k})||}) \leq \frac{\limsup\limits_{k\to\infty} \varepsilon_k}{\liminf\limits_{k\to\infty} ||F(u_{n_k})||} \leq \frac{0}{||F(\bar{x})||} = 0$$

Taking the limit as $k \longrightarrow \infty$ *in* (31) *we obtain*

$$\liminf_{k\to\infty} \langle F(z), z - u_{n_k}\rangle \geq 0.$$

Therefore,

$$\langle F(z), z - \bar{x}\rangle = \lim_{k\to\infty} \langle F(z), z - u_{n_k}\rangle = \liminf_{k\to\infty} \langle F(z), z - u_{n_k}\rangle \geq 0, \quad \forall \;\; z \in C.$$

Hence,

$$\langle F(z), z - \bar{x}\rangle \geq 0.$$

Thus, it follows from Lemma 9 *that* $\bar{x} \in VI(C, F)$.

Lemma 14. *Suppose that assumptions* (*A*1) − (*A*6) *hold, and the sequences* $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{t_n\}$ *and* $\{x_n\}$, *be generated by Algorithm* 1. *Then* $\{x_n\}$ *is bounded.*

Proof. First, we show that

$$D_f(x^*, z_n) \quad \leq \quad D_f(x^*, u_n) - (1 - \frac{\mu\lambda_n}{\beta\lambda_{n+1}})(D_f(y_n, u_n) + D_f(z_n, y_n)), \;\; \forall \;\; n \geq 0.$$

Let $x^* \in \Gamma$, *then using the definition of Bregman distance* (5),*we have*

$$\begin{aligned}
D_f(x^*, z_n) &= D_f(x^*, \nabla f^*(\nabla f(y_n) - \lambda_n(Fy_n - Fu_n)) \\
&= f(x^*) - \langle \nabla f(y_n) - \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle - f(z_n) \\
&= f(x^*) - \langle \nabla f(y_n), x^* - z_n\rangle + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle - f(z_n) \\
&= f(x^*) + \langle \nabla f(y_n), z_n - x^*\rangle + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle - f(z_n) \\
&= f(x^*) - \langle \nabla f(y_n), x^* - y_n\rangle - f(y_n) + \langle \nabla f(y_n), x^* - y_n\rangle + f(y_n) \\
&\quad + \langle \nabla f(y_n), z_n - x^*\rangle + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle - f(z_n) \\
&= D_f(x^*, y_n) - f(z_n) + f(y_n) + \langle \nabla f(y_n), x^* - y_n\rangle + \langle \nabla f(y_n), z_n - x^*\rangle \\
&\quad + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle \\
&= D_f(x^*, y_n) - f(z_n) + f(y_n) + \langle \nabla f(y_n), z_n - x^* + x^* - y_n\rangle \\
&\quad + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle \\
&= D_f(x^*, y_n) - f(z_n) + f(y_n) + \langle \nabla f(y_n), z_n - y_n\rangle \\
&\quad + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle \\
&= D_f(x^*, y_n) - D_f(z_n, y_n) + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle. \qquad (32)
\end{aligned}$$

We have from equation (7) *that*

$$D_f(x^*, y_n) - D_f(z_n, y_n) \quad = \quad D_f(x^*, u_n) - D_f(z_n, u_n)$$

$$+\langle \nabla f(u_n) - \nabla f(y_n), x^* - z_n\rangle. \quad (33)$$

Substituting (33) *into* (32), *we obtain*

$$\begin{aligned} D_f(x^*, z_n) &\le D_f(x^*, u_n) - D_f(z_n, u_n) + \langle \nabla f(u_n) - \nabla f(y_n), x^* - z_n\rangle \\ &\quad +\langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle. \end{aligned} \quad (34)$$

Observe from (6) *that*

$$D_f(z_n, u_n) = D_f(z_n, y_n) + D_f(y_n, u_n) + \langle \nabla f(y_n) - \nabla f(u_n), z_n - y_n\rangle. \quad (35)$$

Now , combining (34) *and* (35), *we have*

$$\begin{aligned} D_f(x^*, z_n) &\le D_f(x^*, u_n) - [D_f(z_n, y_n) + D_f(y_n, u_n) + \langle \nabla f(y_n) - \nabla f(u_n), z_n - y_n\rangle] \\ &\quad +\langle \nabla f(u_n) - \nabla f(y_n), x^* - z_n\rangle + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle \\ &= D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \langle \nabla f(u_n) - \nabla f(y_n), z_n - y_n\rangle \\ &\quad +\langle \nabla f(u_n) - \nabla f(y_n), x^* - z_n\rangle + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle \\ &= D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad +\langle \nabla f(u_n) - \nabla f(y_n), z_n - y_n + x^* - z_n\rangle + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle \\ &= D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad +\langle \nabla f(u_n) - \nabla f(y_n), x^* - y_n\rangle + \langle \lambda_n(Fy_n - Fu_n), x^* - z_n\rangle \\ &= D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad +\langle \nabla f(u_n) - \nabla f(y_n), x^* - y_n\rangle - \langle \lambda_n(Fy_n - Fu_n), z_n - y_n + y_n - x^*\rangle \\ &= D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \langle \nabla f(u_n) - \nabla f(y_n), x^* - y_n\rangle \\ &\quad -\langle \lambda_n(Fy_n - Fu_n), z_n - y_n\rangle - \langle \lambda_n(Fy_n - Fu_n), y_n - x^*\rangle \\ &= D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) - \langle \nabla f(y_n) - \nabla f(u_n), y_n - x^*\rangle \\ &\quad -\langle \lambda_n(Fy_n - Fu_n), z_n - y_n\rangle - \langle \lambda_n(Fy_n - Fu_n), y_n - x^*\rangle \\ &= D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) - \langle \lambda_n(Fy_n - Fu_n), z_n - y_n\rangle \\ &\quad -\langle \lambda_n(Fy_n - Fu_n) - (\nabla f(y_n) - \nabla f(u_n)), y_n - x^*\rangle. \end{aligned} \quad (36)$$

Using the definition of $\{y_n\}$ *in Algorithm* 1 *and Lemma* 3(i), *we have*

$$\langle \nabla f(u_n) - \lambda_n(Fu_n) - \nabla f(y_n), x^* - y_n\rangle \le 0. \quad (37)$$

Since $x^* \in VI(C, F)$ *and* $y_n \in C$, *we have* $\langle Fx^*, y_n - x^*\rangle \ge 0$. *Also, considering the fact that* F *is pseudomonotone implies that*

$$\langle Fy_n, y_n - x^*\rangle \ge 0. \quad (38)$$

Thus, combining (37) *and* (38), *we obtain*

$$\langle \lambda_n(Fy_n - Fu_n) - (\nabla f(y_n) - \nabla f(u_n)), y_n - x^*\rangle \ge 0. \quad (39)$$

By applying (39) *in* (36), *we get*

$$\begin{aligned} D_f(x^*, z_n) &\leq D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad -\langle \lambda_n(Fy_n - Fu_n), z_n - y_n\rangle. \end{aligned} \tag{40}$$

Using Cauchy Schwartz inequality, (25) *and* (8), *we have*

$$\begin{aligned} D_f(x^*, z_n) &\leq D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad +\langle \lambda_n(Fy_n - Fu_n), y_n - z_n\rangle \\ &\leq D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad +\frac{\lambda_n}{\lambda_{n+1}}\lambda_{n+1}||Fy_n - Fu_n||||y_n - z_n|| \\ &\leq D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad +\frac{\lambda_n \mu}{\lambda_{n+1}}||y_n - u_n||||y_n - z_n|| \end{aligned}$$

$$\begin{aligned} D_f(x^*, z_n) &\leq D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad +\frac{\lambda_n}{\lambda_{n+1}} \times \frac{\mu}{2}(||y_n - u_n||^2 + ||y_n - z_n||^2) \\ &\leq D_f(x^*, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) \\ &\quad +\frac{\lambda_n}{\lambda_{n+1}} \times \frac{\mu}{2} \times \frac{2}{\beta}(D_f(y_n, u_n) + D_f(y_n, z_n)) \\ &= D_f(x^*, u_n) - (1 - \frac{\lambda_n \mu}{\lambda_{n+1}\beta})(D_f(z_n, y_n) + D_f(y_n, u_n)). \end{aligned} \tag{41}$$

Applying Lemma 12 *(2), since* $\lim_{n\to\infty} \lambda_n$ *exists and that* $\mu \in (0, \beta)$, *then* $\lim_{n\to\infty}(1 - \frac{\lambda_n \mu}{\lambda_{n+1}\beta}) = 1 - \frac{\mu}{\beta} > 0$. *This implies that, there exists a positive integer* $N_0 > 0$ *such that for all* $n > N_0$, $(1 - \frac{\lambda_n \mu}{\lambda_{n+1}\beta}) > 0$. *Hence, from* (41), *we have*

$$D_f(x^*, z_n) \leq D_f(x^*, u_n) \tag{42}$$

From the definition of Bregman distance (5) *and* $\{u_n\}$ *in Algorithm* 1 *,we have*

$$\begin{aligned} D_f(x^*, u_n) &= D_f(x^*, \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})))) \\ &= f(x^*) - \langle \nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})), x^* - u_n\rangle - f(u_n) \\ &= D_f(x^*, x_n) + \langle \nabla f(x_n), x^* - x_n\rangle + f(x_n) - \langle \nabla f(x_n), x^* - u_n\rangle \\ &\quad -\langle \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})), x^* - u_n\rangle - f(u_n) \\ &= D_f(x^*, x_n) + f(x_n) + \langle \nabla f(x_n), (x^* - x_n - (x^* - u_n))\rangle \\ &\quad -\langle \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})), x^* - u_n\rangle - f(u_n) \\ &= D_f(x^*, x_n) + f(x_n) + \langle \nabla f(x_n), u_n - x_n\rangle \end{aligned}$$

$$\begin{aligned} & -\langle\theta_n(\nabla f(x_n)-\nabla f(x_{n-1})),x^*-u_n\rangle - f(u_n) \\ = \;& D_f(x^*,x_n)-[f(u_n)-f(x_n)-\langle\nabla f(x_n),u_n-x_n\rangle] \\ & -\langle\theta_n(\nabla f(x_n)-\nabla f(x_{n-1})),x^*-u_n\rangle \\ = \;& D_f(x^*,x_n)-D_f(u_n,x_n)-\langle\theta_n(\nabla f(x_n)-\nabla f(x_{n-1})),x^*-u_n\rangle \end{aligned} \tag{43}$$

Now, applying Cauchy Schwartz inequality and (8), *we obtain*

$$\begin{aligned} & -\langle\theta_n(\nabla f(x_n)-\nabla f(x_{n-1})),x^*-u_n\rangle \leq \theta_n||\nabla f(x_n)-\nabla f(x_{n-1})||||x^*-u_n|| \\ = \;& \theta_n||\nabla f(x_n)-\nabla f(x_{n-1})||||x^*-u_n|| \\ \leq \;& \theta_n||\nabla f(x_n)-\nabla f(x_{n-1})||[\frac{1}{2}(||x^*-u_n||^2+1^2)] \\ = \;& \frac{\theta_n}{2}||\nabla f(x_n)-\nabla f(x_{n-1})||[||x^*-x_n||^2+||x_n-u_n||^2+1] \\ \leq \;& \frac{\theta_n}{2}||\nabla f(x_n)-\nabla f(x_{n-1})||[2||x^*-x_n||^2+2||x_n-u_n||^2+1] \\ \leq \;& \frac{\theta_n}{2}||\nabla f(x_n)-\nabla f(x_{n-1})||[\frac{4}{\beta}D_f(x^*,x_n)+\frac{4}{\beta}D_f(x_n,u_n)+1] \\ \leq \;& \frac{2\theta_n}{\beta}||\nabla f(x_n)-\nabla f(x_{n-1})||D_f(x^*,x_n) \\ & +\frac{2\theta_n}{\beta}||\nabla f(x_n)-\nabla f(x_{n-1})||D_f(x_n,u_n)+\frac{\theta_n}{2}||\nabla f(x_n)-\nabla f(x_{n-1})||. \end{aligned} \tag{44}$$

Recall from the definition of $\bar{\theta_n}$ *in Algorithm* 1 *that*

$$\theta_n||\nabla f(x_n)-\nabla f(x_{n-1})|| \;\;\leq\;\; \psi_n \tag{45}$$

Now, applying (45) *we have from* (44) *the following*

$$\begin{aligned} & -\langle\theta_n(\nabla f(x_n)-\nabla f(x_{n-1})),x^*-u_n\rangle \\ & \leq \frac{2\psi_n}{\beta}D_f(x^*,x_n)+\frac{2\psi_n}{\beta}D_f(x_n,u_n)+\frac{\psi_n}{2} \end{aligned} \tag{46}$$

Substitute (46) *into* (43), *we get*

$$\begin{aligned} D_f(x^*,u_n) \;\leq\;& D_f(x^*,x_n)-D_f(x_n,u_n)+\frac{2\psi_n}{\beta}D_f(x^*,x_n) \\ & +\frac{2\psi_n}{\beta}D_f(x_n,u_n)+\frac{\psi_n}{2} \\ =\;& (1+\frac{2\psi_n}{\beta})D_f(x^*,x_n)-(1-\frac{2\psi_n}{\beta})D_f(x_n,u_n)+\frac{\psi_n}{2} \\ \leq\;& (1+\frac{2\psi_n}{\beta})D_f(x^*,x_n)+\frac{\psi_n}{2} \end{aligned} \tag{47}$$

Let $x^*\in\Gamma$ *and* T_j *Bregman* $(\nu_j,0)$ *- demigeneralized for all* $1\leq j\leq M$ *and applying Lemma* 4 , *we obtain*

$$D_f(x^*,w_n) \;\;=\;\; D_f(x^*,\nabla f^*\sum_{j=1}^{M}\varphi_j((1-\beta_n)\nabla f(z_n)+\beta_n\nabla f(T_jz_n)))$$

$$
\begin{aligned}
&= D_f(x^*, \nabla f^* \sum_{j=1}^{M} \varphi_j \nabla f \nabla f^*((1-\beta_n)\nabla f(z_n) + \beta_n \nabla f(T_j z_n))) \\
&\leq \sum_{j=1}^{M} \varphi_j D_f(x^*, \nabla f^*((1-\beta_n)\nabla f(z_n) + \beta_n \nabla f(T_j z_n))) \\
&= \sum_{j=1}^{M} \varphi_j [V_f(x^*, (1-\beta_n)\nabla f(z_n) + \beta_n \nabla f(T_j z_n))] \\
&= \sum_{j=1}^{M} \varphi_j [f(x^*) - \langle x^*, (1-\beta_n)\nabla f(z_n) + \beta_n \nabla f(T_j z_n)\rangle \\
&\quad + f^*((1-\beta_n)\nabla f(z_n) + \beta_n \nabla f(T_j z_n))] \\
&= \sum_{j=1}^{M} \varphi_j [f(x^*) - \langle x^*, (1-\beta_n)\nabla f(z_n)\rangle - \langle x^*, \beta_n \nabla f(T_j z_n)\rangle \\
&\quad + f^*((1-\beta_n)\nabla f(z_n)) + f^*(\beta_n \nabla f(T_j z_n))] \\
&= \sum_{j=1}^{M} \varphi_j [f(x^*) - (1-\beta_n)\langle x^*, \nabla f(z_n)\rangle - \beta_n \langle x^*, \nabla f(T_j z_n)\rangle \\
&\quad + (1-\beta_n) f^*(\nabla f(z_n)) + \beta_n f^*(\nabla f(T_j z_n))] \\
&= \sum_{j=1}^{M} \varphi_j [\beta_n f(x^*) + (1-\beta_n) f(x^*) - (1-\beta_n)\langle x^*, \nabla f(z_n)\rangle \\
&\quad - \beta_n \langle x^*, \nabla f(T_j z_n)\rangle + (1-\beta_n) f^*(\nabla f(z_n)) + \beta_n f^*(\nabla f(T_j z_n))]
\end{aligned}
$$

$$
\begin{aligned}
D_f(x^*, w_n) &\leq \sum_{j=1}^{M} \varphi_j [(1-\beta_n) D_f(x^*, z_n) + \beta_n D_f(x^*, T_j z_n)] \\
&\leq \sum_{j=1}^{M} \varphi_j [(1-\beta_n) D_f(x^*, z_n) + \beta_n D_f(x^*, z_n)] \\
&= \sum_{j=1}^{M} \varphi_j (D_f(x^*, z_n)) \\
&= D_f(x^*, z_n) \qquad (48)
\end{aligned}
$$

Using Lemma 1 *and the fact that* B_i *is Bregman inverse strongly monotone mapping for all* $1 \leq i \leq N$ *and condition (iv), we have*

$$
D_f(x^*, t_n) = D_f(x^*, \nabla f^*(\eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)))
$$

$$\begin{aligned}
&= V_f(x^*, \eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)) \\
&= f(x^*) - \langle x^*, \eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n) \rangle \\
&\quad + f^*(\eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)) \\
&\leq \eta_n f(x^*) + \delta_n f(x^*) + \xi_n f(x^*) + \sum_{i=1}^{N} \sigma_{i,n} f(x^*) - \eta_n \langle x^*, \nabla f(z_n) \rangle \\
&\quad - \delta_n \langle x^*, \nabla f(u_n) \rangle - \xi_n \langle x^*, \nabla f(w_n) \rangle - \sum_{i=1}^{N} \sigma_{i,n} \langle x^*, \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n) \rangle \\
&\quad + \eta_n f^*(\nabla f(z_n)) + \delta_n f^*(\nabla f(u_n)) + \xi_n f^*(\nabla f(w_n)) \\
&\quad + \sum_{i=1}^{N} \sigma_{i,n} f^*(\nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)) \\
&= \eta_n (f(x^*) - \langle x^*, \nabla f(z_n) \rangle + f^*(\nabla f(z_n))) + \delta_n (f(x^*) \\
&\quad - \langle x^*, \nabla f(u_n) \rangle + f^*(\nabla f(u_n))) \\
&\quad + \xi_n (f(x^*) - \langle x^*, \nabla f(w_n) \rangle + f^*(\nabla f(w_n))) \\
&\quad + \sum_{i=1}^{N} \sigma_{i,n} (f(x^*) - \langle x^*, \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n) \rangle + f^*(\nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n))) \\
&= \gamma_n D_f(x^*, z_n) + \delta_n D_f(x^*, u_n) + \xi_n D_f(x^*, w_n) + \sum_{i=1}^{N} \sigma_{i,n} D_f(x^*, Q_{\tau_n} B^f_{i,\tau_n} w_n) \\
&\leq \gamma_n D_f(x^*, u_n) + \delta_n D_f(x^*, u_n) + \xi_n D_f(x^*, z_n) + \sum_{i=1}^{N} \sigma_{i,n} D_f(x^*, w_n) \\
&\leq \gamma_n D_f(x^*, u_n) + \delta_n D_f(x^*, u_n) + \xi_n D_f(x^*, u_n) + \sum_{i=1}^{N} \sigma_{i,n} D_f(x^*, z_n)
\end{aligned}$$

$$\begin{aligned}
D_f(x^*, t_n) &\leq \gamma_n D_f(x^*, u_n) + \delta_n D_f(x^*, u_n) + \xi_n D_f(x^*, u_n) + \sum_{i=1}^{N} \sigma_{i,n} D_f(x^*, u_n) \\
&= D_f(x^*, u_n). \qquad (49)
\end{aligned}$$

Hence

$$D_f(x^*, t_n) \leq D_f(x^*, u_n). \qquad (50)$$

From the definition of $\{x_{n+1}\}$ *in Algorithm* 1*, we obtain*

$$D_f(x^*, x_{n+1}) = D_f(x^*, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(J_{r_n} t_n)))$$

$$\begin{aligned}
&= V_f(x^*, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(J_{r_n} t_n)) \\
&= \alpha_n V_f(x^*, \nabla f(u)) + (1-\alpha_n) V_f(x^*, J_{r_n} t_n) \\
&= \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, J_{r_n} t_n) \\
&\leq \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, t_n) \\
&\leq \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, w_n) \\
&\leq \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, z_n) \\
&\leq \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, u_n) \qquad (51)
\end{aligned}$$

Substituting (47) *into* (51)*, we have*

$$\begin{aligned}
D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, u) + (1-\alpha_n)[(1+\frac{2\psi_n}{\beta})D_f(x^*, x_n) + \frac{\psi_n}{2}] \\
&\quad -((1-\alpha_n)(1-\frac{2\psi_n}{\beta})D_f(x_n, u_n)) \\
&\leq \alpha_n D_f(x^*, u) + (1-\alpha_n)[(1+\frac{2\psi_n}{\beta})D_f(x^*, x_n) + \frac{\psi_n}{2}] \qquad (52)
\end{aligned}$$

Observe from assumption (*A*6)(*ii*) *that* $\frac{\psi_n}{\alpha_n} \longrightarrow 0$ *as* $n \longrightarrow \infty$ *for any* $\Im \in (0, \frac{\beta}{2})$*, there exists* n_0 *such that* $\psi_n < \Im\alpha_n$ *for all* $n \geq n_0$*. Therefore, for some* $M^* = \frac{\psi_n}{2} > 0$*, we have from* (52) *that*

$$\begin{aligned}
D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, u) + (1-\alpha_n)(1+\frac{2\psi_n}{\beta})D_f(x^*, x_n) + \frac{\psi_n}{2} \\
&\leq \alpha_n D_f(x^*, u) + (1-\alpha_n+\alpha_n\Im)D_f(x^*, x_n) + \alpha_n M^* \\
&= (1-\alpha_n(1-\Im))D_f(x^*, x_n) + \alpha_n(1-\Im)\frac{D_f(x^*, u)+M^*}{(1-\Im)} \\
&= \max\{D_f(x^*, x_n), \frac{D_f(x^*, u)+M^*}{(1-\Im)}\} \\
&\vdots \\
&\leq \max\{D_f(x^*, x_N), \frac{D_f(x^*, u)+M^*}{(1-\Im)}\}. \qquad (53)
\end{aligned}$$

By mathematical induction, we obtain

$$D_f(x^*, x_n) \leq \max\{D_f(x^*, x_N), \frac{D_f(x^*, u)+M^*}{(1-\Im)}\}, \ \forall\, n \geq N. \qquad (54)$$

Thus, the sequence $\{D_f(x^*, x_n)\}$ *is bounded. Therefore, by Lemma* 8*, we have that the sequence* $\{x_n\}$ *is bounded. Consequently,* $\{u_n\}$*,* $\{y_n\}$*,* $\{z_n\}$*,* $\{t_n\}$ *and* $\{w_n\}$ *are also bounded.*

Theorem 1. *Suppose that assumptions* $(A1)-(A6)$ *holds, and the sequence* $\{\alpha_n\} \subset (0,1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*. Let* $\{x_n\}$ *be the sequence generated by Algorithm* 1*. Then* $\{x_n\}$ *converges strongly to a solution*

$$x^* = Proj^f_{VI(C,F)\ A^{-1}(0)\bigcap(\bigcap_{j=1}^M F(T_j))\bigcap(\bigcap_{i=1}^N (B_i+G)^{-1}0)} u.$$

Proof. Let $x^* = Proj^f_{VI(C,F)\bigcap\ A^{-1}(0)\bigcap(\bigcap_{j=1}^M F(T_j))\bigcap(\bigcap_{i=1}^N (B_i+G)^{-1}0)} u$. From Lemma 3, we have

$$\langle \nabla f(u) - \nabla f(x^*), z - x^* \rangle \leq 0, \qquad \forall\ z \in VI(C,F)$$

From Lemma 14, we have that, there exists $N_0 \geq 0$, such that for all $n \geq N_0$,

$$D_f(x^*, z_n) \leq D_f(x^*, u_n)$$

and for any $\Im \in (0, \frac{\beta}{2})$, there exists n_0 such that $\psi_n < \Im\alpha_n$ for all $n \geq n_0$. Thus, for some $M^* = \frac{\psi_n}{2} > 0$, we obtain

$$\begin{aligned} D_f(x^*, u_n) &\leq (1 + \frac{2\psi_n}{\beta})D_f(x^*, x_n) - (1 - \frac{2\psi_n}{\beta})D_f(x_n, u_n) + \frac{\psi_n}{2} \\ &\leq (1 + \frac{2\psi_n}{\beta})D_f(x^*, x_n) + \frac{\psi_n}{2} \\ &= (1 + \alpha_n\Im)D_f(x^*, x_n) + \alpha_n M^* \end{aligned} \tag{55}$$

Furthermore, we estimate $D_f(x^*, x_{n+1})$ using (55), (17), Lemma 4 and inequality (19) of Lemma 3 for every $n \geq N_0$ as follows

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(J_{r_n} t_n))) \\ &= V_f(x^*, \alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(J_{r_n} t_n)) \\ &\leq V_f(x^*, \alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(J_{r_n} t_n) - \alpha_n(\nabla f(u) - \nabla f(x^*))) \\ &\quad -\langle -\alpha_n(\nabla f(u) - \nabla f(x^*)), \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(J_{r_n} t_n)) - x^* \rangle \\ &= D_f(x^*, \nabla f^*(\alpha_n \nabla f(x^*) + (1 - \alpha_n)\nabla f(J_{r_n} t_n))) \\ &\quad +\alpha_n \langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^* \rangle \\ &\leq D_f(x^*, x^*) + (1 - \alpha_n)D_f(x^*, J_{r_n} t_n) + \alpha_n \langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)D_f(x^*, t_n) + \alpha_n \langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n)[(1 + \alpha_n\Im)D_f(x^*, x_n) + \alpha_n M^*] + \alpha_n \langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^* \rangle \\ &\leq (1 - \alpha_n(1 - \Im))D_f(x^*, x_n) \\ &\quad +\alpha_n(1 - \Im)[\frac{1}{(1 - \Im)}(\langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^* \rangle + \frac{\psi_n}{\alpha_n})]. \end{aligned} \tag{56}$$

Next, applying Lemma 1 and the fact that B_i is Bregman inverse strongly monotone mapping for all $1 \leq i \leq N$ and condition (iv), we obtain

$$\begin{aligned} D_f(x^*, t_n) &= D_f(x^*, \nabla f^*(\eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n))) \\ &= V_f(x^*, \eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)) \end{aligned}$$

$$
\begin{aligned}
&= f(x^*) - \langle x^*, \eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)\rangle \\
&\quad + f^*(\eta_n \nabla f(z_n) + \delta_n \nabla f(u_n) + \xi_n \nabla f(w_n) + \sum_{i=1}^{N} \sigma_{i,n} \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)) \\
&\leq \eta_n f(x^*) + \delta_n f(x^*) + \xi_n f(x^*) + \sum_{i=1}^{N} \sigma_{i,n} f(x^*) - \eta_n \langle x^*, \nabla f(z_n)\rangle \\
&\quad - \delta_n \langle x^*, \nabla f(u_n)\rangle - \xi_n \langle x^*, \nabla f(w_n)\rangle - \sum_{i=1}^{N} \sigma_{i,n} \langle x^*, \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)\rangle \\
&\quad + \eta_n f^*(\nabla f(z_n)) + \delta_n f^*(\nabla f(u_n)) + \xi_n f^*(\nabla f(w_n)) \\
&\quad + \sum_{i=1}^{N} \sigma_{i,n} f^*(\nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)) \\
&= \eta_n (f(x^*) - \langle x^*, \nabla f(z_n)\rangle + f^*(\nabla f(z_n))) + \delta_n (f(x^*) \\
&\quad - \langle x^*, \nabla f(u_n)\rangle + f^*(\nabla f(u_n))) \\
&\quad + \xi_n (f(x^*) - \langle x^*, \nabla f(w_n)\rangle + f^*(\nabla f(w_n))) \\
&\quad + \sum_{i=1}^{N} \sigma_{i,n} (f(x^*) - \langle x^*, \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)\rangle + f^*(\nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n))) \\
&\quad - \gamma_n \xi_n \rho^*_r(||\nabla f(z_n) - \nabla f(w_n)||) - \gamma_n \delta_n \rho^*_r(||\nabla f(z_n) - \nabla f(u_n)||) \\
&\quad - \sum_{i=1}^{N} \gamma_n \sigma_{i,n} \rho^*_r(||\nabla f(z_n) - \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)||) \\
&\leq \gamma_n D_f(x^*, z_n) + \delta_n D_f(x^*, u_n) + \xi_n D_f(x^*, w_n) + \sum_{i=1}^{N} \sigma_{i,n} D_f(x^*, Q_{\tau_n} B^f_{i,\tau_n} w_n) \\
&\quad - \gamma_n \xi_n \rho^*_r(||\nabla f(z_n) - \nabla f(w_n)||) - \gamma_n \delta_n \rho^*_r(||\nabla f(z_n) - \nabla f(u_n)||) \\
&\quad - \sum_{i=1}^{N} \gamma_n \sigma_{i,n} \rho^*_r(||\nabla f(z_n) - \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)||) \\
&\leq \gamma_n D_f(x^*, u_n) + \delta_n D_f(x^*, u_n) + \xi_n D_f(x^*, z_n) + \sum_{i=1}^{N} \sigma_{i,n} D_f(x^*, w_n) \\
&\quad - \gamma_n \xi_n \rho^*_r(||\nabla f(z_n) - \nabla f(w_n)||) - \gamma_n \delta_n \rho^*_r(||\nabla f(z_n) - \nabla f(u_n)||) \\
&\quad - \sum_{i=1}^{N} \gamma_n \sigma_{i,n} \rho^*_r(||\nabla f(z_n) - \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)||) \\
&\leq \gamma_n D_f(x^*, u_n) + \delta_n D_f(x^*, u_n) + \xi_n D_f(x^*, u_n) + \sum_{i=1}^{N} \sigma_{i,n} D_f(x^*, z_n) \\
&\quad - \gamma_n \xi_n \rho^*_r(||\nabla f(z_n) - \nabla f(w_n)||) - \gamma_n \delta_n \rho^*_r(||\nabla f(z_n) - \nabla f(u_n)||)
\end{aligned}
$$

$$-\sum_{i=1}^{N}\gamma_n\sigma_{i,n}\rho_r^*(||\nabla f(z_n)-\nabla fQ_{\tau_n}B^f_{i,\tau_n}(w_n)||)$$

$$\begin{aligned} D_f(x^*,t_n) &\leq \gamma_n D_f(x^*,u_n)+\delta_n D_f(x^*,u_n)+\xi_n D_f(x^*,u_n)+\sum_{i=1}^{N}\sigma_{i,n}D_f(x^*,u_n) \\ &\quad -\gamma_n\xi_n\rho_r^*(||\nabla f(z_n)-\nabla f(w_n)||)-\gamma_n\delta_n\rho_r^*(||\nabla f(z_n)-\nabla f(u_n)||) \\ &\quad -\sum_{i=1}^{N}\gamma_n\sigma_{i,n}\rho_r^*(||\nabla f(z_n)-\nabla fQ_{\tau_n}B^f_{i,\tau_n}(w_n)||) \\ &= D_f(x^*,u_n)-\gamma_n\xi_n\rho_r^*(||\nabla f(z_n)-\nabla f(w_n)||)-\gamma_n\delta_n\rho_r^*(||\nabla f(z_n)-\nabla f(u_n)||) \\ &\quad -\sum_{i=1}^{N}\gamma_n\sigma_{i,n}\rho_r^*(||\nabla f(z_n)-\nabla fQ_{\tau_n}B^f_{i,\tau_n}(w_n)||). \end{aligned} \tag{57}$$

Thus

$$D_f(x^*,t_n) \leq D_f(x^*,u_n)-\gamma_n\xi_n\rho_r^*(||\nabla f(z_n)-\nabla f(u_n)||). \tag{58}$$

Following a similar computation, we obtain

$$D_f(x^*,t_n) \leq D_f(x^*,u_n)-\gamma_n\delta_n\rho_r^*(||\nabla f(z_n)-\nabla f(u_n)||). \tag{59}$$

and

$$\begin{aligned} D_f(x^*,t_n) &\leq D_f(x^*,u_n)-\sum_{i=1}^{N}\gamma_n\sigma_{i,n}\rho_r^*(||\nabla f(z_n)-\nabla fQ_{\tau_n}B^f_{i,\tau_n}(w_n)||) \\ &\leq D_f(x^*,u_n)-\gamma_n\sigma_{i,n}\rho_r^*(||\nabla f(z_n)-\nabla fQ_{\tau_n}B^f_{i,\tau_n}(w_n)||), \end{aligned} \tag{60}$$

for each $i\in\{1,2,...,N\}$.
We can obtain from the estimation of $D_f(x^*,x_{n+1})$, (57), (58) and (60) the following

$$\begin{aligned} D_f(x^*,x_{n+1}) &= D_f(x^*,\nabla f^*(\alpha_n\nabla f(u)+(1-\alpha_n)\nabla f(J_{r_n}t_n))) \\ &= V_f(x^*,\alpha_n\nabla f(u)+(1-\alpha_n)\nabla f(J_{r_n}t_n)) \\ &\leq \alpha_n D_f(x^*,u)+(1-\alpha_n)D_f(x^*,t_n) \\ &= \alpha_n D_f(x^*,u)+(1-\alpha_n)[D_f(x^*,u_n)-\gamma_n\xi_n\rho_r^*(||\nabla f(z_n)-\nabla f(w_n)||)] \\ &= \alpha_n D_f(x^*,u)+(1-\alpha_n)[(1+\alpha_n\Im)D_f(x^*,x_n)-(1-\alpha_n\Im)D_f(x_n,u_n) \\ &\quad +\alpha_n M^*-\gamma_n\xi_n\rho_r^*(||\nabla f(z_n)-\nabla f(w_n)||)] \\ &\leq \alpha_n D_f(x^*,u)+(1+\alpha_n\Im)D_f(x^*,x_n)-(1-\alpha_n\Im)D_f(x_n,u_n) \\ &\quad +\alpha_n M^*-\gamma_n\xi_n\rho_r^*(||\nabla f(z_n)-\nabla f(w_n)||). \end{aligned} \tag{61}$$

Similar computation gives

$$D_f(x^*,x_{n+1}) \leq \alpha_n D_f(x^*,u)+(1-\alpha_n)D_f(x^*,t_n)$$

$$
\begin{aligned}
&= \alpha_n D_f(x^*, u) + (1-\alpha_n)[D_f(x^*, u_n) - \gamma_n \delta_n \rho_r^*(||\nabla f(z_n) - \nabla f(u_n)||)] \\
&= \alpha_n D_f(x^*, u) + (1-\alpha_n)[(1+\alpha_n \Im) D_f(x^*, x_n) - (1-\alpha_n \Im) D_f(x_n, u_n) \\
&\quad + \alpha_n M^* - \gamma_n \delta_n \rho_r^*(||\nabla f(z_n) - \nabla f(u_n)||)] \\
&\leq \alpha_n D_f(x^*, u) + (1+\alpha_n \Im) D_f(x^*, x_n) - (1-\alpha_n \Im) D_f(x_n, u_n) \\
&\quad + \alpha_n M^* - \gamma_n \delta_n \rho_r^*(||\nabla f(z_n) - \nabla f(u_n)||). \qquad (62)
\end{aligned}
$$

Also,

$$
\begin{aligned}
D_f(x^*, x_{n+1}) &\leq \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, t_n) \\
&= \alpha_n D_f(x^*, u) + (1-\alpha_n)[D_f(x^*, u_n) \\
&\quad - \sum_{i=1}^{N} \gamma_n \sigma_{i,n} \rho_r^*(||\nabla f(z_n) - \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)||)] \\
&= \alpha_n D_f(x^*, u) + (1-\alpha_n)[(1+\alpha_n \Im) D_f(x^*, x_n) - (1-\alpha_n \Im) D_f(x_n, u_n) \\
&\quad + \alpha_n M^* - \sum_{i=1}^{N} \gamma_n \sigma_{i,n} \rho_r^*(||\nabla f(z_n) - \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)||)] \\
&\leq \alpha_n D_f(x^*, u) + (1+\alpha_n \Im) D_f(x^*, x_n) - (1-\alpha_n \Im) D_f(x_n, u_n) \\
&\quad + \alpha_n M^* - \gamma_n \sigma_{i,n} \rho_r^*(||\nabla f(z_n) - \nabla f Q_{\tau_n} B^f_{i,\tau_n}(w_n)||) \qquad (63)
\end{aligned}
$$

for each $i \in \{1, 2, ..., N\}$. Now, suppose that there exists a subsequence $\{D_f(x^*, x_{n_k})\}$ of $\{D_f(x^*, x_n)\}$ such that

$$
\liminf_{k\to\infty} \{D_f(x^*, x_{n_k+1}) - D_f(x^*, x_{n_k})\} \geq 0.
$$

From (61), we denotes Ψ_{n_k} as follows:

$$
\Psi_{n_k} := (1-\alpha_{n_k}\Im) D_f(x_{n_k}, u_{n_k}) + \gamma_{n_k} \xi_{n_k} \rho_r^*(||\nabla f(z_{n_k}) - \nabla f(w_{n_k})||) \qquad (64)
$$

Now, we obtain from (61) that

$$
\begin{aligned}
&\limsup_{k\to\infty} \{(1-\alpha_{n_k}\Im) D_f(x_{n_k}, u_{n_k}) + \gamma_{n_k} \xi_{n_k} \rho_r^*(||\nabla f(z_{n_k}) - \nabla f(w_{n_k})||)\} \\
&\qquad \leq \limsup_{k\to\infty} \{(1+\alpha_{n_k}\Im) D_f(x^*, x_{n_k}) - D_f(x^*, x_{n_k+1}) + \alpha_{n_k} D_f(x^*, u) + \alpha_{n_k} M^*\} \\
&\leq \limsup_{k\to\infty} \{D_f(x^*, x_{n_k}) - D_f(x^*, x_{n_k+1})\} + \limsup_{k\to\infty} \alpha_{n_k} M^* \\
&\leq -\liminf_{k\to\infty} \{D_f(x^*, x_{n_k+1}) - D_f(x^*, x_{n_k})\} \\
&\leq 0. \qquad (65)
\end{aligned}
$$

Hence, $\limsup_{k\to\infty} \Psi_{n_k} \leq 0$, which implies $\lim_{k\to\infty} \Psi_{n_k} = 0$. Then, it follows from the definition of Ψ_{n_k} that

$$
\lim_{k\to\infty} D_f(x_{n_k}, u_{n_k}) = 0. \qquad (66)
$$

Also, applying the conditions on the parameter sequences γ_n, ξ_n and δ_n, we obtain from (61), (62) and (63) the following

$$\lim_{k\to\infty}(\gamma_{n_k}\xi_{n_k}\rho_r^*(||\nabla f(z_{n_k}) - \nabla f(w_{n_k})||)) = \lim_{k\to\infty}(\gamma_{n_k}\delta_{n_k}\rho_r^*(||\nabla f(z_{n_k}) - \nabla f(u_{n_k})||)) = \lim_{k\to\infty}(\gamma_{n_k}\sigma_{i,n_k}\rho_r^*(||\nabla f(z_{n_k}) - \nabla f Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}}(w_{n_k})||)) = 0. \quad (67)$$

Thus, by the property of ρ_r^*, we have from (67) the following

$$\lim_{k\to\infty}||\nabla f(z_{n_k}) - \nabla f(w_{n_k})|| = \lim_{k\to\infty}||\nabla f(z_{n_k}) - \nabla f(u_{n_k})|| = \lim_{k\to\infty}||\nabla f(z_{n_k}) - \nabla f Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}}(w_{n_k})|| = 0. \quad (68)$$

Since ∇f^* is uniformly norm to norm continuous on bounded subsets of E^*, we obtain from (68) that

$$\lim_{k\to\infty}||z_{n_k} - w_{n_k}|| = \lim_{k\to\infty}||z_{n_k} - u_{n_k}|| = \lim_{k\to\infty}||z_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| = 0. \quad (69)$$

Using Lemma 6 and Lemma 7, we obtain from (66) that

$$\lim_{k\to\infty}||x_{n_k} - u_{n_k}|| = 0. \quad (70)$$

We also have from (70) and (69) that

$$\begin{aligned} ||z_{n_k} - x_{n_k}|| &= ||z_{n_k} - u_{n_k} + u_{n_k} - x_{n_k}|| \\ &\leq ||y_{n_k} - u_{n_k}|| + ||u_{n_k} - x_{n_k}|| \longrightarrow 0, \ \ as \ \ k \longrightarrow \infty. \end{aligned} \quad (71)$$

Hence, from (71), we obtain

$$\lim_{k\to\infty}||z_{n_k} - x_{n_k}|| = 0. \quad (72)$$

Again, we have from (69) and (72) that

$$\begin{aligned} ||w_{n_k} - x_{n_k}|| &= ||w_{n_k} - z_{n_k} + z_{n_k} - x_{n_k}|| \\ &\leq ||w_{n_k} - z_{n_k}|| + ||z_{n_k} - x_{n_k}|| \longrightarrow 0, \ \ as \ \ k \longrightarrow \infty. \end{aligned} \quad (73)$$

Thus, we get from (73) that

$$\lim_{k\to\infty}||w_{n_k} - x_{n_k}|| = 0. \quad (74)$$

Furthermore, we have from (69), that

$$\begin{aligned} ||w_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| &= ||w_{n_k} - z_{n_k} + z_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| \\ &\leq ||w_{n_k} - z_{n_k}|| + ||z_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| \longrightarrow 0, \end{aligned} \quad (75)$$

as $k \longrightarrow \infty$. Thus, we have from (75), that

$$\lim_{k\to\infty} ||w_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| = 0. \tag{76}$$

Also, from (76) and (74), we obtain the following

$$\begin{aligned} ||x_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| &= ||x_{n_k} - w_{n_k} + w_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| \\ &\leq ||x_{n_k} - w_{n_k}|| + ||w_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| \longrightarrow 0, \end{aligned} \tag{77}$$

as $k \longrightarrow \infty$. Hence, we have from (77), that

$$\lim_{k\to\infty} ||x_{n_k} - Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}} w_{n_k}|| = 0. \tag{78}$$

Let $v_{n_k} = J_{r_{n_k}} t_{n_k}$. Then, using the definition of $\{x_{n+1}\}$ in Algorithm 1 and condition $(A6)(ii)$, we have

$$x_{n_k+1} = \nabla f^*(\alpha_{n_k} \nabla f(u) + (1-\alpha_{n_k}) \nabla f(v_{n_k}))$$

$$\nabla f(x_{n_k+1}) = \alpha_{n_k} \nabla f(u) + (1-\alpha_{n_k}) \nabla f(v_{n_k})$$

$$\nabla f(x_{n_k+1}) - \nabla f(v_{n_k}) = (\alpha_{n_k} \nabla f(u) + (1-\alpha_{n_k}) \nabla f(v_{n_k})) - \nabla f(v_{n_k})$$

$$\begin{aligned} ||\nabla f(x_{n_k+1}) - \nabla f(v_{n_k})|| &= ||\alpha_{n_k} \nabla f(u) + (1-\alpha_{n_k}) \nabla f(v_{n_k}) \\ &\quad -(\alpha_{n_k} \nabla f(v_{n_k}) + (1-\alpha_{n_k}) \nabla f(v_{n_k}))|| \\ &= ||\alpha_{n_k} \nabla f(u) - \alpha_{n_k} \nabla f(v_{n_k}) \\ &\quad +(1-\alpha_{n_k}) \nabla f(v_{n_k}) - (1-\alpha_{n_k}) \nabla f(v_{n_k}))|| \\ &= \alpha_{n_k} ||\nabla f(u) - \nabla f(v_{n_k})|| \end{aligned}$$

Now, using the fact that $\underset{n\to\infty}{lim}\, \alpha_n = 0$, we obtain

$$\underset{k\to\infty}{lim} ||\nabla f(x_{n_k+1}) - \nabla f(v_{n_k})|| = 0. \tag{79}$$

Since f is uniformly Fréchet differentiable, then ∇f^* is uniformly norm to norm continuous on bounded subsets of E^*, we have from (79) that

$$\underset{n\to\infty}{lim} ||x_{n_k+1} - v_{n_k}|| = 0. \tag{80}$$

Again, using the definition of $\{x_{n+1}\}$ in Algorithm 1 and Lemma 2. Since $v_{n_k} = J_{r_{n_k}} t_{n_k}$, we have

$$D_f(J_{r_{n_k}} t_{n_k}, t_{n_k}) \leq D_f(x^*, t_{n_k}) - D_f(x^*, J_{r_{n_k}} t_{n_k})$$

$$\begin{aligned}
D_f(v_{n_k},t_{n_k}) &\leq D_f(x^*,t_{n_k}) - D_f(x^*,v_{n_k}) \\
&= D_f(x^*,t_{n_k}) - D_f(x^*,x_{n_k+1}) + D_f(x^*,x_{n_k+1}) - D_f(x^*,v_{n_k}) \\
&\leq D_f(x^*,u_{n_k}) - D_f(x^*,x_{n_k+1}) + D_f(x^*,x_{n_k+1}) - D_f(x^*,v_{n_k}) \\
&\leq (1+\frac{2\psi_{n_k}}{\beta})D_f(x^*,x_{n_k}) + \frac{\psi_{n_k}}{2} - D_f(x^*,x_{n_k+1}) \\
&\quad +\alpha_{n_k}D_f(x^*,u) + (1-\alpha_{n_k})D_f(x^*,v_{n_k}) - D_f(x^*,v_{n_k}) \\
&= (1+\alpha_{n_k}\Im)D_f(x^*,x_{n_k}) + \alpha_{n_k}M^* - D_f(x^*,x_{n_k+1}) \\
&\quad +\alpha_{n_k}D_f(x^*,u) + (1-\alpha_{n_k})D_f(x^*,v_{n_k}) - D_f(x^*,v_{n_k}) \\
&= (D_f(x^*,x_{n_k}) - D_f(x^*,x_{n_k+1})) \\
&\quad +\alpha_{n_k}[D_f(x^*,u) + \Im D_f(x^*,x_{n_k}) - D_f(x^*,v_{n_k}) + M^*] \qquad (81)
\end{aligned}$$

Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that

$$\liminf_{k\to\infty}\{x_{n_k+1} - x_{n_k}\} \geq 0.$$

Thus, taking limit on both sides of (81) as $k \longrightarrow \infty$ and the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we obtain

$$\begin{aligned}
D_f(v_{n_k},t_{n_k}) &= (D_f(x^*,x_{n_k}) - D_f(x^*,x_{n_k+1})) \\
&\quad +\alpha_{n_k}[D_f(x^*,u) + \Im D_f(x^*,x_{n_k}) - D_f(x^*,v_{n_k}) + M^*] \longrightarrow 0, \qquad (82)
\end{aligned}$$

as $k \longrightarrow \infty$. Thus,

$$\lim_{k\to\infty} D_f(v_{n_k},t_{n_k}) = 0. \qquad (83)$$

Applying Lemma 7, we have from (83) that

$$\lim_{k\to\infty} ||v_{n_k} - t_{n_k}|| = 0. \qquad (84)$$

Now, using (80) and (84), we get

$$\begin{aligned}
\lim_{k\to\infty} ||x_{n_k+1} - t_{n_k}|| &= ||x_{n_k+1} - v_{n_k} + v_{n_k} - t_{n_k}|| \\
&\leq ||x_{n_k+1} - v_{n_k}|| + ||v_{n_k} - t_{n_k}|| \longrightarrow 0, \qquad (85)
\end{aligned}$$

as $k \longrightarrow \infty$. Hence,

$$\lim_{k\to\infty} ||x_{n_k+1} - t_{n_k}|| = 0. \qquad (86)$$

From the definition of $\{t_{n_k}\}$ in Algorithm 1, (70), (72), (74), and (78), we have

$$\nabla f t_{n_k} = \eta_{n_k}\nabla f(z_{n_k}) + \delta_{n_k}\nabla f(u_{n_k}) + \xi_{n_k}\nabla f(w_{n_k}) + \sum_{i=1}^{N}\sigma_{i,n_k}\nabla f Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}}(w_{n_k})$$

$$\begin{aligned}||\nabla f t_{n_k} - \nabla f x_{n_k}|| &= ||\eta_{n_k}\nabla f(z_{n_k}) + \delta_{n_k}\nabla f(u_{n_k}) + \xi_{n_k}\nabla f(w_{n_k}) \\ &+ \sum_{i=1}^{N}\sigma_{i,n_k}\nabla f Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}}(w_{n_k}) - \nabla f(x_{n_k})|| \\ &= \eta_{n_k}||\nabla f(z_{n_k}) - \nabla f(x_{n_k})|| + \delta_{n_k}||\nabla f(u_{n_k}) - \nabla f(x_{n_k})|| \\ &+\xi_{n_k}||\nabla f(w_{n_k}) - \nabla f(x_{n_k})|| \\ &+ \sum_{i=1}^{N}\sigma_{i,n_k}||\nabla f Q_{\tau_{n_k}} B^f_{i,\tau_{n_k}}(w_{n_k}) - \nabla f(x_{n_k})|| \longrightarrow 0, \end{aligned} \tag{87}$$

as $k \longrightarrow \infty$. Hence, we obtain from (87) the following

$$\lim_{k\to\infty}||\nabla f(t_{n_k}) - \nabla f(x_{n_k})|| = 0. \tag{88}$$

Since ∇f^* is uniformly norm to norm continuous on bounded subsets of E^*, we have from (88) that

$$\lim_{k\to\infty}||t_{n_k} - x_{n_k}|| = 0. \tag{89}$$

Again, we obtain from (84) and (89)that

$$\begin{aligned}\lim_{k\to\infty}||v_{n_k} - x_{n_k}|| &= ||v_{n_k} - t_{n_k} + t_{n_k} - x_{n_k}|| \\ &\leq ||v_{n_k} - t_{n_k}|| + ||t_{n_k} - x_{n_k}|| \longrightarrow 0, \end{aligned} \tag{90}$$

as $k \longrightarrow \infty$. Hence, from (90), we obtain

$$\lim_{k\to\infty}||v_{n_k} - x_{n_k}|| = 0. \tag{91}$$

Also, we obtain from the estimation of $D_f(x^*, x_{n+1})$, (47) and (41) that

$$\begin{aligned}D_f(x^*, x_{n_k+1}) &= D_f(x^*, \nabla f^*(\alpha_{n_k}\nabla f(u) + (1-\alpha_{n_k})\nabla f(J_{r_{n_k}} t_{n_k}))) \\ &= V_f(x^*, \alpha_{n_k}\nabla f(u) + (1-\alpha_{n_k})\nabla f(J_{r_{n_k}} t_{n_k})) \\ &\leq \alpha_{n_k} D_f(x^*, u) + (1-\alpha_{n_k}) D_f(x^*, t_{n_k}) \\ &\leq \alpha_n D_f(x^*, u) + (1-\alpha_{n_k}) D_f(x^*, w_{n_k}) \\ &\leq \alpha_{n_k} D_f(x^*, u) + (1-\alpha_{n_k}) D_f(x^*, z_{n_k}) \\ &= \alpha_{n_k} D_f(x^*, u) + (1-\alpha_{n_k})[D_f(x^*, u_{n_k}) \\ &\quad -(1 - \frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}\beta})(D_f(z_{n_k}, y_{n_k}) + D_f(y_{n_k}, u_{n_k}))] \\ &= \alpha_{n_k} D_f(x^*, u) + (1-\alpha_{n_k})[(1+\alpha_{n_k}\Im) D_f(x^*, x_{n_k}) \\ &\quad -(1-\alpha_{n_k}\Im) D_f(x_{n_k}, u_{n_k}) \\ &\quad +\alpha_{n_k} M^* - (1 - \frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}\beta})(D_f(z_{n_k}, y_{n_k}) + D_f(y_{n_k}, u_{n_k}))]\end{aligned}$$

$$\leq \alpha_{n_k} D_f(x^*, u) + (1 + \alpha_{n_k}\Im) D_f(x^*, x_{n_k}) - (1 - \alpha_{n_k}\Im) D_f(x_{n_k}, u_{n_k}) + \alpha_{n_k} M^* - (1 - \frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}\beta})(D_f(z_{n_k}, y_{n_k}) + D_f(y_{n_k}, u_{n_k})) \quad (92)$$

Now, we obtain from (92) that

$$\begin{aligned}(1 - \alpha_{n_k}\Im) D_f(x_{n_k}, u_{n_k}) + (1 - \frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}\beta})(D_f(z_{n_k}, y_{n_k}) + D_f(y_{n_k}, u_{n_k})) \\ \leq (1 + \alpha_{n_k}\Im) D_f(x^*, x_{n_k}) - D_f(x^*, x_{n_k+1}) + \alpha_{n_k} D_f(x^*, u) + \alpha_{n_k} M^* \\ = D_f(x^*, x_{n_k}) - D_f(x^*, x_{n_k+1}) + \alpha_{n_k}(\Im D_f(x^*, x_{n_k}) + D_f(x^*, u) + M^*) \longrightarrow 0, \quad (93)\end{aligned}$$

as $k \longrightarrow \infty$.

$$\lim_{k\to\infty} D_f(z_{n_k}, y_{n_k}) = \lim_{k\to\infty} D_f(y_{n_k}, u_{n_k}) = \lim_{k\to\infty} D_f(x_{n_k}, u_{n_k}) = 0. \quad (94)$$

Applying Lemma 7, we obtain the following (94) that

$$\lim_{k\to\infty} ||z_{n_k} - y_{n_k}|| = \lim_{k\to\infty} ||y_{n_k} - u_{n_k}|| = \lim_{k\to\infty} ||x_{n_k} - u_{n_k}|| = 0. \quad (95)$$

We also have from (95) that

$$\begin{aligned} ||y_{n_k} - x_{n_k}|| &= ||y_{n_k} - u_{n_k} + u_{n_k} - x_{n_k}|| \\ &\leq ||y_{n_k} - u_{n_k}|| + ||u_{n_k} - x_{n_k}|| \longrightarrow 0, \quad as \ \ k \longrightarrow \infty. \quad (96)\end{aligned}$$

Hence, from (96), we obtain

$$\lim_{k\to\infty} ||y_{n_k} - x_{n_k}|| = 0. \quad (97)$$

Using the definition of $\{w_{n_k}\}$ in Algorithm 1, and the fact that T_j is Bregman $(\nu_j, 0)$ - demigeneralized mapping for all $1 \leq j \leq M$, we obtain

$$\begin{aligned}\langle z_{n_k} - x^*, \nabla f(z_{n_k}) - \nabla f(w_{n_k})\rangle &= \langle z_{n_k} - x^*, \nabla f(z_{n_k}) - \nabla f \nabla f^*(\sum_{j=1}^{M} \varphi_j((1 - \beta_{n_k})\nabla f(z_{n_k}) \\ &\quad + \beta_{n_k} \nabla f(T_j z_{n_k}))\rangle \\ &= \sum_{j=1}^{M} \varphi_j \langle z_{n_k} - x^*, \nabla f(z_{n_k}) - ((1 - \beta_{n_k})\nabla f(z_{n_k}) \\ &\quad + \beta_{n_k} \nabla f(T_j z_{n_k}))\rangle \\ &= \sum_{j=1}^{M} \varphi_j \langle z_{n_k} - x^*, \nabla f(z_{n_k}) - \nabla f(z_{n_k}) + \beta_{n_k} \nabla f(z_{n_k}) \\ &\quad - \beta_{n_k} \nabla f(T_j z_{n_k})\rangle\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{M} \varphi_j \langle z_{n_k} - x^*, \beta_{n_k} \nabla f(z_{n_k}) - \beta_{n_k} \nabla f(T_j z_{n_k}) \rangle \\
&= \sum_{j=1}^{M} \varphi_j \beta_{n_k} \langle z_{n_k} - x^*, \nabla f(z_{n_k}) - \nabla f(T_j z_{n_k}) \rangle \\
&\geq \sum_{j=1}^{M} \varphi_j \beta_{n_k} (1-\nu_j) D_f(z_{n_k}, T_j z_{n_k}) \\
&\geq \sum_{j=1}^{M} \varphi_j a (1-\nu_j) D_f(z_{n_k}, T_j z_{n_k}), \qquad (98)
\end{aligned}$$

for all $x^* \in \bigcap_{j=1}^{M} F(T_j)$. Hence, we have from (98) that

$$\begin{aligned}
\sum_{j=1}^{M} \varphi_j a (1-\nu_j) D_f(z_{n_k}, T_j z_{n_k}) &\leq \langle z_{n_k} - x^*, \nabla f(z_{n_k}) - \nabla f(w_{n_k}) \rangle \\
&\leq ||z_{n_k} - x^*|| ||\nabla f(z_{n_k}) - \nabla f(w_{n_k})||. \qquad (99)
\end{aligned}$$

Thus, we have from (69), conditions $(i), (iii)$ and $1 - \nu_j > 0$ that

$$\lim_{k\to\infty} D_f(z_{n_k}, T_j z_{n_k}) = 0, \qquad (100)$$

for all $1 \leq j \leq M$. Since the function f is totally convex on bounded sets of E, also by applying Lemma 6 and Lemma 7, we have from (100) that

$$\lim_{k\to\infty} ||z_{n_k} - T_j z_{n_k}|| = 0, \forall\, j \in \{1, 2, ..., M\}. \qquad (101)$$

Applying (88) and (86), we have

$$\begin{aligned}
\lim_{k\to\infty} ||x_{n_k+1} - x_{n_k}|| &= ||x_{n_k+1} - t_{n_k} + t_{n_k} - x_{n_k}|| \\
&\leq ||x_{n_k+1} - t_{n_k}|| + ||t_{n_k} - x_{n_k}|| \longrightarrow 0, \qquad (102)
\end{aligned}$$

as $k \longrightarrow \infty$. Thus

$$\lim_{k\to\infty} ||x_{n_k+1} - x_{n_k}|| = 0. \qquad (103)$$

Since $\{x_{n_k}\}$ is bounded and E is reflexive, then, there exists a subsequence $\{x_{n_{k_s}}\} \subset \{x_{n_k}\}$ such that $x_{n_{k_s}} \rightharpoonup p^* \in E$, which implies by (72) and (74) that $w_{n_{k_s}} \rightharpoonup p^*$ and $z_{n_{k_s}} \rightharpoonup p^*$ as $s \longrightarrow \infty$. Hence, by demiclosedness of $(I - T_j)$ at zero for each $j \in 1, 2, ..., M$ together with (101), it follows that $p^* \in \bigcap_{j=1}^{M} F(T_j)$. Furthermore, by Lemma 13 and (95), we conclude that $p^* \in VI(C, F)$. Now combining (74) and (76), we have that $p^* \in \hat{F}(Q_{\tau_{n_k}} B_{i,\tau_{n_k}})$ for all $i \in 1, 2, ..., N$. Again, we have from Lemma 1 that $\hat{F}(Q_{\tau_{n_k}} B_{i,\tau_{n_k}}) = F(Q_{\tau_{n_k}} B_{i,\tau_{n_k}}) =$

$(G+B_i)^{-1}(0)$, for all $i \in 1,2,...,N$. Thus $p^* \in \bigcap_{i=1}^{N}(G+B_i)^{-1}(0)$. Next, we show that $p^* \in A^{-1}0$. Using $r_n \geq c$, we have from (84) the following

$$\lim_{k\to\infty}\frac{1}{r_{n_k}}||v_{n_k}-t_{n_k}||=0. \tag{104}$$

Thus, applying $A_{r_{n_k}}$, the Yosida approximation of A, we obtain

$$\lim_{k\to\infty}||A_{r_{n_k}}t_{n_k}||=\lim_{k\to\infty}\frac{1}{r_{n_k}}||v_{n_k}-t_{n_k}||=0. \tag{105}$$

Since, $A_{r_{n_k}}t_{n_k} \in Av_{n_k}$, for $(s,w^*) \in A$, we have from the monotonicity of A that $\langle s - v_{n_k}, w^* - A_{r_{n_k}}t_{n_k}\rangle \geq 0$ for all $k \in \mathbb{N}$. Also, we have from (84), (91) and (89) that $||v_{n_k}-p^*|| \longrightarrow 0$ and $||t_{n_k}-p^*|| \longrightarrow 0,\ \ as\ k \longrightarrow\ \infty$. Therefore, $\langle s-p^*, w^*\rangle \geq 0$. From the monotonicity of A, we have that $p^* \in A^{-1}0$. Hence, $p^* \in \Gamma$.
Next, we show that $\{x_{n_k}\}$ converges strongly to a point $x^* = Proj_{\Gamma}^{f}u$. Thus, we have from (56) that $x^* = Proj_{\Gamma}^{f}u$.

$$\begin{aligned} D_f(z^*,x_{n+1}) \ &\leq \ (1-\alpha_n(1-\Im))D_f(x^*,x_n) \\ &\quad +\alpha_n(1-\Im)[\frac{1}{(1-\Im)}(\langle\nabla f(u)-\nabla f(x^*),x_{n+1}-x^*\rangle+\frac{\psi_n}{\alpha_n})]. \end{aligned} \tag{106}$$

Since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_j}}\} \subset \{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup p^*$ and

$$\begin{aligned} \limsup_{k\to\infty}\langle\nabla f(u)-\nabla f(x^*),x_{n_k}-x^*\rangle \ &= \ \lim_{j\to\infty}\langle\nabla f(u)-\nabla f(x^*),x_{n_{k_j}}-x^*\rangle \\ &= \ \langle\nabla f(u)-\nabla f(x^*),p^*-x^*\rangle \end{aligned} \tag{107}$$

Thus, from Lemma 3 and (107), we have

$$\limsup_{k\to\infty}\langle\nabla f(u)-\nabla f(x^*),x_{n_k}-x^*\rangle \ = \ \langle\nabla f(u)-\nabla f(x^*),p^*-x^*\rangle \leq 0. \tag{108}$$

Now, since (108) and (103) hold, we obtain the following

$$\limsup_{j\to\infty}\langle\nabla f(u)-\nabla f(x^*),x_{n_{k_j}+1}-x^*,\rangle = \lim_{j\to\infty}\langle\nabla f(u)-\nabla f(x^*),x_{n_{k_j}}-x^*\rangle \leq 0. \tag{109}$$

Therefore, applying Lemma 10 and (109) in (106), it follows that $D_f(x^*,x_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Also, using Definition (3) since we know that

$$\frac{\beta}{2}||x_n-x^*||^2 \leq D_f(x_n,x^*) \longrightarrow 0\ \ as,\ \ n \longrightarrow \infty.$$

Hence, $\{x_n\} \longrightarrow x^*$, where $x^* = Proj_{\Gamma}^{f}u$.

Corollary 1. *Let E be a real reflexive Banach space, $F : E \longrightarrow E^*$ be a monotone and Lipschitz continuous operator, $\{T_j\}_{j=1}^M$ be a finite family of Bregman quasi nonexpansive mapping. Let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function satisfying assumption $(A5)$. Let $\{u_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{\alpha_n\} \subset (0,1)$ satisfy $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ be sequences satisfying assumptions $(A1) - (A6)$ of algorithm (1). Suppose*

$$x^* = Proj^f_{VI(C,F)\ A^{-1}(0) \bigcap (\bigcap_{j=1}^M F(T_j)) \bigcap (\bigcap_{i=1}^N (B_i+G)^{-1}0)} u.$$

Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to a solution

$$x^* = Proj^f_{VI(C,F)\ A^{-1}(0) \bigcap (\bigcap_{j=1}^M F(T_j)) \bigcap (\bigcap_{i=1}^N (B_i+G)^{-1}0)} u.$$

Proof. *Note that, in this case the weak sequential continuity of F in assumption (A2) of Algorithm 1 has to be droped since it follows from the monotonicity of F and (28) that*

$$\begin{aligned} \frac{1}{\lambda_{n_k}} \langle \nabla f(u_{n_k}) - \nabla f(y_{n_k}), z - y_{n_k} \rangle + \langle F(u_{n_k}), y_{n_k} - u_{n_k} \rangle &\leq \langle F(u_{n_k}), z - u_{n_k} \rangle \\ &\leq \langle F(z), z - u_{n_k} \rangle \end{aligned} \tag{110}$$

Furthermore, passing limit as $k \longrightarrow \infty$ in inequality (110) and applying the fact that $||u_{n_k} - y_{n_k}|| \longrightarrow 0$, as $k \longrightarrow \infty$ and since ∇f is uniformly norm to norm continuous on bounded subsets of E, then $\lim\limits_{k\to\infty} ||\nabla f(y_{n_k}) - \nabla f(u_{n_k})|| = 0$, we obtain

$$\langle F(z), z - u^* \rangle \geq 0, \ \ \forall \ \ z \in C.$$

Again, Since T_j is a finite family of Bregman quasi nonexpansive mappings, then T_j is $(0,0)$-Bregman demigeneralized mappings. Hence, it follows from Theorem (1) that the sequence $\{x_n\}$ converges strongly to a solution

$$x^* = Proj^f_{VI(C,F)\ A^{-1}(0) \bigcap (\bigcap_{j=1}^M F(T_j)) \bigcap (\bigcap_{i=1}^N (B_i+G)^{-1}0)} u.$$

4. Application

4.1. Application to a convex minimization problem (CMP)

Let E be a Banach space and C be a nonempty, closed and convex subset of E and $f : E \longrightarrow (-\infty, +\infty]$ be a proper, convex and lower semi continuous function. Consider the following convex minimization problem:

$$Find \ \ x^* \in E \quad such\ that \qquad f(x^*) = \min_{y \in C} f(y). \tag{111}$$

The above problem (111) can be reformulated as:

$$Find \ \ x^* \in E \quad such\ that \qquad 0 \in \partial f(x^*), \tag{112}$$

where ∂f is the subdifferential of f defined by

$$\partial f(x^*) = \{x \in E^* : \langle x, y - x^* \rangle \leq f(y) - f(x^*),\ \forall\ x^* \in E\}.$$

Since the subdifferential ∂f is a maximal monotone operator whenever f is a proper, convex and lower semi continuous function. Hence, by setting $A, G = \partial f$ of assumption $A1$ in Theorem 1, we obtain a strong convergence result for approximating a solution of convex minimization problem (111).

Remark 3. *(a) We dispensed the sets C_n and Q_n in the algorithms introduced by both Ogbuisi and Izuchukwu [37]; Orouji et al. [30] and yet obtained strong convergence theorem.*

(b) The prototypes for the sequences $\{\eta_n\}, \{\delta_n\}, \{\xi_n\}$ and $\{\sigma_{i,n}\}$ for our work are as follows:
$\eta_n := \frac{1}{3n+1}$; $\delta_n := \frac{n}{3n+1}$; $\xi_n := \frac{n}{3n+1}$; $\sigma_{i,n} := \frac{1}{2^i}[(\frac{n}{3n+1})(\frac{2^N}{2^N-1})]$, $\forall i \in \mathbb{N}$ *and* $\forall n \in \mathbb{N}$.

5. Numerical Illustration and Comparison

In this section, we present a numerical experiment to compare the proposed Algorithm 1 with the shrinking projection method of Orouji et al. [30]. We work in $E = \mathbb{R}^2$ with the Euclidean norm and quadratic Bregman generator $f(x) = \frac{1}{2}\|x\|^2$. The maximal monotone operators are chosen as $A(x) = x$ and $G(x) = 2x$, yielding resolvents

$$J_r = (I + rA)^{-1} = \frac{1}{1+r}I, \qquad Q_\tau = (I + \tau G)^{-1} = \frac{1}{1+2\tau}I.$$

The variational inequality operator is $F(x) = x$, and the demigeneralized mapping is $T(x) = \frac{1}{2}x$. Initial points are $x_0 = (2, -1)$ and $x_1 = (1, 1)$, with anchor $u = 0$. Control sequences are set as

$$\alpha_n = \frac{1}{n+1}, \quad \psi_n = \frac{1}{(n+1)^2}, \quad \theta = 0.9, \quad \mu = 0.5, \quad \lambda_1 = 0.8, \quad \beta_n = 0.3,$$

Other components:

- VI operator: $F(x) = x$ (monotone, Lipschitz, pseudomonotone).
- Demigeneralized map single $T(x) = \frac{1}{2}x$.
- Feasible set$C = \mathbb{R}^2$ (so $Proj_C^f = \text{Id}$).
- Initial points $x_0 = (2, -1)$, $x_1 = (1, 1)$, anchor $u = 0$.
- Control parameters: Step sizes
  $\alpha_n = \frac{1}{n+1}$, $\psi_n = \frac{1}{(n+1)^2}$, $\lambda_1 = 0.8$, $\mu = 0.5$, $\theta = 0.9$, $\beta_n = 0.3$.

- Resolvent parameters
- $r_n = 1 \Rightarrow J_{r_n} = \frac{1}{2}I$; $\tau_n = 1 \Rightarrow Q_{\tau_n} = \frac{1}{3}I$.
- Convex weights $\eta_n = \frac{1}{3n+1}$, $\delta_n = \frac{n}{3n+1}$, $\xi_n = \frac{n}{3n+1}$, for $N = 1$: $\sigma_{1,n} = \frac{n}{3n+1}$. Check: $\eta_n + \delta_n + \xi_n + \sigma_{1,n} = 1$.

Our Proposed Algorithm 1 updates

- Inertial Momentum

$$\theta_n = \min\left\{\frac{\psi_n}{\|x_n - x_{n-1}\|}, \theta\right\}, \quad u_n = x_n + \theta_n(x_n - x_{n-1}).$$

- Forward pseudogradient

$$y_n = (I - \lambda_n)u_n, \qquad \lambda_{n+1} = \min\{\mu, \lambda_n\} \Rightarrow \lambda_n \equiv 0.5 \ (n \geq 2).$$

- Demigeneralized averaging

$$z_n = (1 - \lambda_n)y_n + \lambda_n u_n, \qquad w_n = (1 - \beta_n)z_n + \beta_n T(z_n) = \left(1 - \tfrac{\beta_n}{2}\right)z_n = 0.85\, z_n.$$

- Aggregation with resolvents

$$t_n = \eta_n z_n + \delta_n u_n + \xi_n w_n + \sigma_{1,n}\, Q_{\tau_n} B^f_{1,\tau_n}(w_n) = \eta_n z_n + \delta_n u_n + \xi_n w_n + \sigma_{1,n}\left(\tfrac{1}{3}w_n\right),$$

$$x_{n+1} = (1 - \alpha_n)J_{r_n} t_n = (1 - \alpha_n)\left(\tfrac{1}{2}t_n\right).$$

Algorithm 4 Shrinking projection method updates

- Demimetric forward step

$$y_n = (1 - \lambda_n)x_n + \lambda_n T(x_n) = \left(1 - \tfrac{\lambda_n}{2}\right)x_n.$$

- Bregman inverse strongly monotone block with G-resolvent

$$z_n = \sigma_1\, Q_{\eta_n} B^f_{1,\eta_n}(y_n) = \sigma_1\left(\tfrac{1}{1+2\eta_n}\right)y_n.$$

- Prox step:

$$u_n = J_{r_n} z_n = \tfrac{1}{2}z_n.$$

We define the error at iteration n as $e_n = \|x_n - x^\star\|$, where $x^\star = 0$ is the exact solution. Table 1 reports the error norms for both algorithms up to 20 iterations, while Figures 1 illustrate their convergence behavior: the left graph shows the error norms on a linear scale, and the right graph presents the same data on a logarithmic scale to highlight relative contraction rates. Both algorithms converge strongly to the solution, with the shrinking projection method Algorithm 4 exhibiting faster error reduction in the early iterations, whereas the proposed Algorithm 1 demonstrates smoother, stabilized decay. The log-scale plot emphasizes the relative contraction rates, with the shrinking projection showing steeper slopes initially. These results confirm the theoretical convergence properties and illustrate the trade-offs between aggressive contraction and stabilized multi-step aggregation.

Table 1: Error norms $e_n = \|x_n - x^\star\|$ for 20 iterations of both algorithms.

Iteration n	Proposed Algorithm e_n	Shrinking Projection e_n
1	1.414	1.414
2	1.384	0.471
3	0.885	0.290
4	0.559	0.206
5	0.355	0.155
6	0.224	0.122
7	0.142	0.099
8	0.091	0.083
9	0.059	0.072
10	0.038	0.064
11	0.025	0.057
12	0.016	0.051
13	0.011	0.046
14	0.007	0.042
15	0.005	0.038
16	0.003	0.035
17	0.002	0.032
18	0.001	0.030
19	0.001	0.028
20	0.0007	0.026

6. Conclusion

In this section, using Bregman distance technique, we introduce a new accelerated extrapolation Tseng's algorithm with self - adaptive step size for approximating a common element in the set of solutions of pseudomonotone variational inequality problems, zeros of maximal and Bregman inverse strongly monotone mappings and the set of common fixed points of a finite family of Bregman demigeneralized mappings in a real reflexive Banach space. Furthermore, we prove a strong convergence theorem to a solution of the stated problem without prior knowledge of the Lipschitz constant of the operator under some mild assumptions with application. Finally, we give numerical example to demonstrate

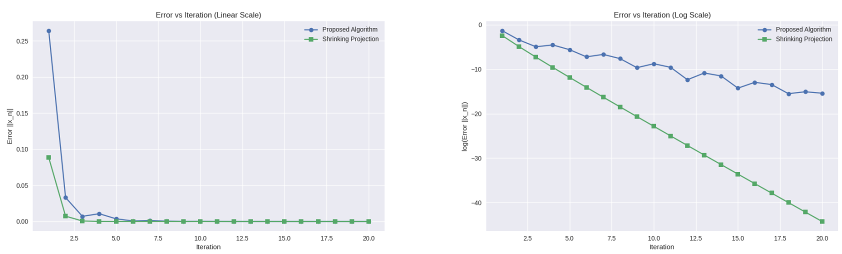

Figure 1: The error norm vs iteration (linear and logarithmic scales) plotting of Comparison of Algorithm 4 and Algorithm 1.

the effectiveness of our algorithm over the recently announced results in the literature. Our result generalize and improve the results in [30, 37]. **Competing Interests:** The authors declare that there are no competing interests surrounding the research work carried out herein.

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