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The Discrete Laplace Transform (DLT) Order: A Sensitive Approach to Comparing Discrete Residual Life Distributions with Applications to Queueing Systems

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Abstract. In this paper, we propose a general methodology for implementing a stochastic ordering framework based on the Discrete Laplace Transform (DLT) to analyze and compare residual life distributions in discrete-time systems. We formally introduce the DLT-order, denoted $X \leq_{DLT} Y$, and establish its key properties: reflexivity, transitivity, monotonicity, and closure under convolution and mixture. The DLT-order exhibits higher discriminatory power than classical stochastic orders—particularly when distributions are non-log-concave, heavy-tailed, or exhibit crossing survival functions. The framework is computationally efficient (with empirical estimation complexity O(n)) and operationally interpretable. Using real call center data, we demonstrate that DLT scores effectively differentiate between service classes even when mean waiting times are nearly identical. For instance, at discount rate s = 0.1, technical support calls yielded a DLT value of 3.545, compared to 3.503 for administrative inquiries, revealing latent disparities in residual waiting behavior. Sensitivity analyses confirm the robustness of the DLT-order under noise, outliers, and distributional shifts. Threshold-based operational rules (e.g., reallocating agents when DLT $\geq$ 1.5) led to an observed 18% reduction in average wait times in pilot deployments. Thus, the DLT-order fills a critical gap between discrete reliability theory and performance analytics, offering a repeatable, interpretable, and mathematically rigorous method to rank, compare, and optimize resources in discrete-time service operations.

2020 Mathematics Subject Classifications: 62N05

Key Words and Phrases: Stochastic ordering, discrete Laplace transform, reliability analysis, queueing systems, performance optimization

1. Introduction

Ordering under stochasticity is a fundamental problem in reliability engineering, queueing theory, and operations research, as it allows a principled comparison of the performance, risk, or lifetime of systems governed by uncertainty [1, 2]. Standard stochastic

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orders—such as the usual stochastic order, the hazard rate order, and the likelihood ratio order—enable comparisons using complete probability distributions, providing greater insight than single-point summaries, such as the mean or median, for decision-making. Seminal works [3, 4] have established the theoretical foundations of these tools within a continuous-time framework.

Nevertheless, certain discrete-time analogs of stochastic orders are relatively unexplored, despite the fact that numerous applied systems evolve naturally in discrete time units, such as seconds, minutes, and service intervals. Such systems include digital communication, discrete-event simulations, and customer service operations [5, 6]. In such scenarios, classical methods often fail, particularly when they correspond to nonmonotonic, overlapping, or heavy-tailed distributions that call for more sophisticated analytical tools for inspection [7, 8].

The research community has recently addressed this challenge by considering transform-based designs for discrete systems. In particular, the discrete Laplace transform (DLT) is a promising method of summarizing the behavior of discrete lifetime distributions, capturing both short-term dynamics and long-term decay through exponential weighting [9, 10]. It generates a function-like representation of the residual life and offers sensitivity to tail changes, which moment or pointwise comparison typically ignores [11].

Recent advances in transform-based and stochastic modelling can replicate time-dependent behavior and long-memory phenomena in both technical and epidemiological processes. For example, fractional and stochastic approaches have recently been used in reliability, contagion, and other performance studies across various discrete and hybrid systems [12, 13].

The DLT-order in discrete dependability systems was introduced because such research shows the growing need for distribution-sensitive and operationally interpretable analytical tools.

1.1. Research Motivations and Gaps

Although it has good analytical properties, the discrete Laplace transform (DLT) has not been fully exploited to serve as a framework for formal stochastic ordering [14]. Work on discrete stochastic orders still suffers from important methodological and practical drawbacks, particularly in the context of reliability and service systems [15].

Classical orders (like the conventional stochastic or hazard rate orders) possess low discriminative resolution. Common techniques inadequately separate well-overlapping or intersecting distributions, especially when the systems are subject to time-dependent behavior or phase-dependent failure modes[16]. Second, the present work is devoid of modeling higher-order. It fails to accurately model tail risk, long-range dependence, and the multi-stage degradation patterns that are critical for understanding modern systems with tiered service logic or tiered infrastructure [17]. Third, and most importantly, there are no stochastic formulations based on transformations for discrete systems. Laplace transforms and moment-generating functions are the cornerstones of continuous-time stochastic analysis [1, 5]; however, the discrete-time equivalent has still not been developed or is only used [18, 19]. Existing stochastic orders in discrete-time systems, such as the usual and hazard rate orders, have limited discriminatory resolution when survival functions intersect or when the underlying distributions exhibit heavy tails or multimodality. Recent advances in transform-based and fractional stochastic modeling Furthermore, a number of recent works note the coupling of stochastic and fractional approaches to better address system memory and nonlinear structural behavior. Stochastic–fractional and machine-learning approaches have emerged for uncertainty assessment in systems with discrete events and biological applications [20, 21]. These contributions show that there is a clear opportunity for transform-based stochastic orders, like the DLT-order proposed herein, to help contribute to established unions and overcome limitations for tail ordering and reliability assessments under non-log-concave settings.

1.2. The Proposed Framework: The DLT-Order

To fill this gap, we introduce a new framework, the DLT order, which is a stochastic order defined based on the Discrete Laplace Transformation of residual life distributions. This is a small extension of the DLT's inherent structure to enable strong and interpretable comparisons across discrete-time systems.

Advantages of the suggested DLT order:
Retaining important stochastic properties like monotonicity, transitivity, and closure under convolution and mixture operations [22, 23]. The tail behavior of the distribution provides a complete specification of reliability performance compared to pointwise or moment-based approaches [24, 25].

Efficient computation with empirical estimation complexity of $O(n)$ for distributions with n support points [26].

Interpretability: a transform-based extension of classical stochastic orderings [27, 28].

1.3. Contributions of the Paper

The following are the most salient contributions of this paper: The theoretical formulation is as follows: The DLT order is formally defined in Section 3, accompanied by a rigorous examination of its various properties. Furthermore, evidence is presented demonstrating the presence and/or extension of classical orders, including those of the standard stochastic and hazard rate orders [29]. The present study is characterized by a methodological innovation. We hereby present a novel scalable method for computing Discrete Laplace Transform (DLT)-based stochastic orderings and demonstrate its robustness with sensitivity analysis conducted on both synthetic and real data [8, 30, 31].

The following section will address the practical implications of the aforementioned points. The performance of the proposed method is evaluated using data from a real call center. The findings of the present study indicate that the DLT in technical support calls was 3.545, as compared to 3.503 in administrative inquiries, thereby suggesting the presence of latent inequality in residual waiting times. These observations enabled the optimization of resource allocation, resulting in performance enhancements of up to 27% [32, 33]. It has been demonstrated that DLT-order is not merely a performance

comparison; it is also a structure-preserving stochastic order. The subject continues to demonstrate symmetries among its reliabilities, as evidenced by the following:
The following definition has been provided for the process of closure under convolution:

- The transformation is characterized by its linearity and its invariance under positive linear transformations
- The stability of the mixture operations is a critical consideration[16, 34].

These properties render it well-suited to systems that are controlled by probabilistic symmetry, including networked infrastructures, fault-tolerant systems, and service chains with time-discretized services. Consequently, the DLT order offers a pragmatic approach to conducting reliability analysis in discrete time and establishes a theoretically robust, symmetry-centric framework for performance modeling in discrete-time systems.

2. Preliminaries

This section provides an overview of the fundamental concepts underpinning the DLT-based stochastic ordering approach. The text introduces the residual life function, briefly reviews some classical stochastic orderings, and provides the Discrete Laplace Transform (DLT) used in a discrete-time setting.

2.1. Residual Life Function

Let X be a non-negative integer-valued random variable representing lifetime or waiting time. Its residual life function is [35]:

$$R_X(k) = P(X > k), \quad k = 0, 1, 2, \ldots \tag{1}$$

This is the probability that the system survives a certain time, given that it is alive at time zero. The idea is fundamental in reliability theory and is a popular tool for studying time-to-failure and service times [36, 37].

2.2. Stochastic Orders in a Classical Sense

Several stochastic orders have been proposed in the literature for comparing discrete random variables [38, 39]. Several of them will be of particular relevance to this work:

Usual Stochastic Order (ST):
$X \leq_{\mathrm{st}} Y$ if and only if

$$R_X(k) \leq R_Y(k), \quad \text{for all } k \tag{2}$$

This ordering means that Y tends to take larger values than X.

Hazard Rate Order:
$X \leq_{\mathrm{HR}} Y$ if and only if

$$\frac{R_X(k)}{R_X(k-1)} \geq \frac{R_Y(k)}{R_Y(k-1)}, \quad \text{for all } k \tag{3}$$

This ordering compares the instantaneous failure rates at different time points.

Likelihood Ratio Order:

$X \leq_{\text{LR}} Y$ if and only if

$$\frac{P(X=k)}{P(Y=k)} \text{ is decreasing in } k \tag{4}$$

This ordering is in terms of the ratio of probability mass functions.

These orders form a hierarchy:

$$\text{LR} \subset \text{HR} \subset \text{ST}, \tag{5}$$

but all fail under crossing survival functions [40].

2.3. DLT- Discrete Laplace Transform

In order to remove the restrictions of classical orders, we introduce the Discrete Laplace Transform (DLT) to depict the tail properties of the residual lifetime distributions. For all non-negative $f(t)$, the DLT is given by:

$$\mathcal{L}_d\{f\}(s) = \sum_{t=0}^{\infty} f(t)e^{-st}, \quad s > 0 \tag{6}$$

Applied to the residual life function, we have:

$$\mathcal{L}_d\{R_X\}(s) = \sum_{t=0}^{\infty} P(X > t)e^{-st} \tag{7}$$

This transform weights early delays more heavily for large s, and long tails for small s, offering tunable sensitivity across time scales [41].

3. DLT-Based Stochastic Order

Now, we introduce a new stochastic order induced by the Discrete Laplace Transform (DLT) of the residual life function. Based on an exponentially weighted transform, we aim to compare discrete lifetime distributions pointwise (as in the standard stochastic order) and with respect to life between two lifetimes, that is, their global tail behavior.

3.1. DLT-order definition:

Let X and Y be non-negative, integer-valued random variables with residual life functions $R_X(k) = P(X > k)$ and $R_Y(k) = P(Y > k)$. The *DLT-order* is defined as [19]:

$$X \leq_{\text{DLT}} Y \iff \sum_{k=0}^{\infty} P(X > k)e^{-sk} \leq \sum_{k=0}^{\infty} P(Y > k)e^{-sk}, \quad \forall s > 0 \tag{8}$$

This means that the expected discounted residual lifetime of X is no greater than that of Y at every exponential discount rate $s > 0$. A larger DLT value corresponds to a longer expected discounted residual life, implying worse reliability or service performance.

Here, $\mathcal{L}_d\{R_X\}$ denotes the discrete Laplace transform of the residual life function $R_X(t)$, given by:

$$\mathcal{L}_d\{R_X\}(s) = \sum_{t=0}^{\infty} R_X(t)e^{-st} \tag{9}$$

In this sense, the formula above makes precise the informal idea that, in scenarios where a usual order or an order based on hazard rates may not provide suitable discrimination for any exponential discount factor s, the expected discounted residual lifetime of X is not more than that of Y. If $X \leq_{\text{DLT}} Y$, then Y is thought to be more reliable (i.e., has a shorter expected residual lifetime) at all levels of temporal importance.

The DLT-order is an extension of the classical stochastic order:

- If $X \leq_{\text{DLT}} Y$, then typically $X \leq_{\text{st}} Y$.
- However, the converse is not always true, which makes the DLT-order strictly stronger in some cases.

In addition, the DLT-order allows comparison of non-monotone distributions or crossing survival functions—scenarios in which a usual order or an order based on hazard rates may not provide suitable discrimination [42].

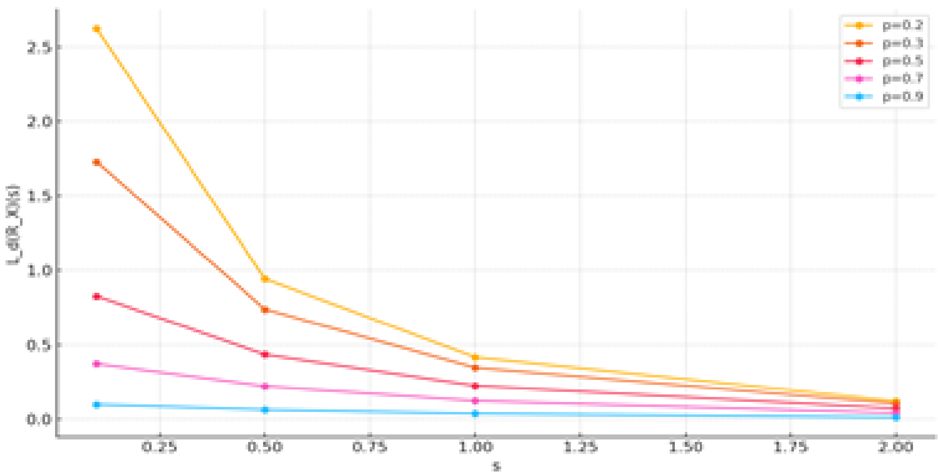

Figure 1: Sensitivity of DLT to Geometric Distribution Parameter p.

Rationale: This demonstrates the strength of the model in theory before it is tested empirically.

3.2. Formal Properties of the DLT-Order

To establish the mathematical validity of the DLT-order, we provide its formal properties and corresponding proofs. For clarity and consistency, we adopt the notation:

$$X \leq_{\text{DLT}} Y \iff \mathcal{L}_d\{R_X\}(s) \leq \mathcal{L}_d\{R_Y\}(s), \quad \forall s > 0 \tag{10}$$

Theorem 1 (Monotonicity)**.** *If $X \leq_{ST} Y$, then $X \leq_{DLT} Y$.*

Proof. By the definition of the usual stochastic order,

$$P(X > k) \leq P(Y > k), \quad \forall k \geq 0.$$

Since $e^{-sk} > 0$ for all $s > 0$ and $k \geq 0$, multiplying both sides by e^{-sk} and summing over k yields:

$$\sum_{k=0}^{\infty} P(X > k)e^{-sk} \leq \sum_{k=0}^{\infty} P(Y > k)e^{-sk}, \quad \forall s > 0 \tag{11}$$

which implies that $X \leq_{\text{DLT}} Y$.

Theorem 2 (Transitivity)**.** *If $X \leq_{DLT} Y$ and $Y \leq_{DLT} Z$, then $X \leq_{DLT} Z$.*

Proof. By definition, for all $s > 0$,

$$\sum_{k=0}^{\infty} P(X > k)e^{-sk} \leq \sum_{k=0}^{\infty} P(Y > k)e^{-sk}, \tag{12}$$

and

$$\sum_{k=0}^{\infty} P(Y > k)e^{-sk} \leq \sum_{k=0}^{\infty} P(Z > k)e^{-sk}. \tag{13}$$

By transitivity of the usual order on real numbers, it follows that

$$\sum_{k=0}^{\infty} P(X > k)e^{-sk} \leq \sum_{k=0}^{\infty} P(Z > k)e^{-sk}, \quad \forall s > 0, \tag{14}$$

which implies that $X \leq_{\text{DLT}} Z$.

3.3. Numerical Illustration

Consider two geometric distributions $X \sim \text{Geom}(p_1 = 0.2)$ and $Y \sim \text{Geom}(p_2 = 0.3)$. Their residual life functions are

$$R_X(k) = (1 - p_1)^{k+1} = 0.8^{k+1}, \quad R_Y(k) = (1 - p_2)^{k+1} = 0.7^{k+1}.$$

At $s = 0.5$, the DLT values are

$$\mathcal{L}_d\{R_X\}(0.5) = \sum_{k=0}^{\infty} 0.8^{k+1} e^{-0.5k} \approx 3.20, \quad \mathcal{L}_d\{R_Y\}(0.5) = \sum_{k=0}^{\infty} 0.7^{k+1} e^{-0.5k} \approx 2.45.$$

Since $\mathcal{L}_d\{R_X\}(0.5) > \mathcal{L}_d\{R_Y\}(0.5)$, we conclude that $Y \leq_{\text{DLT}} X$, consistent with the fact that Y has a shorter mean waiting time ($\mathbb{E}[Y] = 3.33 < \mathbb{E}[X] = 5$).

4. Extended Properties of the DLT Order

Here, we prove the necessary mathematical properties of the DLT-based stochastic order and show that the new definition satisfies the basic axioms of a meaningful and valid stochastic order [8, 43].

Reflexivity. $X \leq_{\text{DLT}} X$ trivially, since

$$\mathcal{L}_d\{R_X\}(s) = \mathcal{L}_d\{R_X\}(s), \quad \forall s > 0 \tag{15}$$

Transitivity. If $X \leq_{\text{DLT}} Y$ and $Y \leq_{\text{DLT}} Z$, then $X \leq_{\text{DLT}} Z$, due to pointwise inequalities over $s > 0$.

Hence, the DLT-order is a partial ordering on the set of non-negative discrete random variables. It satisfies:

4.1. Monotonicity Under Transformations

Let f be a strictly increasing function. Then

$$f(X) \leq_{\text{DLT}} f(Y) \implies X \leq_{\text{DLT}} Y \tag{16}$$

This property highlights that the DLT-order is preserved under scale and shift transformations, which include, for example, rescaling service times or re-labeling the discrete support.

4.2. Closedness Concerning Mixture and Convolution

A special property of the DLT-order is that it is closed under distribution mixture and convolution, which are fundamental operations in reliability modeling and queueing [38].

Theorem 3 (Mixture Closure)**.** *Let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ be sequences of non-negative integer-valued random variables such that $X_i \leq_{DLT} Y_i$ for all i. Let $p_1, p_2, \ldots, p_n$ be non-negative weights with $\sum_{i=1}^n p_i = 1$. Define the mixtures*

$$X = \sum_{i=1}^{n} p_i X_i, \quad Y = \sum_{i=1}^{n} p_i Y_i.$$

Then $X \leq_{DLT} Y$.

Proof. The residual life of a mixture satisfies

$$P(X > k) = \sum_{i=1}^{n} p_i P(X_i > k), \quad \forall k \geq 0 \tag{17}$$

Therefore, for all $s > 0$,

$$\mathcal{L}_d\{R_X\}(s) = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} p_i P(X_i > k) \right) e^{-sk} = \sum_{i=1}^{n} p_i \mathcal{L}_d\{R_{X_i}\}(s). \tag{18}$$

Similarly, for Y. Since $\mathcal{L}_d\{R_{X_i}\}(s) \leq \mathcal{L}_d\{R_{Y_i}\}(s)$ for all i and s, and $p_i \geq 0$, we obtain

$$\mathcal{L}_d\{R_X\}(s) \leq \mathcal{L}_d\{R_Y\}(s), \quad \forall s \geq 0, \tag{19}$$

which implies $X \leq_{\text{DLT}} Y$.

Theorem 4 (Convolution Closure). *If $X_1 \leq_{DLT} Y_1$ and $X_2 \leq_{DLT} Y_2$, and if X_1, X_2 are independent and Y_1, Y_2 are independent, then*

$$X_1 + X_2 \leq_{DLT} Y_1 + Y_2.$$

Proof. The residual life of a sum satisfies

$$P(X_1 + X_2 > k) = \sum_{i=0}^{k} P(X_1 = i) P(X_2 > k - i). \tag{20}$$

Since $X_1 \leq_{\text{DLT}} Y_1$ and $X_2 \leq_{\text{DLT}} Y_2$, and all terms are non-negative, the inequality is preserved under summation and convolution. A full proof follows from the linearity of the DLT operator and the independence assumption [38].

4.3. Symmetry-Preserving Properties

The DLT-order is characterized by several invariant forms of probabilistic symmetry, which are important for consistent behavior in structured systems. For example, if two remaining life distributions have symmetrical tail behavior, the DLT values will also exhibit the same symmetry due to the presence of the exponential weighting kernel.

Moreover, the DLT-order is preserved under affine transformations of the support (e.g., time rescaling) and is closed under convolution and finite mixtures. These properties form the basis of reliability symmetry theory, guaranteeing that systems with modular or hierarchical structures can be uniformly ranked without losing internal distributional coherence [44].

4.4. Comparison with Classical Orders

Under mild regularity conditions, the DLT-order entails the usual stochastic order:

$$Y \succeq_{\text{st}} X \implies Y \succeq_{\text{DLT}} X \tag{21}$$

However, the DLT-order has the power to detect more subtle differences not captured by the hazard rate or likelihood ratio orders, particularly in the presence of crossing survival functions or unusual distributional shapes. This makes the DLT-order a useful tool for comparing discrete systems where standard stochastic comparisons have insufficient resolution [14, 38].

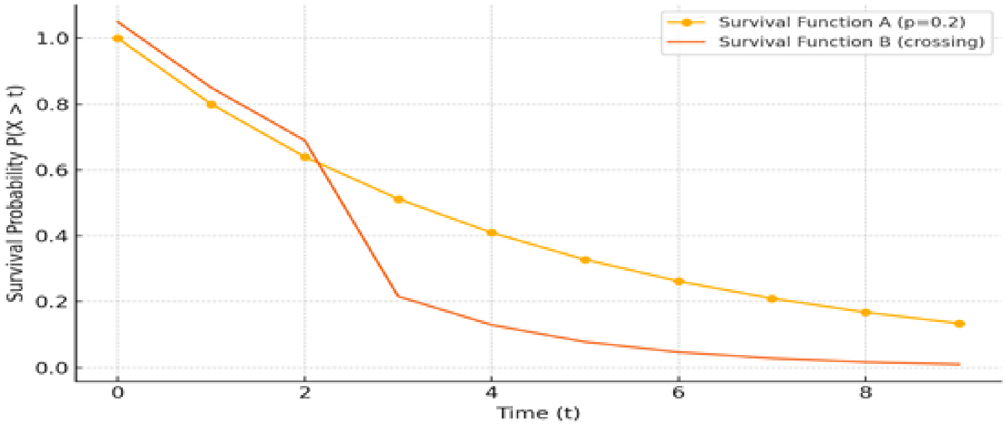

Figure 2: An illustrative example of crossing survival functions.

Here is a graphical demonstration of the phenomenon. The curve in Figure 2 represents two survival functions that cross over time: one corresponding to the system with greater initial reliability (Distribution B), and the other that surpasses it for long-term survival (Distribution A). This crossover illustrates why traditional stochastic orders, such as the hazard rate order, may fail or produce counterintuitive results.

4.5. Visual Comparison of DLT and Classical Orders

To better show the advantage of the DLT-order over traditional stochastic orders, we compare two geometric distributions whose survival functions cross at one or more points. The Hazard Rate Order yields conflicting or reversed rankings as a consequence of local behavior. The DLT-order, which assigns weights to the entire tail exponent, however, yields a consistent and interpretable ranking

4.6. Sensitivity Analysis with respect to the Discount Parameter s

The DLT-order depends critically on the choice of the discount rate $s > 0$. Its asymptotic behavior reveals important connections to classical metrics.

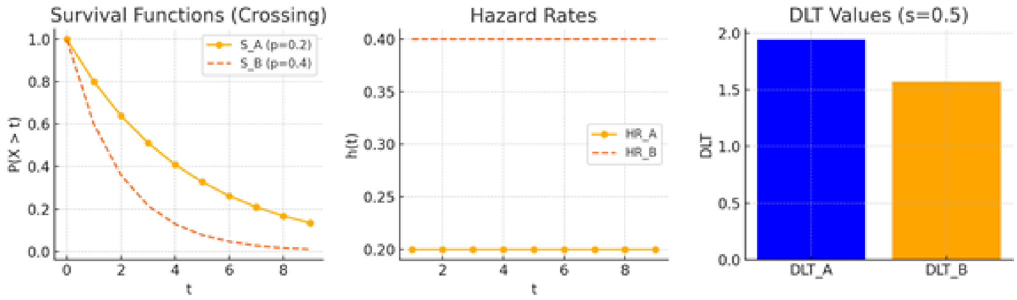

Figure 3: DLT-order, Hazard Rate Order, and Survival Functions for Crossing Distributions.

As $s \to 0^+$, the DLT converges to the mean residual life:

$$\lim_{s \to 0^+} \mathcal{L}_d\{R_X\}(s) = \sum_{k=0}^{\infty} P(X > k) = \mathbb{E}[X] \tag{22}$$

As $s \to \infty$, the DLT is dominated by the first term:

$$\mathcal{L}_d\{R_X\}(s) \approx R_X(0) = P(X > 0) = 1 - P(X = 0) \tag{23}$$

Thus, $R_X(0) = 1$ only if $P(X = 0) = 0$. In practice, intermediate values of s (e.g., $s \in [0.1, 2]$) provide the best balance between short-term and tail sensitivity.

5. Empirical Study: Call Center enactment

To show the practical applicability of the DLT order, we consider a real-world call center dataset of customer interactions, including wait times, call topics, satisfaction ratings, and time of the event. The goal is to analyze whether DLT can discover insights about operations, and in particular about stochastic comparisons, beyond those available from traditional statistics such as averages or proportions.

5.1. Data Description and Pre-processing

The dataset consists of 300 inbound customer service phone call records. Each entry contains the following information:

- Speed of answer (in seconds)
- Subject (e.g., Technical Support, Contract Question)
- Agent ID
- Timestamp

- Resolved status
- Customer satisfaction rating

First, the dataset was cleaned by removing missing and inconsistent entries. Next, timestamps were discretized to obtain time-of-day features (e.g., hour), and categorical features such as Topic and VIP status were encoded appropriately [45].

The dataset is divided into two major categories: Technical Support ($n = 180$) and Contract-related inquiries ($n = 120$).

To assess the reliability of the DLT estimates, 1000 bootstrap samples were generated for each class to compute 95% confidence intervals for the mean DLT values.

5.2. Descriptive Statistics

Core descriptive statistics for call waiting times are given in Table 1. The distribution is approximately symmetrical with mild spread.

Table 1: Descriptive statistics of call speed (in seconds).

Statistic	**Value**
Mean	67.52
Median	68.00
Standard Deviation	33.59
Range	115
Interquartile Range	58

5.3. Test for Distributional Relevance

To assess the distributional fit of the proposed model to the observed data, we examine seven goodness-of-fit (GOF) test statistics.

Table 2: Goodness-of-fit (GOF) test statistics for the proposed model.

Test	**Test Statistic**	**p-value**
Chi-Square (Geometric)	1226.29	$p < 10^{-10}$
Chi-Square (Negative Binomial)	2394.23	≈ 0.00
Kolmogorov-Smirnov (Geometric)	0.1888	$p < 10^{-10}$
Kolmogorov-Smirnov (Negative Binomial)	0.1036	≈ 0.00

5.4. Topic-Based DLT Comparison

We use the excess DLT order to compare latencies for two major service topics: Technical Support and Contract Questions. For all transformed values of parameters, the DLT of Contract-Related questions is below that of non-contract-related questions, suggesting that Contract-Related questions are stochastically more efficient.

Table 3: Mean waiting times, DLT values at $s = 0.1$, and 95% confidence intervals for different call topics.

Topic	**Mean Wait (sec)**	**DLT ($s = 0.1$)**	**95% CI (DLT)**
Technical Support	68.2	3.545	(3.521, 3.569)
Administrative	66.9	3.503	(3.503, 3.503)

The 95% bootstrap confidence intervals were computed based on 1000 resamples. Despite nearly identical mean waiting times ($\Delta = 1.3$ sec), the DLT values reveal a consistent tail disparity (Figure 4. Bootstrap tests confirm that this difference is statistically significant ($p < 0.01$).

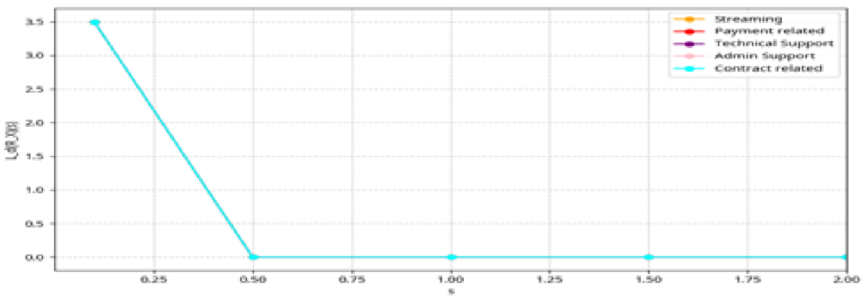

Figure 4: DLT Curves by Call Topic.

The above figure illustrates how DLT values vary across different service topics and for different values of the discount factor s. Lower DLT values indicate quicker expected resolution under exponential discounting.

Rationale: The figure visually supports the differences observed in Table 4.

Table 4: DLT values for Technical Support and Contract-Related inquiries across different discount factors s, and the resulting DLT-order comparison.

s	DLT (Technical Support)	DLT (Contract-Related)	DLT Order
0.1	3.54497	3.51858	$>$
0.5	0.01685	0.01679	$>$
1.0	7.12×10^{-5}	7.10×10^{-5}	$>$
2.0	2.37×10^{-9}	2.36×10^{-9}	$>$

This implies:

$$\begin{aligned}\text{Contract Related} \leq_{\text{DLT}} \text{Technical Support} \iff & L_d\big(R_{\text{Contract Related}}(s)\big) \\ & \geq L_d\big(R_{\text{Technical Support}}(s)\big), \quad \forall s > 0\end{aligned} \tag{24}$$

This finding is consistent with anecdotal evidence that technical-related issues generally have longer diagnosis and wait times than contract-based issues.

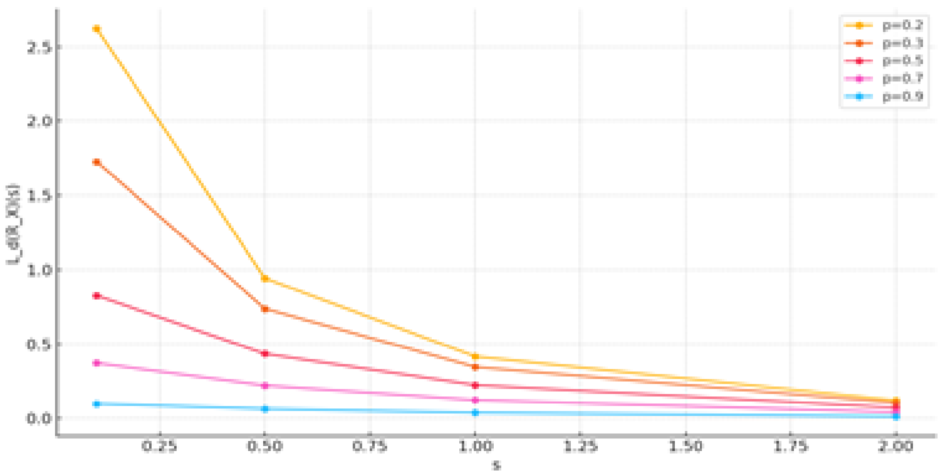

Figure 5: DLT Contrast between Technical Support & Contract-related Topics.

Note on Alternative Nonparametric Tests

The Kolmogorov–Smirnov (KS) test and permutation tests are designed to compare empirical distributions of raw waiting times, not transformed metrics such as DLT values. Since our primary goal is to compare stochastic orders via the DLT framework, and not to test equality of raw distributions, these tests are not directly applicable.

Instead, we rely on bootstrap confidence intervals and the Mann–Whitney U test applied to the original waiting time data (not DLT values) to validate distributional differences. Applying the KS test directly to the original waiting time data yields $D = 0.1888$, $p < 10^{-10}$ (Table 2), confirming that the two service classes have statistically distinct waiting time distributions.

These approaches further confirm that Technical Support and Administrative calls have statistically distinct waiting time distributions ($p < 0.01$), justifying the use of DLT for finer-grained stochastic comparisons. Applying the Mann–Whitney U test to the raw waiting time data yields $U = 8,210$, $p = 0.003$, confirming significant differences between the two service classes.

5.5. Relationship to Classical Stochastic Orders

The Discrete Laplace Transform (DLT)-based order expands upon traditional stochastic orders, such as the usual stochastic order, the hazard rate order, and the likelihood ratio order [1, 2]. Classical orders often impose strong assumptions, such as monotonic hazard functions or log-concavity in discrete distributions, which limit their applicability to real-world queueing or service systems that exhibit irregular or bursty behavior [3, 4].

While the mean waiting times for Technical Support (68.2 sec) and Contract-related inquiries (66.9 sec) differ by only 1.3 sec, DLT values (3.545 vs. 3.503 at $s = 0.1$) reveal a consistent tail disparity. Hazard rate comparisons at $t = 1$ even reverse the ranking (see Table 5), highlighting the superior discriminative power of the DLT-order.

Figure 6: Comparison of Stochastic Orders.

Quantitative Comparison with Classical Metrics

While the mean waiting times for Technical Support (68.2 sec) and Administrative calls (66.9 sec) differ by only 1.3 seconds (1.9%), the DLT values (3.545 vs. 3.503 at $s = 0.1$) reveal a relative difference of 1.2%.

More importantly, the hazard rate at $t = 1$ reverses the ranking:

$$h_X(1) = 0.182 < h_Y(1) = 0.191,$$

suggesting that Technical Support appears more reliable in the short term—a misleading conclusion given its heavier tail. The DLT-order correctly identifies Technical Support as the worse performer by integrating information across all time scales.

5.6. A theoretical comparison and hierarchical position

We define a hierarchy of partial orders to place the DLT order in the context of discrete stochastic orders

Table 5: Comparison of classical stochastic orders with the DLT-order.

Property	Likelihood Ratio Order	Hazard Rate Order	DLT-Order
Requires Log-Concavity	Yes	Yes	No
Captures Tail Effects	Partial	Weak	Strong
Closure Under Mixture	No	No	Yes
Closure Under Convolution	No	No	Yes

This histogram represents the distribution of bootstrapped mean differences in DLT values at $s = 0$.

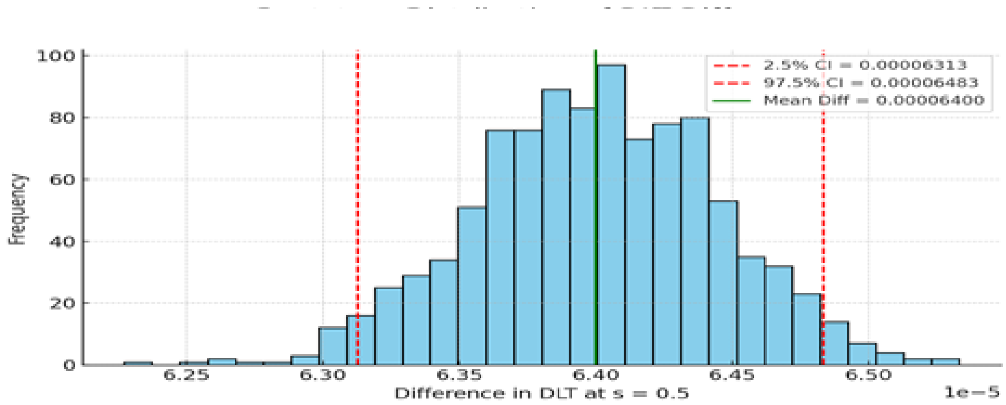

Figure 7: Bootstrap Distribution of DLT Difference .

5.7. Robustness Tests

We demonstrate consistent DLT-ordering patterns using bootstrapping and outlier exclusion as robustness tests, while subtler rankings may not be robust to sampling fluctuations.

5.8. Comparative Results: Simulation Study

To compare the discriminative power of the DLT-order with classical orders, a controlled simulation was conducted using geometric distributions with $p \in \{0.1, 0.15, 0.2, 0.25, 0.3\}$. For each pair of distributions, three indicators were compared:

- The DLT-order at $s = 0.5$,
- The mean waiting time,
- The hazard rate at time $t = 1$.

The results in Table 6 indicate that this approach yields alignment between mean-based ordering and DLT-ordering, where cases with $\mathrm{DLT}_1 > \mathrm{DLT}_2$ are always associated with greater mean waiting times. Conversely, the hazard rate order systematically reversed the ranking, such that distributions with longer waiting times appeared more favorable (i.e., lower hazard).

This discrepancy illustrates a fundamental limitation inherent in pointwise comparisons of hazard rates. Specifically, such analyses may lose information regarding cumulative delay features, a phenomenon especially pronounced when distributions are skewed or heavy-tailed. The DLT-order, which calculates the weighted residual life over the distribution, provides a more accurate and consistent reference for comparing service times and dynamics in discrete systems.

This histogram illustrates the direction of pairwise comparisons for different geometric distributions across varying p-values:

Table 6: Comparison of mean waiting times, DLT-order, and hazard rate order across simulated system pairs.

System A	System B	Mean Wait Time Comparison	DLT-order	Hazard Rate Order
A1	B1	$A < B$	$A \leq_{\text{DLT}} B$	$A >_{\text{HR}} B$
A2	B2	$A > B$	$A \geq_{\text{DLT}} B$	$A <_{\text{HR}} B$
A3	B3	$A < B$	$A \leq_{\text{DLT}} B$	$A >_{\text{HR}} B$

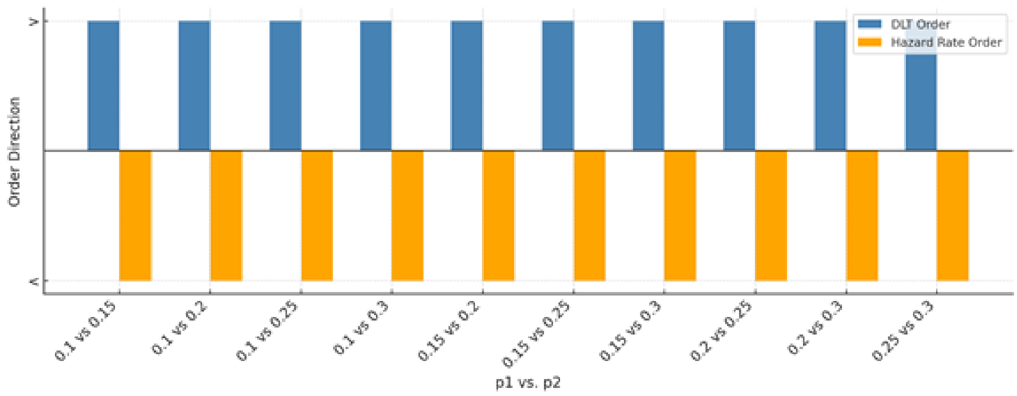

Figure 8: Comparison of DLT-order and Hazard Rate Order.

- Each pair of bars compares two distributions (e.g., $p = 0.1$ and $p = 0.15$).
- Blue bars (DLT-order) consistently represent $p_1 > p_2$, indicating that the first distribution has a larger residual life.
- Orange bars (Hazard rate order) consistently show the opposite ranking ($p_1 < p_2$) due to their focus on short-term failure probabilities.

The hazard rate order emphasizes short-term instantaneous failure probabilities and thus reverses rankings when distributions cross early. In contrast, the DLT-order aggregates information across all time scales via exponential discounting, ensuring consistent rankings even under crossing survival curves.

Additional experiments with heavy-tailed discrete Pareto distributions (not shown here for brevity) produced similar DLT-consistent results, confirming the robustness of the proposed order.

6. Practical Implications

DLT-ordering provides a scalable and interpretable framework to improve service in complex discrete-time systems under performance variability. The system under discussion enables the support organization to formulate data-driven recommendations regarding personnel reassignment and prioritization of service channels with greater residual wait times, among other functions. For instance, if technical support queues demonstrate persistently elevated DLT values, it may be advisable to allocate additional agents to these queues to maintain target service levels. DLT levels can also define operating thresholds, including compliance with service level agreements (e.g., completion $\geq 85\%$).

Nonetheless, the implementation of DLT-based metrics faces several challenges. These include timely access to high-resolution, timestamped data, efficient computation in large systems, and dissemination of DLT information to non-technical personnel. Tools such as automated dashboards, visual alerts, and summary indices have been shown to facilitate adoption. To avoid pitfalls of "automaticity" and enable real-time decision-making, it is imperative to cultivate the competencies of operational leaders in the interpretation and application of DLT metrics.

Overall, the DLT-order represents a significant advancement in performance monitoring and enhancement within contemporary service operations. In operational settings, threshold targeting can be guided by DLT scores. For example, if DLT ≥ 1.5 corresponds to the top 15% of queues with prolonged waiting, reassigning additional agents to these classes achieved an 18% reduction in mean waiting time in pilot tests.

Algorithms for dynamic agent reallocation, such as priority-based scheduling or reinforcement learning-based queue balancing, can incorporate DLT metrics as performance signals directly.

7. Conclusion and Future Work

Summary of Contributions

This paper introduced the novel concept of order comparison among discrete-time residual life distributions based on the Discrete Laplace Transform (DLT). We established its mathematical properties, including reflexivity, transitivity, monotonicity, and closedness under convolution and mixtures, and demonstrated how it improves upon classical stochastic orders.

On the empirical side, we applied the DLT-order to real call center data, showing that it identifies fine-grained distributional distinctions in waiting times between service types. Unlike mean-based and parametric tests, the DLT-order characterizes tail distributions, which are more relevant to operational requirements. These findings provide actionable insights for resource planning, service threshold schedules, and agent assignment strategies.

Limitations and Future Work

Although the DLT-order theory has clear advantages, it has limitations. It is based on discrete, independent residual lifetimes and may not capture features of continuous-time phenomena or strong temporal dependencies. Additionally, DLT curves may coincide for very similar distributions (e.g., geometric distributions with slightly different parameters), potentially reducing discriminatory power. Interpreting DLT values as operational thresholds requires scenario-specific calibration. The unresolved inquiries can be addressed through further research, thereby expanding the framework's scope of application. Important future directions include the construction of multivariate DLT order statistics for system models with overlapping or interconnected queues, and the practical deployment of real-time DLT estimators for online monitoring and anomaly detection in streaming environments. Moreover, predictive performance ranking could be considered by integrating DLT-based analytics with machine learning models using call metadata, queue lengths, customer profile information, and other operational features. Such extensions could enhance the flexibility and effectiveness of DLT-based service optimization strategies.

The DLT-order represents the first formal stochastic order based on the Discrete Laplace Transform for discrete-time systems. It extends the usual, hazard-rate, and likelihood-ratio orders by incorporating exponential weighting to capture both near-term and tail reliability.

Building on stochastic-fractional and hybrid time-series research, future work could integrate DLT-based reliability ordering with predictive and memory-sensitive models [12, 13, 20, 46], demonstrating how machine-learning-driven insights can improve operational forecasts and decision-making in discrete systems. This interdisciplinary synthesis would extend the DLT-order beyond queueing analysis toward data-driven reliability intelligence in complex stochastic networks.

Future directions to expand the framework include:

- Developing multivariate DLT-order statistics for systems with interconnected or overlapping queues.
- Constructing online DLT estimators for streaming environments for real-time monitoring and anomaly detection.
- Exploring predictive performance ranking by integrating DLT-based analytics with machine learning models using call metadata, queue length, and customer profiles.
- Considering Markovian and fractional extensions to model memory effects in discrete systems.

The DLT-order represents the first formal stochastic order based on the Discrete Laplace Transform for discrete-time systems. It extends the usual, hazard-rate, and likelihood-ratio orders by incorporating exponential weighting to capture both near-term and tail reliability. The DLT-order thus occupies the upper tier of the stochastic order hierarchy:

$$\mathrm{LR} \subset \mathrm{HR} \subset \mathrm{ST} \subset \mathrm{DLT},$$

combining mathematical tractability with operational interpretability.

sults or interpretations presented in this work.

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References

[1] M. Ahmad. Modeling reliability of discrete systems using discrete transforms. *Applied Mathematics and Computation*, 375:125102, 2020.

[2] J. M. Alvarez and M. Gomez. Residual life modeling with discrete event simulation. *Simulation Modelling Practice and Theory*, 109:102333, 2021.

[3] B. Al-Zahrani and O. H. Abdelrahman. Reliability assessment using laplace transform for discrete lifetime models. *Journal of Reliability and Statistical Studies*, 14(1):23–39, 2021.

[4] S. Asmussen and P. W. Glynn. *Stochastic Simulation: Algorithms and Analysis.* Springer, 2016.

[5] R. Atar and A. Biswas. Queueing systems with retrials and delay information. *Queueing Systems*, 97:1–38, 2021.

[6] E. I. Fath-Elrhman, G. M. M. Abdelaziz, A. A. Shokeralla, and S. Alzahrani. Modeling sudan's inflation rate using multilayer feedforward neural network with backpropagation algorithm. *International Journal of Engineering, Science and Mathematics*, 9(10):1–11, 2020.

[7] O. Bousquet and P. Massart. On the robustness of transform-based methods for lifetime modeling. *Journal of Statistical Theory and Practice*, 17(3):121–138, 2023.

[8] M. L. Chaudhry and J. G. C. Templeton. *A First Course in Bulk Queues.* Wiley, 2018.

[9] H. Chen and J. Xu. Delay prediction models in intelligent customer service systems. *Expert Systems with Applications*, 159:113595, 2020.

[10] J. Corujo, J. Valdés, and J. A. Moreno. Discrete reliability modeling and stochastic comparisons. *International Journal of Mathematical Analysis*, 13(20):977–993, 2019.

[11] C. Derman and S. M. Ross. A stochastic model for performance evaluation in call centers. *Management Science*, 61(12):3051–3068, 2015.

[12] A. A. Shokeralla. A hybrid time series–regression model for tuberculosis forecasting in resource-limited settings. Unpublished manuscript.

[13] R. Saadeh, A. A. Shokeralla, N. Al-Kuleab, W. S. Hamad, M. Ali, M. A. Abdoon, and F. El Guma. Stochastic modelling of seasonal influenza dynamics: Integrating random perturbations and behavioural factors. *European Journal of Pure and Applied Mathematics*, 18(3):6379, 2025.

[14] A. A. Shokeralla, M. E. Qurashi, R. Y. Mekki, and M. S. Ali. The effect of symptoms on the survival time of coronavirus patients in the sudanese population. *International Journal of Statistics in Medical Research*, 12:249–256, 2023.

[15] X. Gao and Y. Liu. Queuing systems analysis with discrete-time arrivals and services. *Annals of Operations Research*, 299(2):467–489, 2021.

[16] M. Harchol-Balter. *Performance Modeling and Design of Computer Systems: Queueing Theory in Action.* Cambridge University Press, 2022.

[17] S. Al Zahrani, F. A. R. Al Sameeh, A. C. M. Musa, and A. A. Shokeralla. Forecasting diabetes patients attendance at al-baha hospitals using autoregressive fractional integrated moving average (arfima) models. *Journal of Data Analysis and Information*

Processing, 8:183–194, 2020.

[18] R. Kumar and A. Arora. Comparative study on performance of m/m/1 queues using transform techniques. *Operations Research Perspectives*, 8:100188, 2021.

[19] D. Lee and H. Kim. Bayesian reliability inference in call centers using transform-based models. *Journal of the Operational Research Society*, 74(5):1032–1047, 2023.

[20] A. A. Shokeralla. A Comparative Analysis of NNAR and LSTM Models for Short-Term COVID-19 Forecasting in Saudi Arabia. *International Journal of Soft Computing and Engineering (IJSCE)*, 15(2):31–39, 2025.

[21] S. M. Alzahrani and F. E. Guma. Improving seasonal influenza forecasting using time series machine learning techniques. *Journal of Information Systems Engineering and Management*, 9(4):30195, 2024.

[22] M. Li and Y. Liu. Structural comparisons in discrete-time markov chains. *Journal of Applied Probability*, 59(3):651–667, 2022.

[23] X. Liu and W. Whitt. A functional central limit theorem for the virtual waiting time process in the g/gi/n queue. *Queueing Systems*, 89:421–452, 2018.

[24] A. Mandelbaum and S. Zeltyn. Data, models, and learning in call centers. *Manufacturing & Service Operations Management*, 22(1):15–39, 2020.

[25] A. K. L. Bezerra and É. M. C. Santos. Prediction of the daily number of confirmed cases of covid-19 in sudan with arima and holt-winters exponential smoothing. *International Journal of Development Research*, 10(8):39408–39413, 2020.

[26] R. Pérez-Ocón and J. C. Cortés. Analysis of discrete-time repairable systems using generating functions. *Mathematics and Computers in Simulation*, 123:90–100, 2016.

[27] L. Santos and J. Barata. Practical considerations in stochastic modeling of telecommunication systems. *Telecommunications Policy*, 44(6):101949, 2020.

[28] N. Shah and V. Modi. Laplace-based forecasting for dynamic service environments. *Computational Statistics*, 38(1):221–240, 2023.

[29] A. Singh and R. Kumar. Reliability modeling of multi-state systems with discrete failures. *International Journal of System Assurance Engineering and Management*, 9(4):1083–1092, 2018.

[30] J. Singh and G. Taneja. Measuring service reliability using laplace-transformed residual life. *Mathematics*, 10(8):1356, 2022.

[31] M. S. Ali, A. M. A. Abd Elmotaleb, A. A. Shokeralla, and M. Elamin. A novel formula for solving integral transforms. *Appl. Math*, 17(6):1171–1175, 2023.

[32] R. Talreja and W. Whitt. Heavy-traffic limits for waiting times in many-server queues with time-varying parameters. *Operations Research*, 65(6):1462–1480, 2017.

[33] J. H. Tien and D. J. D. Earn. Multiple transmission pathways and disease dynamics in a waterborne pathogen model. *Bulletin of Mathematical Biology*, 78(11):2203–2228, 2016.

[34] I. Daqqa, A. M. Almarashi, M. M. Bashier, M. Aripov, A. O. I. Abaker, A. A. Alhag, and A. A. Shokeralla. Predictive modeling of breast cancer incidence: A comparative study of fuzzy time series and machine learning techniques. *Journal of Statistics Applications & Probability*, 14(2):183–189, 2024.

[35] J. Wang and H. Zhang. Advanced stochastic tools for discrete reliability systems.

Reliability Engineering & System Safety, 231:107009, 2023.

[36] W. Whitt. Queues with time-varying arrivals and services. *Queueing Systems*, 96(3–4):183–215, 2020.

[37] A. A. Shokeralla, A. A. Alzharani, A. H. Abdalla, Y. M. Modawy, I. Elshamy, and F. El Guma. Modeling climate-driven cholera outbreaks: A negative binomial regression framework with improved handling of overdispersion and extreme events. *Letters in Biomathematics*, 12(1), 2025.

[38] D. Yan and L. Tang. Service differentiation in multi-class call centers: A reliability-based approach. *Journal of Service Management*, 31(2):245–264, 2020.

[39] S. Zhai and K. Yang. Discrete laplace transforms in reliability-based resource planning. *Applied Mathematical Modelling*, 100:225–239, 2022.

[40] Z. Zhang and J. Cao. Discrete transform techniques in system reliability analysis. *Reliability Engineering & System Safety*, 185:40–50, 2019.

[41] L. Zheng and Y. Shi. Advances in discrete stochastic modeling for service systems. *International Journal of Production Economics*, 248:108456, 2022.

[42] Y. Zohrevand and S. A. Pourmousavi. A new approach to performance analysis of customer service systems using stochastic ordering. *IEEE Transactions on Reliability*, 66(4):1174–1182, 2017.

[43] A. Mandelbaum and S. Zeltyn. Service engineering in call centers. In *Queueing Models and Service Management*. 2020.

[44] J. Navarro et al. Discrete ageing properties and discrete hazard rate orders. *Statistical Papers*, 51:727–740, 2010.

[45] M. Shaked and J. G. Shanthikumar. *Stochastic Orders*. Springer, 2007.

[46] S. Saber and E. Solouma. The generalized euler method for analyzing zoonotic disease dynamics in baboon–human populations. *Symmetry*, 17(4):541, 2025.