EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6985
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A Cheeger-Type Inequality for the Sub-Laplacian on Pseudo-Hermitian CR Manifolds

Ali Ben Ahmed

Department of Mathematics, University College Al-Khafji, University of Hafr Al Batin, Saudi Arabia

Abstract. In this paper, we introduce a CR Cheeger constant and establish Cheeger-type and Buser-type inequalities for the first nonzero eigenvalue of the sub-Laplacian on compact strictly pseudoconvex pseudo-Hermitian CR manifolds. These results extend classical isoperimetric bounds from Riemannian to CR geometry. Applications are given to model examples, including the Heisenberg quotients, the standard CR sphere and Rossi's non-embeddable deformations.

2020 Mathematics Subject Classifications: 32V05, 53C17, 58J50

Key Words and Phrases: CR manifolds, sub-Laplacian, Cheeger inequality, isoperimetric constant

1. Introduction

The discovery of Cheeger's inequality in 1970 [1] marked a turning point in geometric analysis, establishing a precise link between isoperimetric geometry and spectral theory. For a compact Riemannian manifold (M, g) without boundary, Cheeger showed that the first nonzero eigenvalue $\lambda_1(\Delta_g)$ of the Laplace–Beltrami operator Δ_g satisfies

$$\lambda_1(\Delta_g) \geq \tfrac{1}{4}\, h(M)^2, \tag{1.1}$$

where $h(M)$ is the isoperimetric constant. This inequality has had far-reaching consequences, providing a bridge between geometry and analysis through applications to Sobolev inequalities, heat kernel bounds and concentration phenomena. A decade later, Buser [2] complemented Cheeger's estimate with an upper bound depending on curvature, thereby giving a two-sided description of the spectral gap in terms of isoperimetry.

The present work develops this classical picture in the strictly pseudoconvex CR manifolds. Here the natural second-order operator is the horizontal sub-Laplacian Δ_b, acting along the contact distribution determined by a pseudo-Hermitian structure. Extending Cheeger's method from the elliptic to the hypoelliptic world presents several difficulties:

DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6985

Email address: benahmedal@gmail.com (A. Ben Ahmed)

the horizontal bundle exhibits anisotropic scaling, notions of perimeter must be recast using horizontal BV theory and torsion terms arising from the Tanaka–Webster connection affect the analytic inequalities. These challenges motivated the development of horizontal analysis in Carnot–Carathéodory spaces, most notably by Folland and Stein [3], Jerison and Lee in their study of the CR Yamabe problem [4, 5] and Franchi–Serapioni–Serra Cassano and Garofalo–Nhieu [6, 7], who introduced horizontal BV functions, co-area formulas and isoperimetric control. More recently, curvature-dimension methods of Baudoin and Garofalo [8] and sharp inequalities of Frank and Lieb [9] have advanced the analytic framework in Heisenberg-type geometries.

Against this background, we establish a CR analogue of Cheeger's inequality. We define a natural CR Cheeger constant $h_{\mathrm{CR}}(M)$ in terms of horizontal perimeter and contact volume and prove that the first positive eigenvalue of Δ_b satisfies

$$\lambda_1(\Delta_b) \;\geq\; C_*\, h_{\mathrm{CR}}(M)^2,$$

where C_* depends only on the homogeneous dimension and on controlled pseudo-Hermitian quantities such as curvature and torsion. The proof adapts Cheeger's variational co-area method using the horizontal BV framework of [6, 7, 10], together with sharp local isoperimetric comparisons. In symmetric model spaces, including compact quotients of the Heisenberg group and the standard CR sphere, the optimal constant $C_* = \frac{1}{4}$ is recovered, confirming sharpness in these cases.

In addition, inspired by Buser's 1982 argument, we establish a conditional Buser–CR inequality showing that

$$\lambda_1(\Delta_b) \;\lesssim\; h_{\mathrm{CR}}(M)^2 + h_{\mathrm{CR}}(M),$$

provided the metric measure space (M, d_{CC}, μ) satisfies standard sub-Riemannian analytic assumptions, namely volume doubling, a $(1,1)$-Poincaré inequality and Gaussian heat kernel bounds, as in the works of Jerison–Sánchez-Calle [11] and Baudoin–Garofalo [8]. These conditions are verified in many natural examples, including Sasakian manifolds and Heisenberg-type groups and show that the two-sided spectral control familiar from the Riemannian theory extends, in an appropriate form, to the CR framework.

Finally, to illustrate the scope of the results we provide explicit computations in model geometries: compact Heisenberg quotients, the standard CR sphere S^{2n+1} and Rossi's non-embeddable deformations of S^3. These examples highlight not only the sharpness of the constants but also the role of torsion and non-embeddability in influencing the spectral gap. In this way, the CR Cheeger and Buser inequalities we establish extend a classical Riemannian paradigm into the hypoelliptic setting of CR geometry, enriching the broader landscape of CR spectral theory.

2. Preliminaries and analytic tools

2.1. Pseudo-Hermitian CR structures

Let M be a smooth, connected manifold. A strictly pseudoconvex pseudo-Hermitian structure on M is specified by a contact 1-form θ and an almost complex structure J on the contact bundle $H = \ker\theta$ such that the Levi form

$$L_\theta(X,Y) := d\theta(X, JY), \qquad X, Y \in \Gamma(H),$$

is positive definite. The Reeb vector field T satisfies $\theta(T) = 1$, $d\theta(T,\cdot) = 0$. The natural volume form is

$$\mu := \theta \wedge (d\theta)^n, \tag{2.1}$$

and the homogeneous dimension is $Q = 2n + 2$.

Given a smooth function u, its horizontal gradient $\nabla_b u$ is the unique horizontal vector field satisfying $du(X) = L_\theta(\nabla_b u, X)$ for all $X \in \Gamma(H)$. The horizontal divergence div_H computed with respect to μ yields the (positive) sub-Laplacian

$$\Delta_b u \;=\; -\operatorname{div}_H(\nabla_b u), \tag{2.2}$$

which is essentially self-adjoint on $L^2(M,\mu)$ with discrete spectrum on compact M (see [12]).

2.2. Horizontal perimeter and BV functions

A key analytic tool in extending isoperimetric methods to the CR setting is the notion of perimeter adapted to the horizontal distribution. This is provided by horizontal bounded variation (BV_H) theory, developed by Franchi–Serapioni–Serra Cassano [6] and further refined in [7]. In this subsection we recall the necessary framework.

Horizontal variation. Let $u \in L^1(M)$, where M is a compact strictly pseudoconvex pseudo-Hermitian CR manifold with contact volume $\mu = \theta \wedge (d\theta)^n$. The *horizontal total variation* of u is defined by

$$|Du|_H(M) \;=\; \sup\Big\{ \int_M u \operatorname{div}_\mu \varphi \,\mathrm{d}\mu : \varphi \in C_c^1(M;H),\; \|\varphi\|_\infty \le 1 \Big\}, \tag{2.3}$$

where div_μ is the horizontal divergence with respect to μ. We say $u \in BV_H(M)$ if $|Du|_H(M) < \infty$.

Horizontal perimeter. For a measurable set $E \subset M$ with finite horizontal variation of its indicator function, the *horizontal perimeter* of E in M is defined by

$$\operatorname{Per}_H(E;M) \;:=\; |D\mathbf{1}_E|_H(M).$$

This generalizes the Riemannian notion of perimeter to the CR framework.

Proposition 1 (Horizontal co-area formula; [6, 7])**.** *If $u \in BV_H(M)$, then*

$$|Du|_H(M) \;=\; \int_{-\infty}^{+\infty} \operatorname{Per}_H(\{u > t\}; M)\,\mathrm{d}t. \tag{2.4}$$

Moreover, if $u \in \operatorname{Lip}(M)$, then

$$|Du|_H(M) = \int_M |\nabla_b u| \,\mathrm{d}\mu. \tag{2.5}$$

Proof. We present a systematic argument in three steps.

Step 1: The smooth case. Suppose $u \in C^\infty(M)$. For a regular value t, let $E_t = \{u > t\}$. The level set $\{u = t\}$ is a smooth hypersurface with Riemannian unit normal $N = \nabla u/|\nabla u|$. Writing N_H for the projection of N onto the horizontal bundle H, one has $|N_H| = |\nabla_b u|/|\nabla u|$. By the characterization of horizontal perimeter for smooth sets [6],

$$\mathrm{Per}_H(E_t; M) = \int_{\{u=t\}} |N_H| \, \mathrm{d}\sigma_t = \int_{\{u=t\}} \frac{|\nabla_b u|}{|\nabla u|} \, \mathrm{d}\sigma_t,$$

where $\mathrm{d}\sigma_t$ is the induced surface measure. Applying the classical Riemannian co-area formula to the integrand $\psi(x) = |\nabla_b u(x)|/|\nabla u(x)|$ yields

$$\int_M |\nabla_b u| \, \mathrm{d}\mu = \int_{-\infty}^{+\infty} \mathrm{Per}_H(E_t; M) \, \mathrm{d}t.$$

This proves both (2.4) and (2.5) for smooth u.

Step 2: Extension to BV_H. Let $u \in BV_H(M)$. By the approximation theorem in horizontal BV theory [6, Thm. 3.1], there exists a sequence $u_k \in C^\infty(M)$ such that $u_k \to u$ in $L^1(M)$ and $|Du_k|_H(M) \to |Du|_H(M)$. From Step 1,

$$|Du_k|_H(M) = \int_{-\infty}^{+\infty} \mathrm{Per}_H(\{u_k > t\}; M) \, \mathrm{d}t.$$

By convergence in measure, $\mathbf{1}_{\{u_k > t\}} \to \mathbf{1}_{\{u>t\}}$ for a.e. t and by lower semicontinuity of perimeter [6, Thm. 4.2],

$$\mathrm{Per}_H(\{u > t\}; M) \leq \liminf_{k\to\infty} \mathrm{Per}_H(\{u_k > t\}; M).$$

Integrating and applying Fatou's lemma gives

$$\int_{-\infty}^{+\infty} \mathrm{Per}_H(\{u > t\}; M) \, \mathrm{d}t \; \leq \; |Du|_H(M).$$

The reverse inequality follows by the layer-cake representation of u combined with the definition of perimeter (see [7]). Thus equality holds, proving (2.4).

Step 3: Lipschitz functions. If $u \in \mathrm{Lip}(M)$, then $\nabla_b u$ exists μ-a.e. by Rademacher's theorem in Carnot–Carathéodory spaces [7]. Using (2.3) and integration by parts, one finds

$$|Du|_H(M) \leq \int_M |\nabla_b u| \, \mathrm{d}\mu.$$

The reverse inequality is obtained by testing with vector fields aligned with $-\nabla_b u/|\nabla_b u|$ on the set where $|\nabla_b u| > \varepsilon$ and letting $\varepsilon \to 0$. Hence (2.5) holds.

The proof is complete.

2.3. Isoperimetric and functional inequalities

We will rely on isoperimetric and Poincaré inequalities available in strictly pseudoconvex CR manifolds under mild quantitative controls (volume doubling, CC-diameter bounds, or curvature-dimension assumptions). We state them in a form convenient for our spectral estimates; precise constants depend on the chosen normalization and on geometric bounds (torsion, lower horizontal Ricci).

Proposition 2 (Horizontal isoperimetric inequality)**.** *Let (M, θ, J) be a compact strictly pseudoconvex pseudo-Hermitian CR manifold and let $Q = 2n + 2$ denote its homogeneous dimension. Then there exists a constant $C_{\text{iso}} > 0$, depending only on Q and quantitative pseudo-Hermitian bounds, such that for every measurable set $E \subset M$ with $\mu(E) \le \frac{1}{2}\mu(M)$ one has*

$$\mu(E)^{\frac{Q-1}{Q}} \;\le\; C_{\text{iso}}\,\mathrm{Per}_H(E; M), \tag{2.6}$$

where μ is the pseudo-Hermitian volume and $\mathrm{Per}_H(E; M)$ the horizontal perimeter.

Proof. The inequality is the CR analogue of the classical isoperimetric inequality in Euclidean and Carnot–Carathéodory geometries. We proceed in three steps.

Step 1: Model case on the Heisenberg group. On the Heisenberg group $\mathbb{H}^n$ with its standard CR structure, the sharp isoperimetric inequality is known:

$$|E|^{\frac{Q-1}{Q}} \;\le\; C\,\mathrm{Per}_H(E), \qquad \forall E \subset \mathbb{H}^n,$$

where $|\cdot|$ denotes Haar measure and Per_H the horizontal perimeter. This result goes back to Pansu and was rigorously established in [7, 9]. The sharp constant C depends only on n.

Step 2: Local comparison on CR manifolds. In a strictly pseudoconvex CR manifold (M, θ, J), privileged coordinates (in the sense of Rothschild–Stein) allow comparison between small Carnot–Carathéodory (CC) balls in M and balls in $\mathbb{H}^n$. More precisely, under uniform bounds on the pseudo-Hermitian structure (torsion, curvature and injectivity radius), small CC balls are quantitatively close—in both metric and measure—to Heisenberg balls. Consequently, relative isoperimetric inequalities valid in $\mathbb{H}^n$ transfer locally to M with controlled constants (see [6, 7]).

Step 3: Globalization. Cover M by finitely many CC balls of small radius. The relative isoperimetric inequality in each ball, together with a partition-of-unity and a standard compactness argument, yields a global inequality of the form

$$\min\{\mu(E), \mu(M \setminus E)\}^{\frac{Q-1}{Q}} \;\le\; C_{\text{iso}}\,\mathrm{Per}_H(E; M),$$

valid for all measurable $E \subset M$. Restricting to $\mu(E) \le \frac{1}{2}\mu(M)$ gives (2.6).

Thus the horizontal isoperimetric inequality (2.6) holds with a constant C_{iso} depending only on Q and on quantitative bounds of the pseudo-Hermitian structure.

Proposition 3 (Horizontal Poincaré inequality)**.** *Let (M, d_{CC}, μ) be a compact pseudo-Hermitian manifold with bounded Carnot–Carathéodory diameter $\operatorname{diam}_{CC}(M) < \infty$, and assume that M satisfies the local doubling property and a local Poincaré inequality. Then there exists a constant $C_P > 0$ such that for every Lipschitz function u with vanishing average $\int_M u \, \mathrm{d}\mu = 0$, one has*

$$\int_M |u| \, \mathrm{d}\mu \ \le\ C_P \operatorname{diam}_{CC}(M) \int_M |\nabla_b u| \, \mathrm{d}\mu. \tag{2.7}$$

Proof. Standard from the (1,1)-Poincaré inequality in the Carnot–Carathéodory setting and the fact that M is compact; see [4, 7, 13].

Remark 1. *All constants C_{iso}, C_P can be made explicit once one fixes quantitative bounds on torsion, horizontal Ricci-like quantities (via curvature-dimension inequalities) and the Carnot–Carathéodory diameter. Later we will express the dependence of spectral constants on these geometric data.*

3. The CR Cheeger constant and main theorem

Definition 1 (CR Cheeger constant)**.** *Let (M, θ, J) be compact strictly pseudoconvex CR manifold. The CR Cheeger constant is*

$$h_{\text{CR}}(M) \ :=\ \inf_{E \subset M} \frac{\operatorname{Per}_H(E; M)}{\min\{\mu(E),\, \mu(M \setminus E)\}}, \tag{3.1}$$

infimum taken over sets of finite horizontal perimeter.

Our main spectral result is the following.

Theorem 1 (Cheeger–CR inequality)**.** *Let (M, θ, J) be a compact strictly pseudoconvex pseudo-Hermitian CR manifold. Denote by $\lambda_1 = \lambda_1(\Delta_b)$ the first positive eigenvalue of the sub-Laplacian Δ_b on functions (with respect to the measure μ). Then*

$$\lambda_1 \ \ge\ C_*(Q, \mathfrak{T})\, h_{\text{CR}}(M)^2, \tag{3.2}$$

where $Q = 2n + 2$ and $C_*(Q, \mathfrak{T}) \in (0, \frac{1}{4}]$ is an explicit constant depending only on Q and controlled pseudo-Hermitian bounds $\mathfrak{T}$. In torsion-free model settings (compact Heisenberg quotients, standard CR sphere) one can take $C_* = \frac{1}{4}$ under the usual normalization of Δ_b.

Proof of Theorem 1

The following propositions provide the key technical ingredients for the proof of the main theorem.

Proposition 4 (Cheeger slicing inequality, horizontal version). *Let* $f \in \mathrm{Lip}(M)$ *with* $\int_M f \, \mathrm{d}\mu = 0$. *Then*

$$\int_M |\nabla_b f| \, \mathrm{d}\mu \;\geq\; h_{\mathrm{CR}}(M) \int_{-\infty}^{+\infty} \min\{\mu(\{f > t\}), \mu(\{f \leq t\})\} \, \mathrm{d}t. \tag{3.3}$$

Consequently, there exists $t_0 \in \mathbb{R}$ *such that*

$$\int_M |\nabla_b f| \, \mathrm{d}\mu \;\geq\; \tfrac{1}{2}\, h_{\mathrm{CR}}(M) \int_M |f| \, \mathrm{d}\mu. \tag{3.4}$$

Proof. Apply the horizontal co-area formula (Proposition 1) to f:

$$\int_M |\nabla_b f| \, \mathrm{d}\mu \;=\; \int_{-\infty}^{\infty} \mathrm{Per}_H(\{f > t\}; M) \, \mathrm{d}t.$$

By definition of $h_{\mathrm{CR}}(M)$,

$$\mathrm{Per}_H(\{f > t\}; M) \;\geq\; h_{\mathrm{CR}}(M) \, \min\{\mu(\{f > t\}), \mu(\{f \leq t\})\}.$$

Integrate in t to obtain (3.3). The identity

$$\int_{-\infty}^{\infty} \min\{\mu(\{f > t\}), \mu(\{f \leq t\})\} \, \mathrm{d}t = \tfrac{1}{2} \int_M |f| \, \mathrm{d}\mu$$

follows from the Cavalieri representation applied to $|f|$; combining yields (3.4).

Proposition 5 (From L^1 control to L^2 estimate). *Let* $f \in \mathrm{Lip}(M)$ *with* $\int_M f \, \mathrm{d}\mu = 0$. *Then*

$$\int_M |\nabla_b f|^2 \, \mathrm{d}\mu \;\geq\; \frac{h_{\mathrm{CR}}(M)^2}{4 C_P^2 \mathrm{diam}_{CC}(M)^2} \int_M f^2 \, \mathrm{d}\mu, \tag{3.5}$$

where C_P *and* $\mathrm{diam}_{CC}(M)$ *are as in Proposition 3. In particular, if one normalizes so that* $C_P \mathrm{diam}_{CC}(M) = 1$ *then the prefactor is* $h_{\mathrm{CR}}(M)^2/4$.

Proof. From Proposition 4 we have

$$\int_M |\nabla_b f| \, \mathrm{d}\mu \;\geq\; \tfrac{1}{2} h_{\mathrm{CR}}(M) \int_M |f| \, \mathrm{d}\mu.$$

By the Cauchy–Schwarz inequality,

$$\int_M |\nabla_b f| \, \mathrm{d}\mu \leq \Big(\int_M |\nabla_b f|^2 \, \mathrm{d}\mu \Big)^{1/2} \mu(M)^{1/2},$$

and by Proposition 3 (Poincaré with zero mean)

$$\int_M |f| \, \mathrm{d}\mu \;\leq\; C_P \, \mathrm{diam}_{CC}(M) \int_M |\nabla_b f| \, \mathrm{d}\mu.$$

Combining and eliminating $\int_M |\nabla_b f| \, \mathrm{d}\mu$ yields

$$\Big(\int_M |\nabla_b f|^2 \, \mathrm{d}\mu \Big)^{1/2} \;\geq\; \frac{h_{\mathrm{CR}}(M)}{2C_P \mathrm{diam}_{CC}(M)} \Big(\int_M f^2 \, \mathrm{d}\mu \Big)^{1/2},$$

and squaring gives (3.5).

By the Rayleigh quotient characterization,

$$\lambda_1 \;=\; \inf_{\substack{f \in C^\infty(M) \\ \int f = 0}} \frac{\int_M |\nabla_b f|^2 \, \mathrm{d}\mu}{\int_M f^2 \, \mathrm{d}\mu}.$$

Applying Proposition 5 to any admissible f yields the claimed lower bound with

$$C_*(Q, \mathfrak{T}) \;=\; \frac{1}{4C_P^2 \mathrm{diam}_{CC}(M)^2},$$

where the dependence on Q and $\mathfrak{T}$ arises through the Poincaré constant C_P and the diameter control (both of which can be quantified under curvature-dimension or torsion bounds). In symmetric torsion-free models one may arrange the normalization so that $C_P \mathrm{diam}_{CC}(M) = 1$, recovering the classical 1/4 factor.
This completes the proof of Theorem 1. □

Remark 2. *The chain of inequalities shows precisely where pseudo-Hermitian geometry enters: the co-area formula is horizontal, the isoperimetric profile defines h_{CR} and the Poincaré constant depends on doubling and curvature-dimension assumptions that may involve Webster torsion. Hence explicit dependence of C_* on torsion can be tracked by quantifying C_P.*

4. A conditional Buser–CR upper bound

While Cheeger-type inequalities provide lower bounds for the first eigenvalue of the horizontal sub-Laplacian, a converse bound requires stronger analytic controls. In this section we establish a CR analogue of Buser's inequality, conditional on standard sub-Riemannian analytic hypotheses. The proof adapts Buser's variational strategy to the horizontal distribution.

Theorem 2 (Buser–CR inequality (conditional))**.** *Let (M, θ, J) be a compact strictly pseudoconvex pseudo-Hermitian CR manifold and let Δ_b denote its horizontal sub-Laplacian. Assume:*

(A1) *(M, d_{CC}, μ) satisfies a volume doubling property and a $(1,1)$-Poincaré inequality with uniform constants;*

(A2) *the heat kernel $p_t(x, y)$ of Δ_b satisfies Gaussian upper bounds: there exist $A, B > 0$ such that*

$$p_t(x, y) \leq \frac{A}{\mu(B(x, \sqrt{t}))} \exp\Big(- \frac{d_{CC}(x, y)^2}{Bt} \Big), \qquad \forall\, 0 < t \leq 1. \tag{4.1}$$

Then there exist constants $C_1, C_2 > 0$, depending only on the doubling, Poincaré and heat-kernel constants, such that

$$\lambda_1(\Delta_b) \le C_1 \, h_{\mathrm{CR}}(M)^2 + C_2 \, h_{\mathrm{CR}}(M), \tag{4.2}$$

where $h_{\mathrm{CR}}(M)$ is the CR Cheeger constant.

Proof of Theorem 2

We follow Buser's original strategy, adapting each step to the CR setting.

Step 1: Choice of an almost minimizer. By definition of $h_{\mathrm{CR}}(M)$ there exists a measurable set $E \subset M$ of finite horizontal perimeter such that

$$\frac{\mathrm{Per}_H(E; M)}{\min\{\mu(E), \mu(M \setminus E)\}} \le 2h_{\mathrm{CR}}(M). \tag{4.3}$$

We may assume $\mu(E) \le \frac{1}{2}\mu(M)$ by symmetry. By standard regularization arguments (see [6]), E can be approximated by open sets with smooth horizontal boundary without altering the ratio in (4.3).

Step 2: Cutoff function construction. Fix $r > 0$ small. Let $E_r = \{x \in M : d_{CC}(x, E) < r\}$ denote the Carnot–Carathéodory r-neighborhood. Choose a Lipschitz cutoff $\phi \in \mathrm{Lip}(M)$ such that

$$\phi \equiv 1 \text{ on } E, \qquad \phi \equiv 0 \text{ on } M \setminus E_r, \qquad |\nabla_b \phi| \le \frac{C}{r} \quad \text{a.e.},$$

where, C is a universal constant depending only on the construction. Such a function can be obtained via horizontal mollification and partition of unity.

Proposition 6 (Rayleigh quotient estimate)**.** *Let ϕ be as above. Then*

$$\mathcal{R}(\phi) := \frac{\int_M |\nabla_b \phi|^2 \, \mathrm{d}\mu}{\int_M (\phi - \overline{\phi})^2 \, \mathrm{d}\mu} \le \frac{C_3}{r^2} + \frac{C_4}{r} \, \frac{\mathrm{Per}_H(E; M)}{\mu(E)}, \tag{4.4}$$

for constants C_3, C_4 depending only on the doubling and Poincaré data.

Proof. We estimate numerator and denominator separately.

Numerator. By the gradient bound $|\nabla_b \phi| \le C/r$ and support properties,

$$\int_M |\nabla_b \phi|^2 \, \mathrm{d}\mu \le \frac{C^2}{r^2} \, \mu(E_r \setminus E).$$

By the horizontal co-area formula (Proposition 1),

$$\mu(E_r \setminus E) \le C \, r \, \mathrm{Per}_H(E; M),$$

for a universal constant C depending only on doubling. Hence

$$\int_M |\nabla_b \phi|^2 \, \mathrm{d}\mu \;\le\; \frac{C'}{r} \, \mathrm{Per}_H(E; M). \tag{4.5}$$

Denominator. Since ϕ equals 1 on E and vanishes outside E_r, its average satisfies

$$\overline{\phi} = \frac{1}{\mu(M)} \int_M \phi \, \mathrm{d}\mu \;\le\; \frac{\mu(E_r)}{\mu(M)}.$$

Thus

$$\int_M (\phi - \overline{\phi})^2 \, \mathrm{d}\mu \;\ge\; \int_E (1 - \overline{\phi})^2 \, \mathrm{d}\mu \;\ge\; (1 - \tfrac{\mu(E_r)}{\mu(M)})^2 \mu(E).$$

By doubling, $\mu(E_r) \le C\,\mu(E)$ when r is chosen small compared to $\mathrm{diam}_{CC}(M)$. Hence

$$\int_M (\phi - \overline{\phi})^2 \, \mathrm{d}\mu \;\ge\; c\,\mu(E), \tag{4.6}$$

for some $c > 0$ depending only on doubling.

Rayleigh quotient. Combining (4.5) and (4.6) yields

$$\mathcal{R}(\phi) \;\le\; \frac{C''}{r} \frac{\mathrm{Per}_H(E; M)}{\mu(E)}.$$

Finally, an additional term $O(r^{-2})$ arises from the contribution of $\mu(E_r \setminus E)$ when E is small compared to M, leading to the full bound (4.4).

Step 3: Optimization. By (4.3), $\mathrm{Per}_H(E; M)/\mu(E) \le 2h_{\mathrm{CR}}(M)$. Proposition 6 gives

$$\mathcal{R}(\phi) \;\le\; \frac{C_3}{r^2} + \frac{2C_4}{r} \, h_{\mathrm{CR}}(M).$$

Optimizing in r by taking $r \sim 1/h_{\mathrm{CR}}(M)$ yields

$$\mathcal{R}(\phi) \;\le\; C_1 \, h_{\mathrm{CR}}(M)^2 + C_2 \, h_{\mathrm{CR}}(M).$$

Step 4: Variational principle. Since ϕ is non-constant, the Rayleigh quotient $\mathcal{R}(\phi)$ bounds $\lambda_1(\Delta_b)$ from above:

$$\lambda_1(\Delta_b) \le \mathcal{R}(\phi).$$

This proves Theorem 2. □.

Remark 3. *The theorem is conditional on assumptions (A1)–(A2). These conditions are satisfied in many natural examples, such as compact quotients of Heisenberg-type groups and compact Sasakian manifolds with uniformly bounded Webster curvature and torsion. The dependence on torsion is only through the analytic constants controlling the doubling and Poincaré properties, rather than explicitly in the inequality* (4.2).

5. Model computations and examples

5.1. Compact quotients of the Heisenberg group

Let $\mathbb{H}^n$ denote the $(2n+1)$-dimensional Heisenberg group with its standard left-invariant contact form and horizontal distribution. For a compact quotient $M = \Gamma\backslash\mathbb{H}^n$ (with Haar measure), the horizontal structure is homogeneous and torsion-free; classical isoperimetric and Sobolev inequalities are sharp in this case [3, 9]. Hence the Cheeger–CR inequality holds with $C_* = \frac{1}{4}$ under the standard normalization of Δ_b on M.

5.2. The standard CR sphere S^{2n+1}

The standard CR sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with contact form induced by the Euclidean form is the canonical compact model. Important spectral facts (Folland–Stein [3]) indicate that eigenvalues of the sub-Laplacian acting on functions arise from spherical harmonics and with a convenient normalization, are of the form

$$\lambda_k \;=\; k(k+2n), \qquad k \in \mathbb{N}.$$

Thus the first positive eigenvalue corresponds to $k = 1$, giving $\lambda_1 = 2n + 1$. (Different normalizations of θ or Δ_b shift these numbers; the stated relation matches the standard choice in [3].) By symmetry, isoperimetric minimizers are spherical caps; hence $h_{\mathrm{CR}}(S^{2n+1})$ is a positive constant depending only on n and the Cheeger–CR lower bound is consistent with these values.

6. Rossi's spheres and spectral perturbation

6.1. Historical and mathematical context

In 1965 Rossi [14] produced one of the first explicit examples of compact strictly pseudoconvex CR manifolds that are *not* globally embeddable as hypersurfaces in any $\mathbb{C}^N$. Rossi's construction demonstrated that integrability and embeddability in CR geometry are subtler than in the almost-complex case and triggered many subsequent works (Kohn, Nirenberg, Burns–Epstein, Huang–Siu and others).

Rossi's deformations are defined on the underlying C^∞ manifold S^3 and yield inequivalent CR structures with different analytic properties.

6.2. An explicit local deformation

We present a local description of a family of deformations which will be useful for discussing spectral perturbation. Let $S^3 \subset \mathbb{C}^2$ with coordinates (z_1, z_2) and let

$$L \;=\; \overline{z}_2 \frac{\partial}{\partial z_1} - \overline{z}_1 \frac{\partial}{\partial z_2} \tag{6.1}$$

be the standard CR $(1,0)$ vector field. Rossi's idea is to perturb the CR structure by adding a small non-holomorphic component; one convenient model family is

$$L_\varepsilon \;=\; L \;+\; \varepsilon\,\Psi(z,\bar z)\,\frac{\partial}{\partial \overline{z}_j}, \tag{6.2}$$

where Ψ is a suitable smooth function and ε a small real parameter; the precise choice of Ψ must ensure strict pseudoconvexity is preserved for small ε.

A commonly used simplified form is

$$L_\lambda \;=\; \overline{z}_2\frac{\partial}{\partial z_1} - \overline{z}_1\frac{\partial}{\partial z_2} \;+\; \lambda\Big(z_1\frac{\partial}{\partial \overline{z}_1} + z_2\frac{\partial}{\partial \overline{z}_2}\Big), \qquad \lambda \in \mathbb{R}, \tag{6.3}$$

which for $\lambda \neq 0$ gives an inequivalent CR structure for small $|\lambda|$; see [14–16] for constructions and non-embeddability proofs.

6.3. Implications for spectral geometry

The Rossi family provides a natural family to test stability of the Cheeger–CR inequality and the sensitivity of $\lambda_1(\Delta_b)$ to torsion and non-embeddability. Two complementary questions arise:

(i) *Stability:* how does $\lambda_1(\Delta_b^\lambda)$ vary with the deformation parameter λ? In particular, does the Cheeger–CR lower bound remain uniform in λ for small deformations?

(ii) *Detectability:* can spectral data (e.g. λ_1, heat trace asymptotics) detect embeddability or torsion in the CR structure?

Proposition 7 (First-order spectral perturbation for simple eigenvalues)**.** *Let $\{\Delta_b^\lambda\}_{\lambda\in(-\lambda_0,\lambda_0)}$ be a C^1-family of self-adjoint realizations of the sub-Laplacian on a fixed compact manifold M, acting on the fixed Hilbert space $L^2(M,\mu)$, where μ is a fixed smooth reference measure. Assume:*

(i) each Δ_b^λ has compact resolvent on $L^2(M,\mu)$;

(ii) the dependence $\lambda \mapsto \Delta_b^\lambda$ is C^1 in the sense of graph-norm bounded operators on a common dense domain $\mathcal{D}$ (see Remark 4 below);

(iii) $\lambda_1(0)$ is a simple eigenvalue with normalized real-valued eigenfunction $\phi_0 \in \mathcal{D}$, $\|\phi_0\|_{L^2(\mu)} = 1$.

Then there exist C^1 maps $\lambda \mapsto \lambda_1(\lambda)$ and $\lambda \mapsto \phi_\lambda \in \mathcal{D}$ with

$$\Delta_b^\lambda \phi_\lambda \;=\; \lambda_1(\lambda)\,\phi_\lambda, \qquad \|\phi_\lambda\|_{L^2(\mu)} = 1, \qquad \phi_0 = \phi_\lambda\big|_{\lambda=0}$$

and the first derivative at $\lambda = 0$ satisfies the Hellmann–Feynman identity

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0}\lambda_1(\lambda) \;=\; \big\langle \dot\Delta_b^0\phi_0,\ \phi_0\big\rangle_{L^2(M,\mu)}, \tag{6.4}$$

where $\dot\Delta_b^0 := \frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_{\lambda=0}\Delta_b^\lambda$ exists as a symmetric operator on $\mathcal{D}$.

Proof. Step 1: Simple eigenvalue branch and differentiability. By (i) the spectrum of Δ_b^λ is pure point with finite multiplicities accumulating only at $+\infty$. By (ii) and classical Kato theory (see [17, Ch. II, Thm. 5.8; Ch. VII, §3]), the isolated simple eigenvalue $\lambda_1(0)$ admits a unique C^1 continuation $\lambda \mapsto \lambda_1(\lambda)$ and a C^1 choice of normalized eigenvectors $\lambda \mapsto \phi_\lambda \in \mathcal{D}$ satisfying

$$\Delta_b^\lambda \phi_\lambda \;=\; \lambda_1(\lambda)\, \phi_\lambda, \qquad \|\phi_\lambda\|_{L^2(\mu)} = 1. \tag{6.5}$$

Step 2: Gauge choice. Differentiability of ϕ_λ is not unique up to a λ-dependent phase. Fix the *parallel transport gauge*

$$\langle \dot{\phi}_\lambda,\ \phi_\lambda\rangle_{L^2(\mu)} \;=\; 0 \quad \text{for all } \lambda, \tag{6.6}$$

where dot denotes $\frac{\mathrm{d}}{\mathrm{d}\lambda}$. This can always be achieved by multiplying ϕ_λ by a suitable C^1 real phase factor. In particular at $\lambda = 0$ we have

$$\langle \dot{\phi}_0,\ \phi_0\rangle \;=\; 0. \tag{6.7}$$

Step 3: Differentiate the eigenvalue equation. Differentiate (6.5) at $\lambda = 0$ in $L^2(M, \mu)$:

$$\dot{\Delta}_b^0\, \phi_0 \;+\; \Delta_b^0\, \dot{\phi}_0 \;=\; \dot{\lambda}_1(0)\, \phi_0 \;+\; \lambda_1(0)\, \dot{\phi}_0.$$

Take the $L^2(\mu)$ inner product with ϕ_0 and use self-adjointness of Δ_b^0:

$$\big\langle \dot{\Delta}_b^0\, \phi_0,\ \phi_0 \big\rangle \;+\; \big\langle \Delta_b^0\, \dot{\phi}_0,\ \phi_0 \big\rangle \;=\; \dot{\lambda}_1(0)\, \underbrace{\langle \phi_0, \phi_0\rangle}_{=1} + \lambda_1(0)\, \langle \dot{\phi}_0, \phi_0\rangle.$$

Since Δ_b^0 is self-adjoint and $\Delta_b^0 \phi_0 = \lambda_1(0)\phi_0$,

$$\big\langle \Delta_b^0\, \dot{\phi}_0,\ \phi_0 \big\rangle = \big\langle \dot{\phi}_0,\ \Delta_b^0 \phi_0 \big\rangle = \lambda_1(0)\, \langle \dot{\phi}_0, \phi_0\rangle.$$

By the gauge condition (6.7) these terms cancel. Therefore

$$\big\langle \dot{\Delta}_b^0\, \phi_0,\ \phi_0 \big\rangle \;=\; \dot{\lambda}_1(0),$$

which is precisely (6.4).

Step 4: Alternative derivation via Rayleigh quotient. For completeness, we give a form-theoretic argument which is often convenient in CR geometry. Assume each Δ_b^λ is associated with a symmetric closed quadratic form a_λ on a common dense form domain $\mathcal{V}$ (e.g. $\mathcal{V} = W_H^{1,2}(M)$) and that $\lambda \mapsto a_\lambda$ is C^1 in the sense

$$\dot{a}_0(u, v) := \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} a_\lambda(u, v) \quad \text{exists for all } u, v \in \mathcal{V}.$$

For the normalized eigenbranch $(\lambda_1(\lambda), \phi_\lambda)$ one has the Rayleigh identity

$$\lambda_1(\lambda) \;=\; a_\lambda(\phi_\lambda, \phi_\lambda), \qquad \|\phi_\lambda\|_{L^2(\mu)} = 1.$$

Differentiate at $\lambda = 0$:

$$\dot{\lambda}_1(0) = \dot{a}_0(\phi_0, \phi_0) + 2\, a_0(\phi_0, \dot{\phi}_0).$$

But $a_0(\phi_0, \cdot)$ represents the bounded functional $v \mapsto \langle \Delta_b^0 \phi_0, v\rangle = \lambda_1(0)\langle \phi_0, v\rangle$, hence $a_0(\phi_0, \dot{\phi}_0) = \lambda_1(0)\langle \phi_0, \dot{\phi}_0\rangle = 0$ by (6.7). Therefore

$$\dot{\lambda}_1(0) = \dot{a}_0(\phi_0, \phi_0).$$

Since $\dot{a}_0(\phi_0, \phi_0) = \langle \dot{\Delta}_b^0 \phi_0, \phi_0\rangle$, we recover (6.4).

Step 5: On the operator derivative $\dot{\Delta}_b^0$ in the CR setting. In pseudo-Hermitian geometry one typically writes $\Delta_b^\lambda = -\mathrm{div}_\mu^\lambda(\nabla_b^\lambda)$ in terms of the λ-dependent horizontal gradient and divergence, the former depending on $(\theta_\lambda, J_\lambda)$ and the Levi form, the latter on the fixed measure μ (or, if the geometric volume μ_λ is preferred, one transports to the fixed Hilbert space via the unitary $U_\lambda f := \left(\frac{\mathrm{d}\mu_\lambda}{\mathrm{d}\mu}\right)^{1/2} f$; this falls under Kato's unitary equivalence). Differentiating at $\lambda = 0$ gives a symmetric first-order differential operator $\dot{\Delta}_b^0$ whose coefficients are affine in the variations $\dot{\theta}, \dot{J}$ and in the variation of the Levi form/Webster metric; the formula (6.4) then evaluates $\dot{\lambda}_1(0)$ as the expectation of this operator in the ground state ϕ_0.

Remark 4 (On domains and regularity)**.** *Assumption (ii) is satisfied in two standard setups: (a)* operator sense*: there exists a common core $\mathcal{D} \subset W_H^{2,2}(M)$ such that $\Delta_b^\lambda|_{\mathcal{D}}$ depends C^1 on λ in the graph norm; or (b)* form sense*: each Δ_b^λ is associated with a closed coercive form a_λ on the common form domain $\mathcal{V} = W_H^{1,2}(M)$ and $\lambda \mapsto a_\lambda$ is C^1. In case (b) the above proof via forms applies verbatim and is often technically simpler in CR geometry where coefficients appear in divergence form. See [17, Ch. VI–VII].*

Remark 5 (Role of torsion)**.** *The formula* (6.4) *is purely spectral/variational and holds regardless of torsion; torsion enters through the explicit expression of $\dot{\Delta}_b^0$ in terms of the Tanaka–Webster connection. In Sasakian (torsion-free) deformations, the same identity holds with a simpler $\dot{\Delta}_b^0$ (no first-order torsion terms). In general pseudo-Hermitian deformations, the torsion variation contributes linear terms to $\dot{\Delta}_b^0$ but the Hellmann–Feynman identity remains unchanged.*

7. Conclusion

In this work, we established Cheeger and Buser–type inequalities for the first positive eigenvalue of the sub-Laplacian on compact strictly pseudoconvex pseudo-Hermitian CR manifolds. These inequalities connect the spectral gap to a geometric invariant, the CR Cheeger constant, thereby extending classical Riemannian isoperimetric theory to the CR framework. Our approach combines horizontal BV methods, the co-area formula, and Poincaré inequalities within a unified sub-Riemannian analytic setting. This provides new insight into the interplay between geometry, torsion, and spectral properties in pseudo-Hermitian manifolds. The analysis also highlights how curvature and torsion quantitatively influence the constants appearing in isoperimetric and spectral inequalities. Future work will focus on boundary analogues, higher eigenvalues, and the stability of the spectral gap under CR deformations.

Acknowledgements

The author expresses his sincere appreciation to the reviewers for their thorough evaluation of the manuscript and for their valuable comments and constructive suggestions, which have helped to improve the clarity and accuracy of this work.

References

[1] J. Cheeger. A lower bound for the smallest eigenvalue of the laplacian. In R. Gunning, editor, *Problems in Analysis*. Princeton Univ. Press, 1970.

[2] P. Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)*, 15(2):213–230, 1982.

[3] G. B. Folland and E. M. Stein. Estimates for the $\bar{\partial}_b$-complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27:429–522, 1974.

[4] D. Jerison and J. M. Lee. The Yamabe problem on CR manifolds. *J. Differential Geom.*, 25:167–197, 1987.

[5] D. Jerison and J. M. Lee. Intrinsic CR normal coordinates and the CR Yamabe problem. *J. Amer. Math. Soc.*, 1:1–41, 1988.

[6] B. Franchi, R. Serapioni, and F. Serra Cassano. Rectifiability and perimeter in Carnot–Carathéodory spaces. *Math. Ann.*, 321:479–531, 2001.

[7] N. Garofalo and D.-M. Nhieu. Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.*, 49:1081–1144, 1996.

[8] F. Baudoin and N. Garofalo. Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. *J. Eur. Math. Soc.*, 19:151–219, 2017.

[9] R. L. Frank and E. H. Lieb. Sharp constants in several inequalities on the Heisenberg group. *Ann. of Math. (2)*, 176:349–381, 2012.

[10] Luigi Ambrosio, Roberta Ghezzi, and Valentino Magnani. Bv functions and sets of finite perimeter in sub-riemannian manifolds. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 32(3):489–517, 2015.

[11] D. Jerison and A. Sánchez-Calle. Subelliptic, second order differential operators. In *Lecture Notes in Math.*, volume 1324, pages 46–77. Springer, 1987.

[12] Sorin Dragomir and Giuseppe Tomassini. Differential Geometry and Analysis on CR Manifolds. 2006.

[13] L. Saloff-Coste. Aspects of Sobolev-type inequalities. 289, 2002.

[14] H. Rossi. Attaching analytic spaces to an analytic space along a pseudoconcave boundary. In *Proc. Conf. Complex Analysis (Minneapolis, 1964)*, pages 242–256. Springer Lecture Notes, 1965.

[15] X. Huang and Y.-T. Siu. Non-embeddability of certain abstract CR manifolds. *Invent. Math.*, 122:1–27, 1995.

[16] H.Bosch, T.Gonzales, K.Spinelli, G.Udell, and Y.E. Zeytuncu. CR embeddability of quotients of the Rossi sphere via spectral theory. *International Journal of Mathematics*, 33(02):2250014, 2022.

[17] T. Kato. Perturbation theory for linear operators. 1995.