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Some Applications of Fuzzy Differential Subordination on Analytic Functions Connected with Lommel Function

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Abstract. The findings of this study are connected with geometric function theory and were acquired by using Fuzzy subordination-based techniques in conjunction with the convolution concept and Lommel Function $\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}}$. The first class introduced and investigated here is a generalized class of analytic functions. It is also shown that for particular choice of parameters for the new generalized class, the class of close-to-convex functions emerges. Using the properties of the convolution and subordination, certain characterization properties of this class are proved involving combinations of the functions from the class. Further, three more classes are defined in connection to this first class, developing new applications of Lommel function by using the fuzzy subordination technique and convolutions.

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1. introduction

Lotfi A. Zadeh [1] established the foundations of fuzzy set theory, which has since become a central tool for handling uncertainty in mathematical analysis. The study of geometric function theory (GFT) has benefited from the contributions of fuzzy set theory and complex analysis since the first work introducing the idea of subordination in fuzzy set theory was published in 2011 [2]. Miller and Mocanu's traditional qualities of subordination [3, 4] served as the inspiration for this concept. Later papers that studied fuzzy differential subordination, which included components from the previously established theory of differential subordination [5–7], followed the study path laid forth by Miller and Mocanu. The idea was immediately embraced by GFT researchers, and all of the conventional research paths in this area were changed to account for the novel fuzzy properties. An essential area of research in GFT is operator-related research. Shortly after the notion was launched, in 2013, such experiments were published to acquire new fuzzy subordination results [8]. We only highlight a few of the numerous publications that have been published in the

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past few years to demonstrate how the body of knowledge on this subject is constantly growing [9–16].

In a related study, Haydar [17] extended these ideas and derived new results for fuzzy differential subordinations, highlighting how the use of diverse operators enriches the subject. A number of researchers have since examined linear operators in this setting, producing a wide body of work on fuzzy third order differential subordination [18, 19]. These contributions represent the first systematic attempts to employ fuzzy sets in the geometric theory of analytic functions.

Let A denote the class of function satisfying f(0) =f$'$(0) − 1 = 0 written as:

$$f(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_\kappa \xi^\kappa. \tag{1}$$

which are analytic and univalent in the open unit disc $\mathcal{U} = \{\xi : |\xi| < 1\}$. If f and g are analytic in $\mathcal{U}$, f is *subordinate* to g, denoted f(ξ) $\prec g(\xi)$, if there exists an analytic function ϖ, with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$ for all $\xi \in \mathcal{U}$, such that f(ξ) =g($\varpi(\xi)$), $\xi \in \mathcal{U}$. If the function g is univalent in $\mathcal{U}$, f(ξ) $\prec$g(ξ) is given as (see [4, 20, 21]):

$$f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(U).$$

For two functions f$_\iota(\xi) \in \mathcal{A}(\iota = 1, 2)$ are given by

$$f_\iota(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa,\iota} \xi^\kappa,$$

we define the convolution of f$_1(\xi)$ and f$_2(\xi)$ as

$$(f_1 * f_2)(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa,1} a_{\kappa,2} \xi^\kappa = (f_2 * f_1)(\xi).$$

The Lommel function is a special type of mathematical function that arises in various areas of applied mathematics and physics. It is often encountered in problems involving wave propagation, optics, and acoustics, particularly when dealing with cylindrical geometries. The Lommel functions are solutions to a specific type of differential equation, known as Bessel's differential equation, which describes the behavior of waves in cylindrical coordinates. These functions are particularly valuable because they allow for the representation of waveforms that are not easily handled by simpler functions, thus providing more accurate models in physical applications.

Geometric properties of several families of special functions are discussed in many articles, especially the Bessel functions (see [22–25]) and the generalized hypergeometric functions (see [26–29]). The theory of Bessel functions contains the first and second class Lommel functions as specific solutions of certain second-order differential equations [30–32]. We now review the Lommel function, which is represented by $L_{\rho,\nu}(\xi)$ and provided by

$$L_{\rho,\nu}(\xi) = \frac{\xi^{\rho+1}}{4} \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa \Gamma\left(\frac{\rho-\nu+1}{2}\right) \Gamma\left(\frac{\rho+\nu+1}{2}\right)}{\Gamma\left(\frac{\rho-\nu+3}{2} + \kappa\right) \Gamma\left(\frac{\rho+\nu+3}{2} + \kappa\right)} \left(\frac{\xi}{2}\right)^{2\kappa} \tag{2}$$

which is a particular solution of the nonhomogeneous Bessel differential equation

$$\xi^2 w''(\xi) + \xi w'(\xi) + \left[\xi^2 - \nu^2\right] w(\xi) = \xi^{\rho+1}, \tag{3}$$

where $\frac{\rho-\nu+1}{2}, \frac{\rho+\nu+1}{2} \in \mathbb{C}\backslash\mathbb{Z}^-; \mathbb{Z}^-=\{-1,-2,...\}$ and Γ stands for Euler gamma function, it is clear that the function $L_{\rho,\nu}$ is analytic for all $\xi \in \mathbb{C}$.
Now, we define $L_{\rho,\nu}(\xi)$ as follows:

$$\mathfrak{L}_{\rho,\nu}(\xi) = 4\,\left(\frac{\rho-\nu+1}{2}\right)\left(\frac{\rho+\nu+1}{2}\right)\,\xi^{\frac{1-\rho}{2}}\,L_{\rho,\nu}(\sqrt{\xi}), \tag{4}$$

and using the shifted factorial $(\mathfrak{y})_n$ defined as

$$(\mathfrak{y})_n = \frac{\Gamma(\mathfrak{y}+n)}{\Gamma(\mathfrak{y})} = \begin{cases} 1, & (n=0, \mathfrak{y} \in \mathbb{C}\backslash\{0\}), \\ \mathfrak{y}(\mathfrak{y}+1)\ldots(\mathfrak{y}+n-1), & (n \in \mathbb{N}, \mathfrak{y} \in \mathbb{C}), \end{cases}$$

then $\mathfrak{L}_{\rho,\nu}(\xi)$ can be represented by the following series representation

$$\mathfrak{L}_{\rho,\nu}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}}{4^{\kappa-1}\left(\frac{\rho-\nu+1}{2}+1\right)_{\kappa-1}\left(\frac{\rho+\nu+1}{2}+1\right)_{\kappa-1}}\xi^{\kappa}, \tag{5}$$

for simplicity, let $\mathfrak{n} = \frac{\rho-\nu+3}{2}$ and $\mathfrak{v} = \frac{\rho+\nu+3}{2}$. Thus the function $\mathfrak{L}_{\mathfrak{n},\mathfrak{v}}$ can be defined as following:

$$\mathfrak{L}_{\mathfrak{n},\mathfrak{v}}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}}{4^{\kappa-1}(\mathfrak{n})_{\kappa-1}(\mathfrak{v})_{\kappa-1}}\xi^{\kappa}, \tag{6}$$

the function $\mathfrak{L}_{\mathfrak{n},\mathfrak{v}}(\xi)$ is analytic for all $\xi \in \mathbb{C}$ and $\mathfrak{n}, \mathfrak{v} \in \mathbb{C}\backslash\mathbb{Z}_0^-$, it is clear that $\mathfrak{L}_{\mathfrak{n},\mathfrak{v}}(\xi) \in \mathcal{A}$
Now, we define the new operator

$$\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}} : \mathcal{A} \longrightarrow \mathcal{A},$$

by means of Hadamard product as following:

$$\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi) := \left(\mathfrak{L}_{\mathfrak{n},\mathfrak{v}} * \mathfrak{f}\right)(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}}{4^{\kappa-1}(\mathfrak{n})_{\kappa-1}(\mathfrak{v})_{\kappa-1}} a_{\kappa}\xi^{\kappa}. \tag{7}$$

Remark 1. *We note that by taking $\mathfrak{v} = 1$ in (7), then we get the operator $\mathfrak{L}\mathfrak{M}_{\mathfrak{n}}$ defined as following:*

$$\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}}{4^{\kappa-1}(\mathfrak{n})_{\kappa-1}(\kappa-1)!} a_{\kappa}\xi^{\kappa}.$$

which is related to Bessel functions of the first kind (see [24]).

The operator $\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}}\mathrm{f}(\xi)$, satisfying

$$\xi\left(\mathfrak{L}\mathfrak{M}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi)\right)' = \mathfrak{n}\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi) - (\mathfrak{n}-1)\mathfrak{L}\mathfrak{M}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi) \tag{8}$$

and

$$\xi\left(\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}+g}\mathfrak{f}(\xi)\right)' = \mathfrak{v}\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi) - (\mathfrak{v}-1)\mathfrak{L}\mathfrak{M}_{\mathfrak{n},\mathfrak{v}+g}\mathfrak{f}(\xi) \tag{9}$$

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2. Definitions and Preliminaries

Definition 1. *[2] A fuzzy set is pair (S, F), where S is a set , $S \neq \phi$ and $F : S \to [0,1]$ a membership function.*

The fuzzy subset is likewise covered by the following idea.

Definition 2. *[2] A fuzzy subset of S is a pair (ℓ, F_ℓ), where the support of the fuzzy set (ℓ, F_ℓ) is defined as $\ell = \{x \in S : 0 < F_\ell(x) \leq 1\} = \sup(\ell, F_\ell)$ and $F_\ell : S \to [0,1]$ is belongs to (ℓ, F_ℓ).*

Definition 3. *[2] Fuzzy subsets (Y_1, F_{Y_1}) and (Y_2, F_{Y_2}) of S are equal iff $Y_1 = Y_2$, whereas $(Y_1, F_{Y_1}) \subseteq (Y_2, F_{Y_2})$ iff $F_{Y_1}(\eta) \leq F_{Y_2}(\eta)$, $\eta \in S$.*

Definition 4. *[2] Let $\mathcal{U} \subset \mathbb{C}$ and ξ_0 are a fixed point in $\mathcal{U}$ and let the functions $\mathfrak{f}, \mu \in \mathcal{H}(\mathcal{U})$. $\mathfrak{f}$ is said to be fuzzy subordinate to μ and write $\mathfrak{f} \prec_F \mu$ or $\mathfrak{f}(\xi) \prec_F \mu(\xi)$ if*

$$\mathfrak{f}(\xi_0) = \mu(\xi_0) \text{ and } F_{\mathfrak{f}(\mathcal{U})}(\mathfrak{f}(\xi)) \leq F_{\mu(\mathcal{U})}(\mu(\xi)), \xi \in \mathcal{U},$$

where

$$\mathfrak{f}(\mathcal{U}) = \sup(\mathfrak{f}(\mathcal{U}), F_{\mathfrak{f}(\mathcal{U})}) = \{\mathfrak{f}(\xi) : 0 < F_{\mathfrak{f}(\mathcal{U})}(\mathfrak{f}(\xi)) \leq 1, \ \xi \in \mathcal{U}$$

and

$$\mu(\mathcal{U}) = \sup(\mu(\mathcal{U}), F_{\mu(\mathcal{U})}) = \{\mu(\xi) : 0 < F_{\mu(\mathcal{U})}(\mu(\xi)) \leq 1, \ \xi \in \mathcal{U}.$$

The following lemma is necessary to validate our research.

Lemma 1. *[6] Let $\beta, \gamma \in \mathbb{C}$. Also let $\mu \in \mathcal{A}$ be convex univalent in $\mathcal{U}$ with $Re[\beta\mu(\xi) + \gamma] > 0$ $(\xi \in \mathcal{U}), \mu(0) = 1$ and $p(\xi) \in \mathcal{A}$ with $p(\xi) = 1 + p_1\xi + p_2\xi^2 + ...$ is analytic in $\mathcal{U}$. If*

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})}\left[p(\xi) + \frac{\xi p'(\xi)}{\beta p(\xi) + \gamma}\right] \leq F_{\mu(\mathcal{U})}\mu(\xi),$$

implies

$$F_{p(\mathcal{U})}p(\xi) \leq F_{\mu(\mathcal{U})}\mu(\xi), \quad \xi \in \mathcal{U}.$$

Lemma 2. *[6] Let $\beta, \mathfrak{y} \in \mathbb{C}$. Also let $\mu \in \mathcal{A}$ be convex univalent in $\mathcal{U}$ with $\mu(0) = 1$ and $Re[\beta\mu(\xi) + \mathfrak{y}] > 0\,(\xi \in \mathcal{U})$, and $q(\xi) \in \mathcal{A}$ with $q(0) = 1$ and $q(\xi) \prec \mu(\xi)\,(\xi \in \mathcal{U})$. If $p(\xi) = 1 + p_1\xi + p_2\xi^2 + ...$ is analytic in $\mathcal{U}$,*

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})}\left[p(\xi) + \frac{\xi p'(\xi)}{\beta q(\xi) + \mathfrak{y}}\right] \leq F_{\mu(\mathcal{U})}\mu(\xi) \qquad (\xi \in \mathcal{U}),$$

then

$$F_{p(\mathcal{U})}p(\xi) \leq F_{\mu(\mathcal{U})}\mu(\xi).$$

where $F : \mathbb{U} \to [0,1]$.

Using the operator $\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathrm{f}(\xi)$, we introduce the class $F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, of analytic functions $\mathfrak{f} = \{\mathfrak{f}_1,\mathfrak{f}_2,...,\mathfrak{f}_\vartheta\}$ on open unit disc $\mathcal{U}$ satisfying $\frac{\xi(\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi))'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)} \prec_F \mu(\xi)(\mathfrak{f}_i \in \mathcal{A},\ i = 1,2,...,\vartheta,\ \xi \in \mathcal{U})$, where $\xi^{-1}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathrm{f}_j(\xi) \neq 0$ and μ is convex univalent in $\mathcal{U}$ with $\mu(0) = 1$. Also we describe $\mathscr{F} = \{\mathscr{F}_1,\mathscr{F}_2,...,\mathscr{F}_\vartheta\}$ where $\mathscr{F}_i(\xi) = \frac{\varsigma+1}{\xi^\varsigma}\int_0^\xi t^{\varsigma-1}\mathfrak{f}_i(t)dt \quad (\varsigma \in \mathbb{C}; Re(\varsigma) > 0; i = 1,2,...,\vartheta)$. and proved that $\mathscr{F} \in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, whenever $\mathfrak{f} \in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. There more such classes denoted by $F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, $F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)$ and $F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)$ are introduced and studied here by Fuzzy subordination method and convolutions.

3. Main Results

Throughout this paper, unless otherwise mentioned, we set $\mathfrak{n},\mathfrak{v} > 1$.

3.1. The class $F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$

Definition 5. *Let* $\mathfrak{f} = \{f_1,f_2,...,f_\vartheta\}, \mathfrak{f}_i \in \mathcal{A}, 1 \leq i \leq \vartheta$ *be such that*

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi\left(\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}\right] \leq F_{\mu(\mathcal{U})}\mu(\xi) \qquad (\xi \in \mathcal{U}; i = 1,2,...,\vartheta),$$

where $\xi^{-1}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}f_j(\xi) \neq 0$ *in* $\mathcal{U}$, μ *is convex univalent in* $\mathcal{U}$ *with* $\mu(0) = 1$. *Then we say that* $\mathfrak{f} = \{\mathfrak{f}_1,\mathfrak{f}_2,....,\mathfrak{f}_\vartheta\} \in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$.

Theorem 1. *Let* $\mathfrak{f} = \{\mathfrak{f}_1,\mathfrak{f}_2,...,\mathfrak{f}_\vartheta\} \in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$ *and* $\mathscr{F}(\xi) = \frac{1}{\vartheta}\sum_{i=1}^{\vartheta}f_i(\xi)$. *Then* $\mathscr{F}(\xi)$ *satisfies:*

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi\left(\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathscr{F}(\xi)\right)'}{\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathscr{F}(\xi)}\right] \leq F_{\mu(\mathcal{U})}\mu(\xi) \qquad (\xi \in \mathcal{U}) \tag{10}$$

Proof. Let $\mathfrak{f} = \{\mathfrak{f}_1,\mathfrak{f}_2,...,\mathfrak{f}_\vartheta\} \in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. Then for any $\xi_0 \in \mathcal{U}$, we have

$$\frac{\xi_0\left(\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi_0)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi_0)} \prec_F \mu(\xi)$$

and hence equals to $\mu(w_i)$(say) for some $w_i \in \mathcal{U}, i = 1,2,...,\vartheta$. Then

$$\frac{\sum_{i=1}^{\vartheta}\xi_0\left(\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi_0)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi_0)} = \sum_{i=1}^{\vartheta}\mu(w_i).$$

Let $\mathfrak{f}(\xi) = \xi + \sum\limits_{\kappa=2}^{\infty} a_\kappa \xi^\kappa$. Then, from (7), we see that

$$\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi) = \mathfrak{f}(\xi) * \left\{ \xi +_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}}{4^{\kappa-1}\,(\mathfrak{n})_{\kappa-1}\,(\mathfrak{v})_{\kappa-1}} \xi^k \right\}$$
$$= \left(\mathfrak{L}_{\mathfrak{n},\mathfrak{v}} * \mathfrak{f}\right)(\xi),$$

where

$$\mathfrak{L}_{\mathfrak{n},\mathfrak{v}}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}}{4^{\kappa-1}\,(\mathfrak{n})_{\kappa-1}\,(\mathfrak{v})_{\kappa-1}} \xi^k. \tag{11}$$

Hence

$$\frac{\xi_0 \left(\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathscr{F}(\xi_0)\right)'}{\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathscr{F}(\xi_0)} = \frac{\xi_0[\mathfrak{L}_{\mathfrak{n},\mathfrak{v}}(\xi) * \sum\limits_{i=1}^{\vartheta} \mathfrak{f}_i(\xi_0)]'}{\mathfrak{L}_{\mathfrak{n},\mathfrak{v}}(\xi) * \sum\limits_{j=1}^{\vartheta} \mathfrak{f}_j(\xi_0)}.$$

Since

$$\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}} \sum_{j=1}^{\vartheta} \mathfrak{f}_j(\xi) = \sum_{j=1}^{\vartheta} \mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi),$$

we have

$$\frac{\xi_0 \left(\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathscr{F}(\xi_0)\right)'}{\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathscr{F}(\xi_0)} = \frac{1}{\vartheta} \left[\frac{\xi_0 \sum\limits_{i=1}^{\vartheta} \left(\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi_0)\right)'}{\frac{1}{\vartheta} \sum\limits_{j=1}^{\vartheta} \mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi_0)} \right]$$
$$= \frac{1}{\vartheta} \sum_{i=1}^{\vartheta} \mu(w_i) = \mu(w_0),$$

for some $w_0 \in \mathcal{U}$, since μ is convex in $\mathcal{U}$.

Theorem 2. *Suppose* $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, ..., \mathfrak{f}_\vartheta\} \in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. *Define*

$$\mathscr{F}_i(\xi) = \frac{\varsigma+1}{\xi^\varsigma} \int_0^\xi t^{\varsigma-1}\mathfrak{f}_i(t)dt \quad (\varsigma \in \mathbb{C}; Re(\varsigma) > 0; i = 1, 2, ..., \vartheta).$$

If μ *is bounded in* $\mathcal{U}$ *and* $Re\{\mu(\xi) + \varsigma\} > 0$, *then* $\mathscr{F} = \{\mathscr{F}_1, \mathscr{F}_2, ..., \mathscr{F}_\vartheta\} \in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$.

Proof. From the definition of $\mathscr{F}_i(\xi)$, it follows that

$$\xi\mathscr{F}_i'(\xi) + \varsigma\mathscr{F}_i(\xi) = (\varsigma+1)\mathfrak{f}_i(\xi),$$

and on taking convolution with $\mathfrak{L}_{\mathfrak{n},\mathfrak{v}}$ given by (11), we obtain

$$\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_i(\xi)]' + \varsigma\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_i(\xi) = (\varsigma+1)\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi), \quad i = 1, 2, ..., \vartheta. \tag{12}$$

Let

$$p_i(\xi) = \frac{\vartheta\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_i(\xi)]'}{\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi)}. \tag{13}$$

From (12), we have

$$\frac{p_i(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi) + \varsigma\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_i(\xi) = (\varsigma+1)\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi). \tag{14}$$

Differentiating (14) with respect to ξ, we obtain

$$\frac{p_i'(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi) + \frac{p_i(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi)]' + \varsigma[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_i(\xi)]' = (\varsigma+1)[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'.$$

From (13), we have

$$p_i'(\xi)\frac{\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi)}{\vartheta} + \frac{p_i(\xi)}{\vartheta}\frac{\sum\limits_{i=1}^{\vartheta}p_i(\xi)\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi)}{\vartheta\xi} + \varsigma\frac{p_i(\xi)\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi)}{\vartheta\xi}$$
$$= (\varsigma+1)[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'.$$

Hence

$$p_i'(\xi) + \frac{p_i(\xi)}{\vartheta\xi}\sum_{i=1}^{\vartheta}p_i(\xi) + \varsigma\frac{p_i(\xi)}{\xi} = \frac{(\varsigma+1)[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi)}.$$

Then

$$\frac{\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta}p_i(\xi)+\varsigma} + p_i(\xi) = \frac{(\varsigma+1)\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi)}.\frac{1}{\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta}p_i(\xi)+\varsigma}$$
$$= \frac{(\varsigma+1)\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\left\{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi).\sum\limits_{i=1}^{\vartheta}p_i(\xi) + \varsigma\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_j(\xi)\right\}}.$$

From (14), we have

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta}p_i(\xi)+\varsigma} + p_i(\xi)\right] = F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{(\varsigma+1)\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}(\varsigma+1)\sum\limits_{i=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)}\right] \leq F_{\mu(\mathcal{U})}\mu(\xi), \tag{15}$$

since $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, ..., \mathfrak{f}_\vartheta\} \in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. Now we can write for any $\xi_0 \in \mathcal{U}$,

$$\frac{\frac{1}{\vartheta}\xi_0 p_i'(\xi_0)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}p_j(\xi_0)+\varsigma} + \frac{1}{\vartheta}p_i(\xi_0) = \frac{1}{\vartheta}\mu(w_i),$$

for some $w_i \in \mathcal{U}$. This is true for $i = 1, 2, ..., \vartheta$. Since μ is convex, there exists a $w_0 \in \mathcal{U}$ such that

$$\frac{\xi_0 Q^{'}(\xi_0)}{Q(\xi_0) + \varsigma} + Q(\xi_0) = \mu(w_0),$$

where $Q(\xi) = \frac{1}{\vartheta} \sum\limits_{i=1}^{\vartheta} p_i(\xi)$. Hence

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})} \left[\frac{\xi Q^{'}(\xi)}{Q(\xi) + \varsigma} + Q(\xi) \right] \leq F_{\mu(\mathcal{U})} \mu(\xi).$$

Since $Re\{\mu\}$ is bounded and $Re\{\mu(\xi) + \varsigma\} > 0$, it follows from Lemma 1 that $Q(\xi) \prec_F \mu(\xi)$ $(\xi \in \mathcal{U})$. From (15), we have

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})} \left[\frac{\xi p_i^{'}(\xi)}{Q(\xi) + \varsigma} + p_i(\xi) \right] \leq F_{\mu(\mathcal{U})} \mu(\xi),$$

where $Q(\xi) \prec_F \mu(\xi)$. Lemma 2 gives $p_i(\xi) \prec_F \mu(\xi)$ $(\xi \in \mathcal{U})$, $i = 1, 2, ..., \vartheta$, that is

$$F_{\psi(\mathbb{C}^2 \times \mathcal{U})} \left[\frac{\xi [\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}} \mathscr{F}_i(\xi)]^{'}}{\frac{1}{\vartheta} \sum\limits_{j=1}^{\vartheta} \mathfrak{LM}_{\mathfrak{n},\mathfrak{v}} \mathscr{F}_j(\xi)} \right] \leq F_{\mu(\mathcal{U})} \mu(\xi).$$

Now

$$\mathscr{F}_i(\xi) = \frac{\varsigma + 1}{\xi^{\varsigma}} \int_0^{\xi} t^{\varsigma - 1} \mathfrak{f}_i(t) dt, \quad \varsigma \in \mathbb{C}, Re\varsigma > 0.$$

It can be proved, easily, that , for every i, $1 \leq i \leq \vartheta$,

$$\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}} \mathscr{F}_i(\xi) = \frac{\varsigma + 1}{\xi^{\varsigma}} \int_0^{\xi} t^{\varsigma - 1} \mathfrak{LM}_{\mathfrak{n},\mathfrak{v}} \mathfrak{f}_i(t) dt,$$

and hence

$$\sum_{i=1}^{\vartheta} \mathfrak{LM}_{\mathfrak{n},\mathfrak{v}} \mathscr{F}_i(\xi) = \frac{\varsigma + 1}{\xi^{\varsigma}} \int_0^{\xi} t^{\varsigma - 1} \sum_{i=1}^{\vartheta} \mathfrak{LM}_{\mathfrak{n},\mathfrak{v}} \mathfrak{f}_i(t) dt$$

$$= \frac{\varsigma + 1}{\xi^{\varsigma}} \int_0^{\vartheta} t^{\varsigma} \mathfrak{g}(t) dt,$$

where $\mathfrak{g}(t) = t^{-1} \sum\limits_{i=1}^{\vartheta} \mathfrak{LM}_{\mathfrak{n},\mathfrak{v}} \mathfrak{f}_i(t) \neq 0$, for $\xi \in \mathcal{U}$. Now define

$$\Omega(\xi) = \sum_{k=1}^{\infty} \frac{\varsigma + 1}{\varsigma + k} \xi^{k-1}, \qquad Re(\varsigma) > 0.$$

Then an easy calculations show that

$$\xi^{-1}\sum_{i=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_i(\xi)=(\Omega*\mathfrak{g})(\xi)\neq 0.$$

Thus $\mathscr{F}=\{\mathscr{F}_1,\mathscr{F}_2,...,\mathscr{F}_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$.

Theorem 3. *If* $\mathfrak{f}=\{f_1,f_2,...,f_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, *and* $Re\{\mu\}$ *is bounded in* $\mathcal{U}$, *then* $\mathfrak{f}=\{f_1,f_2,...,f_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n}+g,\mathfrak{v}}(\vartheta;\mu)$ *holds for* $Re\{\mu(\xi)+(\mathfrak{n}-1)\}>0$ *in* $\mathcal{U}$.

Proof. Let

$$p_i(\xi)=\frac{\vartheta\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi)]^{'}}{\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}\qquad(\xi\in\mathcal{U};i=1,2,...,\vartheta).\tag{16}$$

From (8) and (16), we have

$$\frac{1}{\vartheta}p_i(\xi)\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)=\mathfrak{n}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)-(\mathfrak{n}-1)\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi).\tag{17}$$

Differentiating (17) with respect to ξ, we get

$$\frac{\xi}{\vartheta}p_i^{'}(\xi)\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)+\frac{\xi}{\vartheta}p_i(\xi)\sum_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)]^{'}=\mathfrak{n}\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]^{'}-(\mathfrak{n}-1)\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi)]^{'}.$$

Using (16), we obtain

$$\begin{aligned}&\frac{\xi}{\vartheta}p_i^{'}(\xi)\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)+p_i(\xi)\left[\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)]^{'}+\frac{(\mathfrak{n}-1)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)\right]\\ =\;&\mathfrak{n}\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]^{'}.\end{aligned}$$

Then

$$\begin{aligned}&\frac{\frac{\xi}{\vartheta}p_i^{'}(\xi)\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}{\frac{\xi}{\vartheta}\sum\limits_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)]'+\frac{(\mathfrak{n}-1)}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}+p_i(\xi)\\ =\;&\frac{\mathfrak{n}\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]^{'}}{\frac{\xi}{\vartheta}\sum\limits_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)]'+\frac{(\mathfrak{n}-1)}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}.\end{aligned}$$

Using (8), we have

$$\frac{\frac{\xi}{\vartheta}p_i^{'}(\xi)\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}{\frac{\xi}{\vartheta}\sum\limits_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)]'+\frac{(\mathfrak{n}-1)}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}+p_i(\xi)=\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]^{'}}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_j(\xi)}.\tag{18}$$

On the left side of (18), by using (16), we've

$$\frac{\xi p_i^{'}(\xi)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} p_j(\xi)+(\mathfrak{n}-1)}+p_i(\xi)=\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]^{'}}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_j(\xi)}.$$

Since $\mathfrak{f}=\{\mathfrak{f}_1,\mathrm{f}_2,...,\mathrm{f}_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, then we've

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} p_j(\xi)+(\mathfrak{n}-1)}+p_i(\xi)\right]$$
$$=F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi\,[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_j(\xi)}\right]\leq F_{\mu(\mathcal{U})}\mu(\xi),$$
$$i=1,2,\ldots,\vartheta. \tag{19}$$

Therefore for any $\xi_0\in\mathcal{U}$, we have

$$\frac{\xi_0 p_i^{'}(\xi_0)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} p_j(\xi_0)+(\mathfrak{n}-1)}+p_i(\xi_0)=\frac{1}{\vartheta}\mu(w_i)$$

for some $w_0\in\mathcal{U}$. Since μ is convex, there exists a $w_i\in\mathcal{U}$, such that

$$\frac{\frac{\xi_0}{\vartheta}\sum\limits_{i=1}^{n} p_i^{'}(\xi_0)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} p_j(\xi_0)+(\mathfrak{n}-1)}+\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi_0)=\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mu(w_i)=\mu(w_0).$$

Setting $Q(\xi)=\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta}p_i(\xi)$, we have

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi Q^{'}(\xi)}{Q(\xi)+(\mathfrak{n}-1)}+Q(\xi)\right]\leq F_{\mu(\mathcal{U})}\mu(\xi),$$

which by Lemma 1, implies that $Q(\xi)\prec\mu(\xi)$. From (19), we've

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi p_i^{'}(\xi)}{Q(\xi)+(\mathfrak{n}-1)}+p_i(\xi)\right]\leq F_{\mu(\mathcal{U})}\mu(\xi),$$

where $Q(\xi)\prec_F\mu(\xi)$. The using of Lemma 2 gives $p_i(\xi)\prec_F\mu(\xi)$, which implies that $\mathfrak{f}=\{\mathfrak{f}_1,\mathfrak{f}_2,...,\mathfrak{f}_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n}+g,\mathfrak{v}}(\vartheta;\mu)$.

3.2. The class $F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$

Definition 6. *Let $F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$ denote the class of functions $\mathfrak{f}\in\mathcal{A}$ which satisfies*

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_j(\xi)}\right]\leq F_{\mu(\mathcal{U})}\mu(\xi)\qquad(\xi\in\mathcal{U}),$$

where $\mathfrak{g}=\{\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, μ is convex univalent in $\mathcal{U}$ with $\mu(0)=1$.

Theorem 4. *Let $\mathfrak{f}\in F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. If $Re(\mu)$ is bounded in $\mathcal{U}$ and $Re\{\mu(\xi)+\tau\}>0$, then*

$$\mathscr{F}(\xi)=\frac{\tau+1}{\xi^\tau}\int\limits_0^\xi t^{\tau-1}\mathfrak{f}(t)dt\qquad(\xi\in\mathcal{U};\tau\in\mathbb{C},Re(\tau)>0),$$

also belongs to $F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$.

Proof. Since $\mathfrak{f}\in F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, then there exists $\mathfrak{g}=\{\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, such that

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{g}_j(\xi)}\right]\leq F_{\mu(\mathcal{U})}\mu(\xi)\qquad(\xi\in\mathcal{U}).$$

Let

$$G_i(\xi)=\frac{\tau+1}{\xi^\tau}\int\limits_0^\xi t^{\tau-1}\mathfrak{g}_i(t)dt\qquad(Re\tau>0).$$

Then by Theorem 2, we have $G=\{G_1,G_2,...,G_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. Also let

$$p(\xi)=\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)}\qquad(\xi\in\mathcal{U}).\tag{20}$$

Now, from the definitions of G_i and $\mathscr{F}$, we have

$$\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_i(\xi)]'+\tau\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_i(\xi)=(\tau+1)\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{g}_i(\xi),\tag{21}$$

and

$$\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}_i(\xi)]'+\tau\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}(\xi)=(\tau+1)\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi).\tag{22}$$

From (20), (21) and (22), we have

$$\frac{1}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)+\tau\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}(\xi)=(\tau+1)\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi).\tag{23}$$

Differentiating (23) with respect to ξ, and multiplying the resulting equation by ξ, we have

$$\frac{\xi}{\vartheta}p^{'}(\xi)\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)+\frac{\xi}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)]^{'}+\tau\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathscr{F}(\xi)]^{'}=(\tau+1)\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi)]^{'}. \quad (24)$$

From (20) into (24), we have

$$\frac{\xi}{\vartheta}p^{'}(\xi)\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)+\frac{\xi}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)]^{'}+\tau\frac{p(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)=(\tau+1)\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi)]^{'}.$$

Hence, we get

$$\frac{\frac{\xi}{\vartheta}p^{'}(\xi)\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)]'+\frac{\tau}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)}+p(\xi)$$

$$=\frac{(\tau+1)\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi)]^{'}}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)]'+\frac{\tau}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)}=\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi)]^{'}}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{g}_j(\xi)} \quad \text{(by using (21)).}$$

From the above, we have

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi p^{'}(\xi)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}Q_j(\xi)+\tau}+p(\xi)\right]=F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi)]^{'}}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{g}_j(\xi)}\right]\le F_{\mu(\mathcal{U})}\mu(\xi),$$

where $Q_j(\xi)=\dfrac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)]^{'}}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}G_j(\xi)}$. Now $Q_j(\xi)\prec_F\mu(\xi),\quad j=1,2,...,\vartheta$, since $G=\{G_1,G_2,...,G_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$ and μ is a convex univalent. Since $Re\{\mu(\xi)+\tau\}>0$, an application of Lemma 2 implies that $p(\xi)\prec_F\mu(\xi)$, hence $\mathscr{F}\in F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. This complete the proof.

Theorem 5. *If* $\mathfrak{f}\in F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$ *and* $Re(\mu)$ *is bounded in* $\mathcal{U}$*, then* $\mathfrak{f}\in F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$ *holds for* $Re(\mu(\xi)+(\mathfrak{n}-1))>0$ *in* $\mathcal{U}$*.*

Proof. This theorem's proof is removed since it is similar to that of Theorem 3.

3.3. The class $F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)$

Definition 7. *Let* $F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)$, $\alpha\ge 0$, *denote the class of functions* $\mathfrak{f}\in\mathcal{A}$ *satisfying the condition*

$$\begin{aligned}
& F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[J(\alpha;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta)(\xi)\right] \\
= \quad & F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left\{\alpha\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{g}_j(\xi)}+(1-\alpha)\frac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_j(\xi)}\right\} \\
\leq \quad & F_{\mu(\mathcal{U})}\mu(\xi) \quad (\xi\in\mathcal{U}),
\end{aligned}$$

where $\mathfrak{g}=\{\mathfrak{g}_1,\mathfrak{g}_{2,...,}\mathfrak{g}_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$, $\xi^{-1}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{g}_j(\xi)\neq 0$ in $\mathcal{U}$, μ is convex univalent in $\mathcal{U}$ with $\mu(0)=1$.

Remark 2. *We note that* $F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;0,\mu)=F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$.

Theorem 6. *If* $\mathfrak{f}\in F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)$ *and* $Re\{\mu\}$ *is bounded in* $\mathcal{U}$*, then* $\mathfrak{f}\in F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;0,\mu)=F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$ *hold for* $Re\{\mu(\xi)+(\mathfrak{n}-1)\}\geq 0$.

Proof. For $\alpha=0$, the theorem is trivial and hence we can assume that $\alpha\neq 0$. Let

$$p(\xi)=\frac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_j(\xi)} \qquad (\xi\in\mathcal{U}).$$

Then, a simple calculation reveals that

$$\frac{\xi p'(\xi)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}q_j(\xi)+(\mathfrak{n}-1)}+p(\xi)=\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{g}_j(\xi)},$$

where $q_j(\xi)=\dfrac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_j(\xi)}$. Also $\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}q_j(\xi)\prec_F\mu(\xi)$. Since $\mathfrak{f}\in F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)$, we have

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[J(\alpha;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta)(\xi)\right]=F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\alpha\xi p'(\xi)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}q_j(\xi)+(\mathfrak{n}-1)}+p(\xi)\right]\leq F_{\mu(\mathcal{U})}\mu(\xi).$$

Now an application of Lemma 2 gives $p(\xi)\prec_F\mu(\xi)$ which implies $\mathfrak{f}\in F\aleph_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. This completes the proof.

Theorem 7. *For* $\alpha>\beta\geq 0$*, and* $Re\mu(\xi)$ *is bounded in* $\mathcal{U}$*, then* $F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)\subset F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\beta,\mu)$.

Proof. The case $\beta = 0$ was treated in the previous theorem. Hence we assume that $\beta \neq 0$. Suppose that $\mathfrak{f} \in F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)$. Then

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[J(\alpha;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta)(\xi)\right] \leq F_{\mu(\mathcal{U})}\mu(\xi). \tag{25}$$

Let ξ_1 be any arbitrary point in $\mathcal{U}$. Then

$$J(\alpha;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta)(\xi_1) \in \mu(\mathcal{U}).$$

From Theorem 6, we have

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_j(\xi)}\right] \leq F_{\mu(\mathcal{U})}\mu(\xi). \tag{26}$$

Now

$$J(\beta;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta)(\xi) = (1-\frac{\beta}{\alpha})\frac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_j(\xi)} + \frac{\beta}{\alpha}J(\alpha;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta)(\xi).$$

From (25) and (26) it follows that

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\xi_1[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi_1)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_j(\xi_1)}\right] \leq F_{\mu(\mathcal{U})}\mu(\xi)$$

and

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\alpha\frac{\xi_1[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}(\xi_1)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{g}_j(\xi_1)} + (1-\alpha)\frac{\xi_1[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}(\xi_1)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{g}_j(\xi_1)}\right] \leq F_{\mu(\mathcal{U})}\mu(\xi).$$

Now $\mu(\mathcal{U})$ is convex and $\frac{\beta}{\alpha} < 1$, hence we have

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[J(\beta;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,...,\mathfrak{g}_\vartheta)(\xi)\right] \leq F_{\mu(\mathcal{U})}\mu(\xi).$$

showing that $\mathfrak{f} \in F\wp_{\mathfrak{n},\mathfrak{v}}(\vartheta;\beta,\mu)$. This completes the proof.

3.4. The class $F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu)$

Definition 8. *Let* $F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha,\mu),\ \alpha\geq 0$, *denote the class of functions* $\mathfrak{f}\in\mathcal{A}$ *satisfying*

$$\begin{aligned} F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[J(\alpha;\mathfrak{f};\mathfrak{f}_1,\mathfrak{f}_2,...,\mathfrak{f}_\vartheta)(\xi)\right] &= F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left\{\alpha\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_j(\xi)}+(1-\alpha)\frac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}\right\} \\ &\leq F_{\mu(\mathcal{U})}\mu(\xi).\ (\xi\in\mathcal{U}), \end{aligned}$$

where $\mathfrak{f}=\{\mathrm{f}_1,\mathrm{f}_{2,...,}\mathrm{f}_\vartheta\}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$ and $\xi^{-1}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathrm{f}_j(\xi)\neq 0$ in $\mathcal{U}$, μ is convex univalent in $\mathcal{U}$ with $\mu(0)=1$.

Remark 3. *We note that* $F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;0;\mu)=F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$.

Theorem 8. *If* $\mathfrak{f}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha;\mu)$ *and* $Re\{\mu\}$ *is bounded in* $\mathcal{U}$, *then* $\mathfrak{f}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;0;\mu)=F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$ *hold for* $Re\{\mu(\xi)+(\mathfrak{n}-1)\}\geq 0$.

Proof. For $\alpha=0$, the theorem is trivial and hence we can assume that $\alpha\neq 0$. Let

$$p(\xi)=\frac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}\qquad(\xi\in\mathcal{U}).$$

A quick calculation then reveals that

$$\frac{\xi p'(\xi)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}q_j(\xi)+(\mathfrak{n}-1)}+p(\xi)=\frac{\xi[\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n},\mathfrak{v}}\mathfrak{f}_j(\xi)},$$

where $q_j(\xi)=\dfrac{\xi[\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathfrak{LM}_{\mathfrak{n}+1,\mathfrak{v}}\mathfrak{f}_j(\xi)}$. Also $\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}q_j(\xi)\prec_F\mu(\xi)$. Since $\mathrm{f}(\xi)\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha;\mu)$, we have

$$F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[J(\alpha;\mathfrak{f};\mathfrak{f}_1,\mathfrak{f}_2,...,\mathfrak{f}_\vartheta)(\xi)\right]=F_{\psi(\mathbb{C}^2\times\mathcal{U})}\left[\frac{\alpha\xi p'(\xi)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}q_j(\xi)+(\mathfrak{n}-1)}+p(\xi)\right]\leq F_{\mu(\mathcal{U})}\mu(\xi).$$

Now an application of Lemma 2 gives $p(\xi)\prec_F\mu(\xi)$ which implies $\mathfrak{f}\in F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\mu)$. This completes the proof.

Theorem 9. *For* $\alpha>\beta\geq 0$, *and* $Re\{\mu\}$ *is bounded in* $\mathcal{U}$, *then* $F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\alpha;\mu)\subset F\mathfrak{R}_{\mathfrak{n},\mathfrak{v}}(\vartheta;\beta;\mu)$.

Proof. This theorem's proof is removed since it is similar to that of Theorem 7.

Remark 4. *We can get the same results if we used equation (9).*

4. Conclusion

The new findings of this work are related to new classes of analytic normalized functions in $\mathcal{U}$. The novel results from the investigation reported in this work lead to an advancement in the theory of fuzzy differential subordination to introduce some classes of univalent functions. The Introduction in Section 1 covers the Lommel function $\mathfrak{LM}_{\mathfrak{u},\mathfrak{v}}\mathfrak{f}(\xi)$, the fundamental ideas required for the study, and the rationale behind the topic's investigation. The main finding is presented in Section 2. The current effort offers valuable information to advance the recently initiated research avenues. The outcome of the present investigation could inspire the use of this operator for introducing other new classes of analytic functions . In addition to their theoretical significance, Lommel functions are computationally relevant. Numerical methods for evaluating these functions have been developed, allowing researchers and engineers to apply them effectively in simulations and predictive models. The development of algorithms and software libraries that incorporate the computation of Lommel functions has greatly enhanced our ability to analyze complex systems featuring cylindrical symmetry. Consequently, the Lommel function stands out as a rich area of study, interlinking pure mathematics with practical technological applications, offering insights that could lead to advancements in fields ranging from telecommunications to structural engineering.

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