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Qualitative Study on Semi-Analytical Methods for Solving Nonlinear Time-Fractional partial differential equations

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Abstract. This paper focuses on finding Semi-Analytical solutions of nonlinear partial differential equations of fractional order by using four techniques such as Sumudu decomposition, Natural decomposition, Adomian decomposition and modified Laplace variational iteration methods. The fractional derivatives are described in the Caputo sense. In these methods, the solution manifests as a convergent series with conveniently computable components. Numerical results show that the four approaches are easy to implement and accurate when applied to partial differential equations of fractional order, although there are some distinct differences between the methods studied, which depend on the nature of the equations and the conditions associated with them.

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1. Introduction

Fractional calculus is a field of mathematics study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way frictional are an outgrowth of exponents with integer values. Fractional calculus has been part of mathematics and science literature for more than three centuries. It has been used to model physical and engineering processes that are found to be best described by fractional differential equations [1–4]. Recent research has studied the applications of fractional calculus in several domains, including fractional-order epidemic models [5], fractional-order Optimal Control Problem [6]. Most phenomena in nature are described by nonlinear differential equations. Therefore, scientists in different branches of science try to solve them. But because of the nonlinear part of these groups of equations, finding an exact solution is not easy [7]. Approximation and numerical techniques must be used. The Adomian decomposition method [8–11], the Sumudu decomposition method [12], the natural

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decomposition method [13] and modified Laplace variational iteration method [14] are relatively new approaches to provide an analytical approximation to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians. For nonlinear models, the previous methods have shown dependable results and give analytical approximations solutions that converge very rapidly.

This paper is organized as follows: Section 2 provides a brief overview of Basic Concept of Fraction Calculus. Section 3 presents the description of proposed Transformations Methods and the analysis of these methods given by Section 4. Section 5 includes the numerical experiments and the comparison among these methods. The paper ends with conclusion and final remarks.

2. Basic Concept of Fractional Calculus

This section is devoted to a description of the basic ideas of fractional calculus, which are essential to comprehending complex systems, and explores the fundamental meanings of integrals and fractional derivatives, as well as their different forms and use.

Definition 1. *[8] A real function $f(x), x > 0$, is siaid to be in the space $C_\mu, \mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^m \in C_\mu, m \in N$.*

Definition 2. *[8] The Riemann-Liouville fractional integral operator of order $\alpha \geq \mu$, of function $f \in C_\mu, \mu \geq -1$, is defined as*

$$J^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, & \alpha > 0, x > 0, \\ f(x), & \alpha = 0. \end{cases}$$

We mention only the following Properties of the operator J^α.
For function $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,

(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,

(iii) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative exhibits certain drawbacks for modeling real-world occurrences using fractional differential equations, as noted in [8, 12]. Consequently, we will now provide a modified fractional differentiation operator D^α. presented by Caputo in his study of viscoelasticity theory [15].

Definition 3. *[12] For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha > 0$ is defined as:*

$$D^\alpha f(x,t) = \frac{d^\alpha f(x,t)}{dt^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(x,\tau)}{d\tau^m}, & m-1 < \alpha \leq m \\ \frac{d^m f(x,t)}{dt^m}, & \alpha = m, \in N \end{cases}$$

One may refer to the cited sources for the mathematical characteristics of fractional derivatives and integrals.

In Section 5, we demonstrate the application of semi-analytical methods for solving known and commonly encountered categories of fractional partial differential equations (FPDEs), such as nonlinear time-fractional advection partial differential equation and nonlinear time-fractional hyperbolic partial differential equation.

3. The proposed Transformations Methods

There are several integral transformations available for solving fractional differential equations. The Laplace transform is the most extensively utilized.

The core idea behind the Laplace transforms is mopping of a time-domain function $f(t)$ into an s-domain function $F(s)$ using an integral transformation. By this, differentiation and integration operations in the time domain are made equivalent to multiplication and division by s in the Laplace domain [16, 17]. In the Sumudu transform, the differentiation and integration operations in the t-domain are made equivalent to division and multiplication by u in a u-domain. This makes it possible to treat the variable u and transformed function $f(u)$ as replicas of t and $f(t)$, respectively. It is even possible to express them in the same engineering units as t and $f(t)$ so that the consistency of units in a differential equation describing a physical process can be maintained even after the transformation [16]. The Natural transform is derived from the Fourier integral, and it converges to the Laplace transform and the Sumudu transform [18].

Definition 4. *[19] The Laplace transform of $f(t)$:*

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Definition 5. *[12] The Sumudu transform over the following set of functions*

$$\mathbb{A} = \left\{ f(t) \middle| \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{t_j}},\ \mathit{if}\, t \in (-1)^j \times [0, \infty) \right\},$$

is defined as, for $u \in (\tau_1, \tau_2)$, we have

$$\mathbb{S}[f(t)] = G(u) = \int_0^\infty f(ut)e^{-t}dt = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t)dt.$$

Definition 6. *[18] The Natural transform over the following set of functions*

$$\mathbb{A} = \left\{ f(t) \middle| \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{t_j}},\ \mathit{if}\, t \in (-1)^j \times [0, \infty) \right\},$$

is defined as

$$\mathbb{N}[f(t)] = R(s, u) = \int_0^\infty e^{-st} f(ut)dt, \quad s > 0, u > 0.$$

Remark 1. *If $\mathbb{S}[f(t)] = G(u), L[f(t)] = F(s)$ and $\mathbb{N}[f(t)] = R(s,u)$, then the relationship between Sumudu and Laplace transforms [20]:*
The transformations are connected via a change of variables:

- $G(u) = \frac{1}{u}F\left(\frac{1}{u}\right)$.

Equivalently:

- $F(s) = \frac{1}{s}G\left(\frac{1}{s}\right)$, *where* $\mathrm{s} = \frac{1}{u}$ *and* $u = \frac{1}{\mathrm{s}}$.

Relationship Between Sumudu and Natural Transforms [18]:

$$R(s,u) = \frac{1}{s}G\left(\frac{u}{s}\right).$$

Relationship Between Natural and Laplace Transforms [18]:

$$R(s,u) = \frac{1}{u}\ \mathrm{F}\left(\frac{s}{u}\right).$$

Definition 7. *[19] The Laplace transform of the Caputo fractional derivative is defined as*

$$\mathbb{L}\left[D^{\alpha}f(x)\right] = s^{\alpha}\mathbb{L}[f(x)] - \sum_{k=0}^{m-1} s^{\alpha-k-1}f^{k}(0), \quad m-1 < \alpha \leq m$$

Definition 8. *[12] The Sumudo transform of the Caputo fractional derivative is defined as*

$$\mathbb{S}\left[D_t^{\alpha}f(x,t)\right] = \frac{\mathbb{S}[f(x,t)]}{u^{\alpha}} - \sum_{k=0}^{m-1}\frac{f^{k}(x,0)}{u^{\alpha-k}}, \quad m-1 < \alpha \leq m$$

Definition 9. *[21] The Natural transform of the Caputo fractional derivative is defined as*

$$\mathbb{N}\left[D^{\alpha}f(x)\right] = \frac{s^{\alpha}}{u^{\alpha}}\mathbb{N}[f(x)] - \sum_{k=0}^{m-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{k}(0), \quad m-1 < \alpha \leq m$$

4. Analysis of Methods

The aim of this section is to discuss the use of the transform algorithm for nonlinear partial fractional differential equations.

We consider the following time-fractional partial differential equation [8]:

$$D_t^{\alpha}u(x,t) = f\left(u, u_x, u_{xx}\right) + g(x,t), \quad m-1 < \alpha \leq m, \tag{1}$$

Where $D_t^{\alpha} = \frac{d^{\alpha}}{dt^{\alpha}}$ is the Caputo fractional derivative of order $\alpha, m \in \mathrm{N}, f$ is a nonlinear function and g is the source function. The initial and boundary conditions associated with (1) are of the form

$$\begin{aligned} &u(x,0) = h(x), \quad 0 < \alpha \leq 1 \\ &u(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \quad t > 0, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &u(x,0)=h(x), \frac{du(x,0)}{dt}=k(x), \quad 1<\alpha\leq 2 \\ &u(x,t)\to 0 \quad \text{as} \quad |x|\to\infty, \quad t>0. \end{aligned} \tag{3}$$

The decomposition method requires that the nonlinear fractional differential Eq. (1) be expressed in terms of operator form as

$$D_t^\alpha u(x,t)+Lu(x,t)+Nu(x,t)=g(x,t), \quad x>0, \tag{4}$$

Where L is a linear operator, which might include other fractional derivatives of order less than α, N is a nonlinear operator that might also include other fractional derivatives of order less than α, $g(x,t)$ and D_t^α are defined as in Eq. (1).

4.1. Adomian Decomposition and Transform-Based Variants

Although the Adomian decomposition method, the Sumudu decomposition method, and the natural decomposition method apply different integral transforms in their formulations, the obtained decomposition components and numerical results are identical for all the considered problems.

Adomian Decomposition Method:

Applying the operator J^α, the inverse of the operator D_t^α, to both sides of Eq. (4) yields [8]:

$$u(x,t)=\sum_{k=0}^{m-1}\frac{d^k u}{dt^k}\left(x,0^+\right)\frac{t^k}{k!}+J^\alpha g(x,t)-J^\alpha[Lu(x,t)+Nu(x,t)]. \tag{5}$$

The Adomian decomposition method suggests that the solution $u(x,t)$ be decomposed into an infinite series of components.

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{6}$$

And the nonlinear function in Eq. (5) is decomposed as follows:

$$Nu=\sum_{n=0}^{\infty}A_n, \tag{7}$$

Here A_n are the so-called Adomian polynomials. The general form of the formula for A_n Adomian polynomials is

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{k=0}^{n}\lambda^k u_k\right)\right]_{\lambda=0}. \tag{8}$$

The Adomian polynomial A_n can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [9].

Substituting the series of decompositions (6) and (7) into both sides of (5) gives.

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1} \frac{d^k u}{dt^k}\left(x,0^+\right)\frac{t^k}{k!} + J^\alpha g(x,t) - J^\alpha\left[L\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n\right]. \quad (9)$$

From this equation, the iterates are determined by the following recursive way.

$$\begin{aligned} u_0(x,t) &= \sum_{k=0}^{m-1} \frac{d^k u}{dt^k}\left(x,0^+\right)\frac{t^k}{k!} + J^\alpha g(x,t), \\ u_1(x,t) &= -J^\alpha\left[Lu_0 + A_0\right], \\ u_2(x,t) &= -J^\alpha\left[Lu_1 + A_1\right], \\ &\vdots \\ u_{n+1}(x,t) &= -J^\alpha\left[Lu_n + A_n\right]. \end{aligned} \quad (10)$$

Finally, we approximate the solution $u(x,t)$ by the truncated series.

$$\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t) \quad \text{and} \quad \lim_{N\to\infty}\phi_N(x,t) = u(x,t). \quad (11)$$

Sumudu Decomposition Method:
Applying the Sumudu transform on both sides of Eq. (4) yields [12]

$$\mathbb{S}\left[D_t^\alpha u(x,t)\right] = \mathbb{S}[g(x,t) - Lu(x,t) - Nu(x,t)]. \quad (12)$$

Using the differentiation property of the Sumudu transform for Caputo, we get

$$u^{-\alpha}\mathbb{S}[u(\mathrm{x},\mathrm{t})] - \sum_{\mathrm{k}=0}^{\mathrm{m}-1} u^{-(\alpha-\mathrm{k})}u^{\mathrm{k}}(\mathrm{x}.0) = \mathbb{S}[\mathrm{g}(\mathrm{x},\mathrm{t}) - \mathrm{Lu}(\mathrm{x},\mathrm{t}) - \mathrm{Nu}(\mathrm{x},\mathrm{t})]. \quad (13)$$

Simplify, we get.

$$u(x,t) = \sum_{k=0}^{m-1} u^{(k)} f_k(x) + \mathbb{S}^{-1}\left(u^\alpha \mathbb{S}[g(x,t)]\right) - \mathbb{S}^{-1}\left(u^\alpha \mathbb{S}[\mathrm{Lu}(x,t) + \mathrm{Nu}(x,t)]\right). \quad (14)$$

Substituting the series of decompositions (6) and (7) into both sides of (5) gives,

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) = {} & \sum_{k=0}^{m-1} u^{(k)} f_k(x) + \mathbb{S}^{-1}\left(u^\alpha \mathbb{S}[g(x,t)]\right) \\ & - \mathbb{S}^{-1}\left[u^\alpha \mathbb{S}\left(L\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n\right)\right], \end{aligned} \quad (15)$$

where A_n defined in Eq. (8).

On comparing both sides of the Eq. (15), we get

$$
\begin{aligned}
u_0(x,t) &= \sum_{k=0}^{m-1} u^{(k)} f_k(x) + \mathbb{S}^{-1}\left(u^{\alpha}\mathbb{S}[g(x,t)]\right),\\
u_1(x,t) &= -\mathbb{S}^{-1}\left(u^{\alpha}\mathbb{S}\left[Lu_0 + A_0\right]\right),\\
u_2(x,t) &= -\mathbb{S}^{-1}\left(u^{\alpha}\mathbb{S}\left[Lu_1 + A_1\right]\right),\\
&\vdots\\
u_{n+1}(x,t) &= -\mathbb{S}^{-1}\left(u^{\alpha}\mathbb{S}\left[Lu_n + A_n\right]\right), \quad n \geq 1.
\end{aligned}
\tag{16}
$$

Finally, we approximate the solution $u(x,t)$ by the truncated series.

$$
\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t) \quad \text{and} \quad \lim_{N\to\infty} \phi_N(x,t) = u(x,t). \tag{17}
$$

Natural Decomposition Method:
We apply the Natural transform on both sides of Eq. (4), which yields [13].

$$
\mathbb{N}\left[D_t^{\alpha} u(x,t)\right] = \mathbb{N}[g(x,t) - Lu(x,t) - Nu(x,t)]. \tag{18}
$$

Using the differentiation property of the Natural transform for Caputo, we get

$$
\frac{s^{\alpha}}{u^{\alpha}}\mathbb{N}[u(x,t)] - \sum_{k=0}^{m-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} u^k(x,0) = \mathbb{N}[g(x,t) - Lu(x,t) - Nu(x,t)]. \tag{19}
$$

Simplify, we get

$$
\begin{aligned}
u(x,t) = \sum_{k=0}^{m-1} \frac{s^{-(k+1)}}{u^{-k}} u^k(x,0) &+ \mathbb{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}[g(x,t)]\right)\\
&- \mathbb{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}[Lu(x,t) + Nu(x,t)]\right).
\end{aligned}
\tag{20}
$$

Substitution the decomposition series (6) and (7) into both sides of (20) gives.

$$
\begin{aligned}
\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1} \frac{s^{-(k+1)}}{u^{-k}} u^k(x,0) &+ \mathbb{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}[g(x,t)]\right)\\
&- \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}\left(L\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n\right)\right].
\end{aligned}
\tag{21}
$$

Where A_n defined in Eq. (8).

From this equation, the iterates are determined by the following recursive way

$$
u_0(x,t) = \sum_{k=0}^{m-1} \frac{s^{-(k+1)}}{u^{-k}} u^k(x,0) + \mathbb{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}[g(x,t)]\right),
$$

$$
\begin{aligned}
u_1(x,t) &= -\mathbb{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{N}\left[Lu_0 + A_0\right]\right),\\
u_2(x,t) &= -\mathbb{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{N}\left[Lu_1 + A_1\right]\right),\\
&\vdots\\
u_{n+1}(x,t) &= -\mathbb{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{N}\left[Lu_n + A_n\right]\right), \quad n \geq 1.
\end{aligned} \tag{22}
$$

Finally, we approximate the analytical solution $u(x,t)$ by truncated series:

$$
\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t) \quad \text{and} \quad \lim_{N\to\infty} \phi_N(x,t) = u(x,t).
$$

4.2. Modified Laplace Variational Iteration Method

The new tactic of modified Laplace variation iteration technique is applied as follows. We apply Laplace transform on both sides of Eq. (4) yields [14].

$$
\mathbb{L}\left[D_t^\alpha u(x,t)\right] = \mathbb{L}[g(x,t) - Lu(x,t) - Nu(x,t)]. \tag{23}
$$

Using the differentiation property of Laplace transform for Caputo, we get

$$
s^\alpha \mathbb{L}[u(x,t)] - \sum_{k=0}^{m-1} s^{(\alpha-k-1)} u^k(x,0) = \mathbb{L}[g(x,t) - Nu(x,t)], \tag{24}
$$

$$
u(x,s) = \frac{h(x)}{s} + \frac{k(x)}{s^2} + \frac{\mathbb{L}}{s^\alpha}[g(x,t)] - \frac{\mathbb{L}}{s^\alpha}[Nu(x,t)], \tag{25}
$$

$$
u(x,t) = f(x,t) - \mathbb{L}^{-1}\left[\frac{\mathbb{L}}{s^\alpha}[Nu(x,t)]\right]. \tag{26}
$$

Differentiating the results obtained above with respect to α, we then get the value of the general Lagrange multiplier, for the correction functional iterative formula to equal one.

$$
\frac{du(x,t)}{dt} = \frac{df(x,t)}{dt} - \frac{d}{dt}\mathbb{L}^{-1}\left[\frac{\mathbb{L}}{s^\alpha}[Nu(x,t)]\right]. \tag{27}
$$

Therefore, equation (27) can be put above in the following formula:

$$
u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left[\frac{du_n(x,y)}{dy} - \frac{df(x,y)}{dy} + \frac{d}{dy}\mathbb{L}^{-1}\left[\frac{\mathbb{L}}{s^\alpha}\left[Nu_n(x,y)\right]\right]\right] dy. \tag{28}
$$

The general Lagrange multiplier for equation (28) can be identified optimally via variation theory to get

$$
1 + \lambda|_{y=t} = 0, \quad \lambda|_{y=t} = -1.
$$

Substituting $\lambda = -1$ into equation (28) we get the iterative formula $n = 1, 2, \cdots$ follows:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{du_n(x,y)}{dy} - \frac{df(x,y)}{dy} + \frac{d}{dy}\mathbb{L}^{-1}\left[\frac{\mathbb{L}}{s^\alpha}[Nu_n(x,y)] \right] \right] dy. \quad (29)$$

Equation (29) is the new modified function of the Laplace transform and variational iteration method. Start with the initial iteration $u(x,t) = u(x,0)$. The exact solution is present as the sequent approximation $u_{n+1}(x,t)$, $n = 1, 2, \cdots$. In other words,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

5. Numerical Experiments and Discussion of Results

In this section, we shall illustrate the four techniques by several examples. These examples are somewhat artificial in the sense that the exact answer, for the special cases $\alpha = 1, \alpha = 2$ is known in advance, and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the analytical techniques and to examine the effect of varying the order of the time-fractional derivative on the behavior of the solution. All the results are calculated by using the software MATLAB.

Example 1. *Consider the nonlinear time-fractional advection partial differential equation.*

$$D^\alpha u(x,t) + u(x,t)u_x(x,t) = x + xt^2, \quad t > 0, x \in R, 0 < \alpha \leq 1, \quad (30)$$

subject to the initial condition

$$u(x,0) = 0. \quad (31)$$

The values of $\alpha = 1$ is the only case for which we know the exact solution $u(x,t) = xt$.

To solve the problem using the decomposition method [8], we simply substitute (30) and the initial conditions (31) into (10), to obtain the following recurrence relation.

$$u_0(x,t) = u(x,0) + J^\alpha \left(x + xt^2\right) = x\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),$$

$$u_{n+1}(x,t) = -J^\alpha \left[A_n\right], \quad n \geq 0. \quad (32)$$

Where A_n are the Adomian polynomials for the nonlinear function $N = u(x,t)u_x(x,t)$. The few components of Adomian polynomials, are given by [9]:

$$\begin{aligned} A_0 &= u_0 u_{0x}, \\ A_1 &= u_{0x}u_1 + u_0 u_{1x}, \\ A_2 &= u_{0x}u_2 + u_{1x}u_1 + u_{2x}u_0, \end{aligned} \quad (33)$$

and we can continue the calculations to find A_3 and so on by the same manner.

In view of (32), the first few components of the decomposition series are derived as follows

$$u_0(x,t) = x\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \tag{34}$$

$$u_1(x,t) = -x\left[\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)^2\Gamma(3\alpha+5)}\right], \tag{35}$$

$$u_2(x,t) = 2x\left[\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)} + \dots\right]. \tag{36}$$

And soon, in this manner the rest of components of the decomposition series can be obtained. The first three terms of the decomposition series (6) are given by

$$\begin{aligned} u(x,t) = x\Bigg[&\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} \\ &- \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)^2\Gamma(3\alpha+5)} + \frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \cdots\Bigg]. \end{aligned} \tag{37}$$

To solve the problem using the Sumudu decomposition method, we simply substitute (30) and the initial conditions (31) into (16), to obtain the following recurrence relation

$$u_0(x,t) = \sum_{k=0}^{m-1} u^{(k)} f_k(x) + \mathbb{S}^{-1}\left(u^\alpha \mathbb{S}\left[x + xt^2\right]\right) = x\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),$$

$$u_{n+1}(x,t) = -\mathbb{S}^{-1}\left(u^\alpha \mathbb{S}\left[A_n\right]\right), \quad n \geq 1. \tag{38}$$

The few components of Adomian polynomials show in (33).

In view of (38), the first few components of the decomposition series are derived as in equations (35) and (36) and so on, in this manner the rest of components of the Sumudu decomposition series can be obtained.

The first three terms of the decomposition series are given by u(x, t)as written in equation (37).

To solve the problem using the Natural decomposition method, we simply substitute (30) and the initial conditions (31) into (22), to obtain the following recurrence relation.

$$u_0(x,t) = \sum_{k=0}^{m-1} \frac{s^{-(k+1)}}{u^{-k}} u^k(x,0) + \mathbb{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{N}[g(x,t)]\right) = x\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),$$

$$u_{n+1}(x,t) = -\mathbb{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{N}\left[A_n\right]\right), \quad n \geq 1. \tag{39}$$

In view of (39), the first few components of the decomposition series are derived as in equations (35) and (36) and so on, in this manner the rest of components of the Natural decomposition series can be obtained. The first three terms of the decomposition series are given by u(x, t) as written in equation (37). To solve the problem using the Modified Laplace Variational Iteration Method [14], by enforcement of the Laplace transform to the sides of equation (30) and using the initial conditions (31), we get

$$u(x,s) = \frac{x}{s^{1+\alpha}} + \frac{2x}{s^{3+\alpha}} - \left(\frac{1}{s^{\alpha}} L\left(u(x,t)u_x(x,t)\right)\right). \tag{40}$$

Taking the inverse Laplace transform to the sides of equation (40), we obtain

$$u(x,t) = \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(\alpha+3)} - L^{-1}\left(\frac{1}{s^{\alpha}} L\left(u(x,t)u_x(x,t)\right)\right). \tag{41}$$

Now the new tactic of the modified Laplace variational iteration technique is instituted on (27) from equation (41), and we get

$$\frac{du(x,t)}{dt} = \frac{d}{dt}\left(\frac{xt^{\alpha}}{\Gamma(1+\alpha)}\right) + \frac{d}{dt}\left(\frac{2xt^{\alpha+2}}{\Gamma(\alpha+3)}\right) - \frac{d}{dt}\left(L^{-1}\left(\frac{1}{s^{\alpha}} L\left(u(x,t)u_x(x,t)\right)\right)\right). \tag{42}$$

We simply substitute equation (42) and the initial condition equation (31) into equation (29), by the new modified function, we find

$$\begin{aligned}
u_0(x,t) =& u(x,0) = 0\\
u_1(x,t) =& \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(\alpha+3)}\\
u_2(x,t) =& \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(\alpha+3)} - \left(\frac{\Gamma(1+2\alpha)xt^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(3\alpha+1)}\right) - \frac{4x\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(1+\alpha)\Gamma(\alpha+3)\Gamma(3\alpha+3)}\\
&- \frac{4x\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)^2\Gamma(3\alpha+5)}.
\end{aligned}$$

Tables (1, 2, 3) show the approximate solutions for Eq. (30) obtained for different values of a using the decomposition method, Sumudu decomposition method, Natural decomposition method, Adomian decomposition method, and Modified Laplace Variational Iteration. It is to be noted that only three terms of this decomposition series and the Modified Laplace Variational Iteration were used in evaluating the approximate solutions for obtaining the results. The accuracy can be improved by computing more terms of the approximate solution.

Table 1: Numerical values when $\alpha = 0.5$ for Eq. (30).

t	x	U_{ADM}	$U_{\text{NDM}} - U_{\text{SDM}}$	U_{MLVIM}
0.2	0.25	0.1128438450	0.1128438450	0.1067989518
	0.5	0.2256876900	0.2256876900	0.2135979036
	0.75	0.3385315349	0.3385315349	0.3203968554
	1	0.4513753799	0.4513753799	0.4271958072
0.4	0.25	0.1640101327	0.1640101327	0.1257268756
	0.5	0.3280202655	0.3280202655	0.2514537512
	0.75	0.4920303982	0.4920303982	0.3771806267
	1	0.6560405310	0.6560405310	0.5029075023
0.6	0.25	0.2440592839	0.2440592839	0.1176010458
	0.5	0.4881185678	0.4881185678	0.2352020917
	0.75	0.7321778516	0.7321778516	0.3528031375
	1	0.9762371355	0.9762371355	0.4704041833

Table 2: Numerical values when $\alpha = 0.75$ for Eq. (30).

t	x	U_{ADM}	$U_{\text{NDM}} - U_{\text{SDM}}$	U_{MLVIM}
0.2	0.25	0.0787870110	0.0787870110	0.0784882208
	0.5	0.1575740220	0.1575740220	0.1569764416
	0.75	0.2363610330	0.2363610330	0.2354646624
	1	0.3151480440	0.3151480440	0.3139528832
0.4	0.25	0.1289407026	0.1289407026	0.1245800056
	0.5	0.2578814051	0.2578814051	0.2491600112
	0.75	0.3868221077	0.3868221077	0.3737400168
	1	0.5157628102	0.5157628102	0.4983200224
0.6	0.25	0.1772526142	0.1772526142	0.1544935814
	0.5	0.3545052283	0.3545052283	0.3089871628
	0.75	0.5317578425	0.5317578425	0.4634807443
	1	0.7090104566	0.7090104566	0.6179743257

Table 3: Numerical values when $\alpha = 1$ for Eq. (30).

t	x	U_{ADM}	$U_{\text{NDM}} - U_{\text{SDM}}$	U_{MLVIM}	U_{Exact}
0.2	0.25	0.0500001744	0.0500001744	0.0499892825	0.0500000000
	0.5	0.1000003488	0.1000003488	0.0999785651	0.1000000000
	0.75	0.1500005233	0.1500005233	0.1499678476	0.1500000000
	1	0.2000006977	0.2000006977	0.1999571302	0.2000000000
0.4	0.25	0.1000229939	0.1000229939	0.0996521651	0.1000000000
	0.5	0.2000459878	0.2000459878	0.1993043302	0.2000000000
	0.75	0.3000689818	0.3000689818	0.2989564952	0.3000000000
	1	0.4000919757	0.4000919757	0.3986086603	0.4000000000
0.6	0.25	0.1504123340	0.1504123340	0.1472969143	0.1500000000
	0.5	0.3008246680	0.3008246680	0.2945938286	0.3000000000
	0.75	0.4512370020	0.4512370020	0.4418907429	0.4500000000
	1	0.6016493361	0.6016493361	0.5891876571	0.6000000000

Table (4) shows the absolute error between the exact and approximate solutions for Eq. (30) produced using Adomian Decomposition Method and Modified Laplace Variational Iteration Method. The results are computed after applying three iterations of each method for various values.

Table 4: The absolute error for $\alpha = 1$ for Eq. (30).

t	x	U_{ADM}	U_{MLVIM}
0.2	0.25	0.0000001744	0.0000107175
	0.5	0.0000003488	0.0000214349
	0.75	0.0000005233	0.0000321524
	1	0.0000006977	0.0000428698
0.4	0.25	0.0000229939	0.0003478349
	0.5	0.0000459878	0.0006956698
	0.75	0.0000689818	0.0010435048
	1	0.0000919757	0.0013913397
0.6	0.25	0.0004123340	0.0027030857
	0.5	0.0008246680	0.0054061714
	0.75	0.0012370020	0.0081092571
	1	0.0016493361	0.0108123429

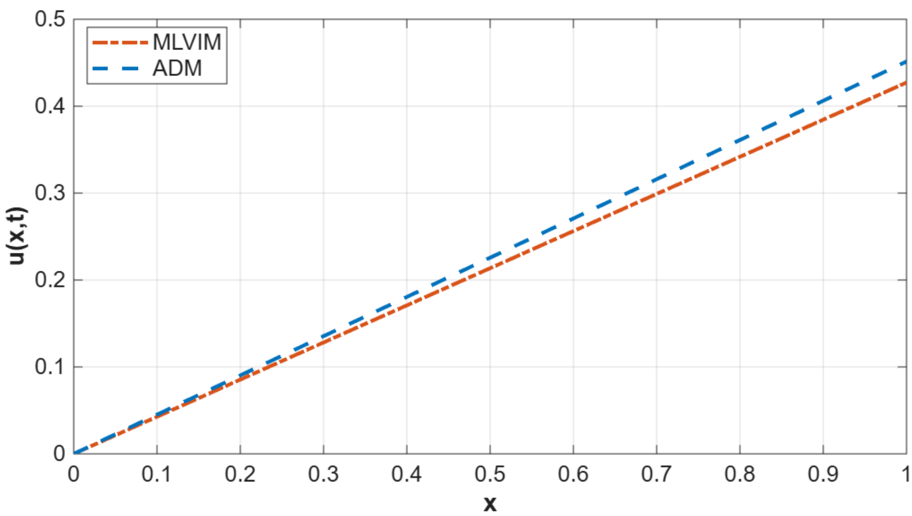

Figure 1: Comparison of ADM and MLVIM solution for the first three approximations $\alpha = 0.5$, $t = 0.2$, with $x = 0 : 1$, for equation (30).

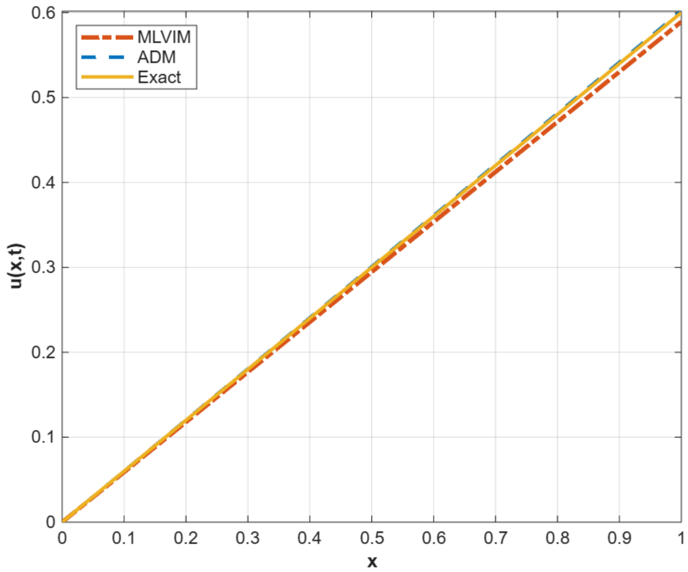

Figure 2: Comparison of ADM and MLVIM solution for the first three approximations $\alpha = 1, t = 0.6$, with $x = 0 : 1$, for equation (30).

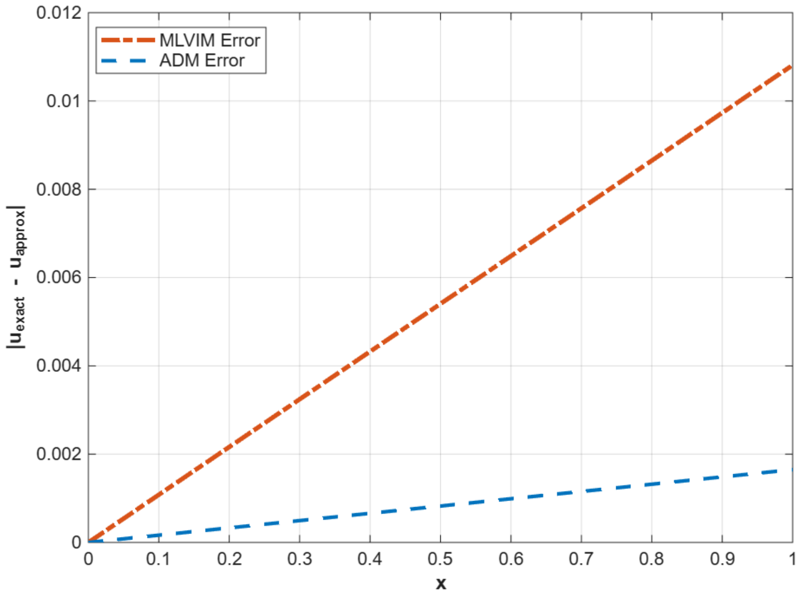

Figure 3: Comparison of the absolute error between the ADM and MLVIM solution for the first three approximations $\alpha = 1, t = 0.6$, with $\mathrm{x} = 0 : 1$, for Eq. (30).

Example 2. *Consider the nonlinear time-fractional hyperbolic partial differential equation.*

$$D^{\alpha}u(x,t) = \frac{d}{dx}\left(u(x,t)\frac{u(x,t)}{dx}\right), \quad t > 0, \quad x \in R, \quad 1 < \alpha \leq 2, \tag{43}$$

subject to the initial condition

$$u(x,0)=x^2, \quad u_t(x,0)=-2x^2. \tag{44}$$

The values of $\alpha=2$ is the only case for which we know the exact solution

$$u(x,t)=\left(\frac{x}{t+1}\right)^2.$$

To solve the problem using the decomposition method [8], we simply substitute (43) and the initial conditions (44) into (10), to obtain the following recurrence relation.

$$\begin{aligned} u_0(x,t) &= u(x,0)+J^{\alpha}\left(x+xt^2\right)=x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \\ u_{n+1}(x,t) &= J^{\alpha}\left[A_n\right], \quad n\geq 0, \end{aligned} \tag{45}$$

Where A_n are the Adomian polynomials for the nonlinear function $N=u(x,t)u_x(x,t)$, The few components of Adomian polynomials, are given by [9]

$$\begin{aligned} A_0 &= u_0u_{0x} \\ A_1 &= u_{0x}u_1+u_0u_{1x}, \\ A_2 &= u_{0x}u_2+u_{1x}u_1+u_{2x}u_0, \end{aligned} \tag{46}$$

and we can continue the calculations to find A_3 and so on by the same manner .

In view of (45), the first few components of the decomposition series are derived as follows $u_0(x,t)=x^2(1-2t)$,

$$u_1(x,t)=6x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{4t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \tag{47}$$

$$\begin{aligned} u_2(x,t)=72x^2 &\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)}-\frac{2\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}\right., \\ &\left.+\frac{8\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}-\frac{16\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\right). \end{aligned} \tag{48}$$

And soon, in this manner the rest of components of the decomposition series can be obtained. The first three terms of the decomposition series (6) are given by.

$$\begin{aligned} u(x,t)=&x^2(1-2t)+6x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{4t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right) \\ &+72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\cdots\right). \end{aligned} \tag{49}$$

To solve the problem using the Sumudu decomposition method [12], we simply substitute (43) and the initial conditions (44) into (16), to obtain the following recurrence relation

$$\begin{aligned} u_0(x,t) &= x^2(1-2t) \\ u_{n+1}(x,t) &= \mathbb{S}^{-1}\left(u^{\alpha}\mathbb{S}\left[A_n\right]\right), \quad n\geq 1. \end{aligned} \tag{50}$$

In view of (50), the first few components of the decomposition series are derived as follows

$$u_0(x,t) = x^2(1-2t),$$

where $u_1(x,t)$ and $u_2(x,t)$ as written in equations (47) and (48) and soon, in this manner the rest of components of the decomposition series can be obtained. The first three terms of the decomposition series (6) are given by

$$u(x,t) = \text{ Eq.(49)}.$$

To solve the problem using the Natural decomposition method, we simply substitute (43) and the initial conditions (44) into (22), to obtain the following recurrence relation

$$\begin{aligned} u_0(x,t) &= x^2(1-2t), \\ u_{n+1}(x,t) &= \mathbb{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{N}\left[A_n\right]\right), \quad n \geq 1. \end{aligned} \tag{51}$$

In view of (51), the first few components of the decomposition series are derived as follows

$$u_0(x,t) = x^2(1-2t),$$

where $u_1(x,t)$ and $u_2(x,t)$ as written in equations (47) and (48) and soon, in this manner the rest of components of the decomposition series can be obtained. The first three terms of the decomposition series (6) are given by $u(x,t)$ as written in equation (49). To solve the problem using the Modified Laplace Variational Iteration Method [14], by enforcement of the Laplace transform to the sides of equation (43) and using the initial conditions (44), we get

$$\begin{aligned} s^\alpha\mathbb{L}[u(x,t)] - s^{\alpha-1}x^2 + 2s^{\alpha-2}x^2 &= \mathbb{L}\left(\frac{d}{dx}\left(u(x,t)\frac{u(x,t)}{dx}\right)\right), \\ \mathbb{L}[u(x,t)] &= \frac{x^2}{s} - \frac{2x^2}{s^2} + s^{-\alpha}\mathbb{L}\left(\frac{d}{dx}\left(u(x,t)\frac{u(x,t)}{dx}\right)\right), \\ u(x,s) &= \frac{x^2}{s} - \frac{2x^2}{s^2} + \left(\frac{1}{s^\alpha}L\left(u(x,t)u_x(x,t)\right)\right). \end{aligned} \tag{52}$$

Taking the inverse Laplace transform to the sides of equation (52), we obtain

$$u(x,t) = x^2 - 2x^2t + \mathbb{L}^{-1}\left(s^{-\alpha}\mathbb{L}\left(\frac{d}{dx}\left(u(x,t)\frac{u(x,t)}{dx}\right)\right)\right). \tag{53}$$

Now the new tactic of the modified Laplace variational iteration technique is instituted on (27) from equation (53), and we get

$$\frac{du(x,t)}{dt} = \frac{d}{dt}\left(x^2 - 2x^2t\right) + \frac{d}{dt}\left(L^{-1}\left(\frac{1}{s^\alpha}L\left(u(x,t)u_x(x,t)\right)\right)\right). \tag{54}$$

We simply substitute equation (54) and the initial condition equation (44) into equation (29); by the new modified function, we find

$$
\begin{aligned}
u_0(x,t) =& x^2(1-2t),\\
u_1(x,t) =& x^2(1-2t) + \frac{6x^2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{24x^2t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{48x^2t^{\alpha+2}}{\Gamma(\alpha+3)},\\
u_2(x,t) =& x^2(1-2t) + 6x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{3t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{4t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{12t^{2\alpha}}{\Gamma(2\alpha+1)}\cdots\right)
\end{aligned}
$$

Tables (5, 6, 7) show the approximate solutions for Eq. (43) obtained for different values of a using the decomposition method Sumudu decomposition method, Natural decomposition method, Adomian decomposition method and Modified Laplace Variational Iteration. It is to be noted that only three terms of this decomposition series and the Modified Laplace Variational Iteration were used in evaluating the approximate solutions for Table (5, 6, 7). The Accuracy can be improved by computing more terms of the approximate solution

Table 5: Numerical values when $\alpha = 1.5$ for Eq. (43).

t	x	U_{ADM}	$U_{\mathrm{NDM}} - U_{\mathrm{SDM}}$	U_{MLVIM}
0.2	0.25	0.0592832468	0.0592832468	0.0596734742
	0.5	0.2371329870	0.2371329870	0.2386938966
	0.75	0.5335492208	0.5335492208	0.5370612674
	1	0.9485319481	0.9485319481	0.9547755866
0.4	0.25	0.0654118621	0.0654118621	0.0708846937
	0.5	0.2616474484	0.2616474484	0.2835387748
	0.75	0.5887067589	0.5887067589	0.6379622433
	1	1.0465897936	1.0465897936	1.1341550992
0.6	0.25	0.0631775739	0.0631775739	0.0846789648
	0.5	0.2527102956	0.2527102956	0.3387158594
	0.75	0.5685981652	0.5685981652	0.7621106836
	1	1.0108411825	1.0108411825	1.3548634375

Table 6: Numerical values when $\alpha = 1.75$ for Eq. (43).

t	x	U_{ADM}	$U_{\text{NDM}} - U_{\text{SDM}}$	U_{MLVIM}
0.2	0.25	0.0487012403	0.0487012403	0.0487471818
	0.5	0.1948049610	0.1948049610	0.1949887272
	0.75	0.4383111623	0.4383111623	0.4387246362
	1	0.7792198441	0.7792198441	0.7799549088
0.4	0.25	0.0437480416	0.0437480416	0.0448869280
	0.5	0.1749921662	0.1749921662	0.1795477122
	0.75	0.3937323741	0.3937323741	0.4039823524
	1	0.6999686650	0.6999686650	0.7181908488
0.6	0.25	0.0381836436	0.0381836436	0.0445168272
	0.5	0.1527345746	0.1527345746	0.1780673088
	0.75	0.3436527928	0.3436527928	0.4006514448
	1	0.610938298	0.610938298	0.7122692353

Table 7: Numerical values when $\alpha = 2$ for Eq. (43).

t	x	U_{ADM}	$U_{\text{NDM}} - U_{\text{SDM}}$	U_{MLVIM}	U_{Exact}
0.2	0.25	0.0433950857	0.0433950857	0.0433999819	0.0434027778
	0.5	0.1735803429	0.1735803429	0.1735999276	0.1736111111
	0.75	0.3905557714	0.3905557714	0.3905998371	0.3906250000
	1	0.6943213714	0.6943213714	0.6943997104	0.6944444444
0.4	0.25	0.0315669714	0.0315669714	0.0317794680	0.0318877551
	0.5	0.1262678857	0.1262678857	0.1271178720	0.1275510204
	0.75	0.2841027429	0.2841027429	0.2860152121	0.2869897959
	1	0.5050715429	0.5050715429	0.5084714881	0.5102040816
0.6	0.25	0.0220044571	0.0220044571	0.0236648775	0.0244140625
	0.5	0.0880178286	0.0880178286	0.0946595101	0.0976562500
	0.75	0.1980401143	0.1980401143	0.2129838978	0.2197265625
	1	0.3520713143	0.3520713143	0.3786380405	0.3906250000

Table (8) shows the absolute error between the exact and approximate solutions for Eq. (43) produced using Adomian Decomposition Method and Modified Laplace Variational Iteration Method. The results are computed after applying three iterations of each method for various values.

Table 8: The absolute error for $\alpha = 2$ for Eq. (43).

t	x	U_{ADM}	U_{MLVIM}
0.2	0.25	0.0000076921	0.0000027959
	0.5	0.0000307682	0.0000111835
	0.75	0.0000692286	0.0000251629
	1	0.0001230730	0.0000447341
0.4	0.25	0.0003207837	0.0001082871
	0.5	0.0012831347	0.0004331484
	0.75	0.0028870530	0.0009745838
	1	0.0051325387	0.0017325935
0.6	0.25	0.0024096054	0.0007491850
	0.5	0.0096384214	0.0029967399
	0.75	0.0216864482	0.0067426647
	1	0.0385536857	0.0119869595

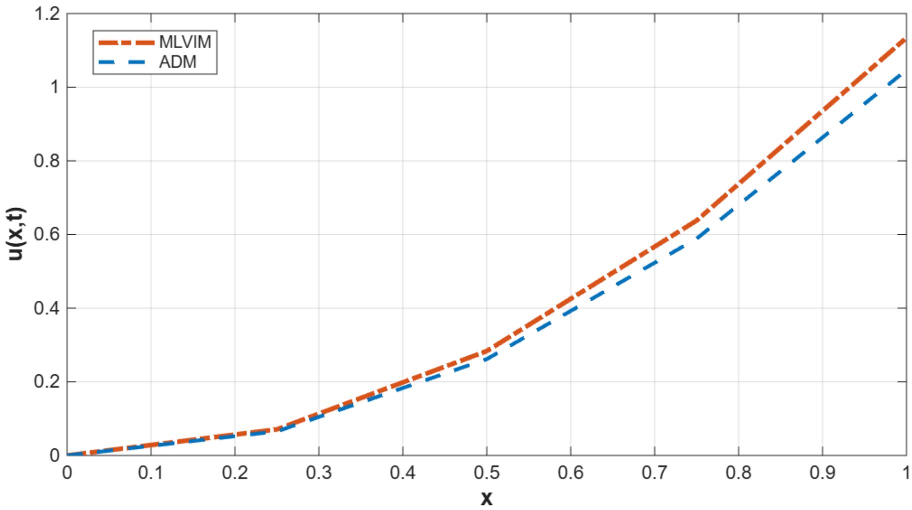

Figure 4: Comparison of ADM and MLVIM solution for the first three approximations $\alpha = 1.5$, $t = 0.4$, with $x = 0 : 1$, for equation (43).

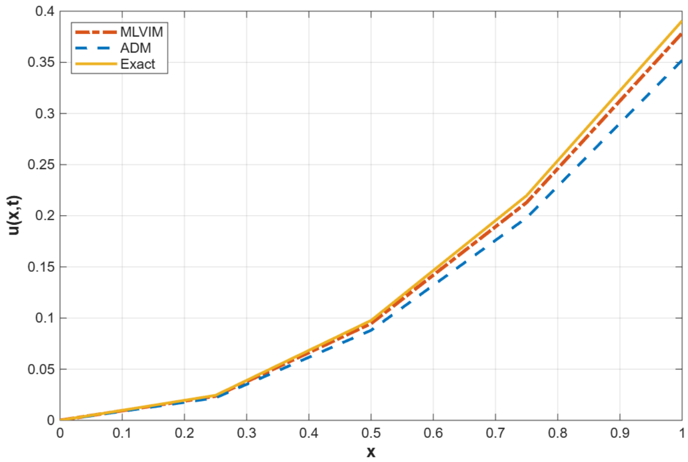

Figure 5: Comparison of ADM and MLVIM solution for the first three approximations $\alpha = 2$, $t = 0.6$, with $x = 0 : 1$, for equation (43).

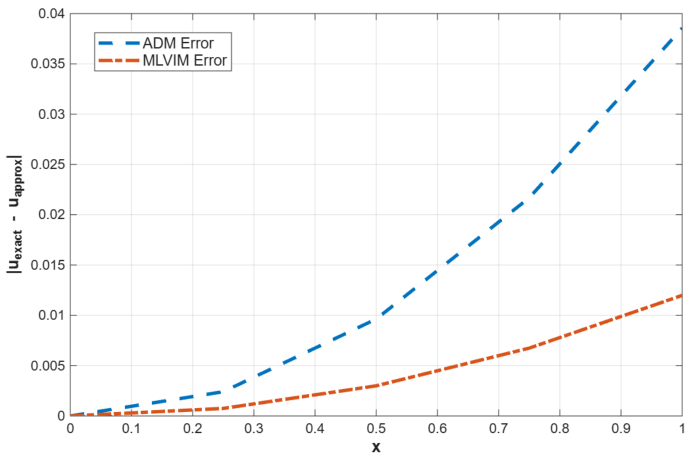

Figure 6: Comparison of the absolute error between the ADM and MLVIM solution for the first three approximations $\alpha = 2, t = 0.6$, with $\text{x} = 0 : 1$, for Eq. (43).

6. Conclusions

The main objective of this article was to investigate an accurate approximate solutions for nonlinear partial differential equations of fractional order. This goal was achieved by applying the modified Laplace variational iteration method and three variants of the Adomian Decomposition Method (ADM): the standard ADM, ADM with the Sumudu transform, and the natural ADM. All these methods yield solutions expressed as convergent series that are straightforward to compute. There are two key observations to highlight. First, all three variants of the Adomian Decomposition Method (ADM)—the standard ADM, ADM with the Sumudu transform, and the natural ADM—yielded identical numerical solutions and convergence rates. This demonstrates their equivalence in both accuracy and efficiency for the cases studied. Therefore, the choice among them can primarily depend on ease of implementation, although differences may emerge for more complex problems. Second, in Example 1, the decomposition method provided a faster-converging approximate solution than the modified Laplace variational iteration method. Conversely, in Example 2, the modified Laplace variational iteration method showed faster convergence toward the exact solution than the decomposition method. Hence, the accuracy and performance of these methods depend on the specific nonlinear fractional differential equation under consideration. Both approaches can thus serve as effective alternatives for solving fractional partial differential equations. These findings are consistent with previously reported results [8, 14]. The future work of this study can be extended in several directions such that application to higher-dimensional problems and extend the methods to equations involving variable fractional orders and coupled fractional operators.

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