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Improving Cardinality Rough Neighborhoods via Grills and Their Applications

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Abstract. In order to solve problems and provide practical solutions, researchers attempt to accurately describe societal difficulties and obstacles. An efficient method for handling complicated real-world data is rough set theory. The method finds confirmed and likely data from subsets using rough approximation operators. In order to increase accuracy while following Pawlak's conventional approximation axioms, earlier research has created rough approximation models based on neighborhood systems. Based on cardinality rough neighborhoods and grills, we present new rough set notions in this study. These models are a suitable approach for a number of scenarios, including computational analysis, comparisons on medical datasets, real-world data analysis challenges, and classification accuracy metrics. We thoroughly examine the fundamental components of these ideas and clarify how they relate to each other and to earlier paradigms. Next, we describe the boundary regions and assess the accuracy of the data using a topological technique. Additionally, we look at how well our models handle heart failure disease in certain individuals and come to the conclusion that the suggested rough set concepts improve upon the characteristics of the earlier approach spaces. Finally, we identify the shortcomings of the current concepts and show their advantages in terms of extending the verified information gleaned from subsets of data while preserving the key elements of Pawlak's original paradigms that were destroyed by the models that went before them.

2020 Mathematics Subject Classifications: 54D80, 54C55, 54A25, 54A05

Key Words and Phrases: Lower/upper approximation, grill, neighborhood, rough set

1. Introduction

Rough sets theory (Rs_st) applications to knowledge discovery entail gathering actual data and using it to create classification models according to the data [1, 2]. Each subset in this theory corresponds to two distinct sets that are derived from an equivalence relation (eqr) and are referred to as the lower (Lw) and upper (Up) approximations (Ap_s). Many researchers have substituted different kinds of relations for the eqr in order to expand the

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applications of Rs_st [3–11]. As a result, equivalence classes were replaced with various neighborhood (nbd) representations of blocks or granular computing to characterize systems of information. These neighborhoods (nbd_s) include union, intersection, and right and left nbd_s [12–15], equal nbd_s [16, 17], rough nbd ideal [18]. The researchers demonstrated their usefulness in handling actual situations involving social difficulties, economics, and medical and in helping decision-makers make wise choices. In other words, they provide a broad framework devoid of limitations on the kind of binary relationships. Notably, Rs_st has demonstrated its usefulness as a crucial instrument for characterizing information content across numerous frameworks and applications in numerous fields [19–26]. In [27], the topological characteristics of Rs_s were examined. As a result, topological and rough set (Rs) theories were combined, and this became the main topic of several works [14, 28–32]. Additionally, this relationship involved topological generalizations, including minimal structures [33], supra topology [34], infra topology [35], nano-topology [36], and bitopology [37]. We direct readers to [38, 39] for a comprehensive summary of the works examining the connections between Rs_st and topology. The symmetry between closure and interior topological operators with Lw and Up-Ap_s, as well, allows us to give abstract conceptions meaningful meanings and employ abstract methods to convey knowledge extracted from systems of data. The concept of two operators, ξ and ζ, is the foundation of grill topological spaces. Choquet was the first to propose this concept [40]. Some similarities between the Choquat notion and ideals, nets, and filters have been found. [41] and [18] have addressed a number of hypotheses and aspects. It contributes to the expansion of the topological form, which is used to measure attributes like love, intelligence, beauty, and educational attainment rather than numbers. Furthermore, it extends the topological structure (tsr) by using the idea of grill alterations in the boundary region (Br), Up, and Lw-Ap_s, opening up new boundaries in nano topological spaces [42]. Grills are effective for analyzing raw datasets, especially for eliminating ambiguity. Consequently, a principal impetus for this endeavor is the implementation of novel grill-based Rs methodologies. To put it another way, it creates a special situation for its generic Rs model counterparts when the grill is the universal set. Because of this, a few intriguing studies examined the Rst that grills describe. This paper is laid out as follows. In order to explain why this study is necessary, we refer to the earlier types of R nbd_s and the key ideas associated with them in the following section. We then divide Sect. 3 into two subsections, each of which has two new Ap spaces based on cardinality rough nbd_s via grills. We present a new kind of Ap space and examine its primary characteristics in Sect. 3.1. To deal with irrational traits and ambiguous situations, In Sect. 3.2, we modified the prior method and demonstrate the advantages of the new one. The objective of Sect. 4 is to investigate the suggested Rs models from a topological perspective. In Sect. 5, we demonstrate the effectiveness of the provided models in handling a medical scenario involving the treatment of heart failure. We also highlight how our method enhances making choices and how we apply a topological approach motivated by this method to determine the most important characteristics or symptoms for making choices. Finally, in Sect. 6 we address the benefits and drawbacks of the current models and highlight the key findings of this work with a proposal for further research, respectively.

2. Preliminaries

This part was devoted to reviewing the key terms and findings and explaining the advantages of hybridizing grills with cardinal rough *nbds* in order to improve accuracy.

2.1. Conventional approximation space

Definition 1. *[1] Let $\aleph$ represent a universe, which is a finite, nonempty set. One way to describe a binary relation on $\aleph$ is as a subcollection of $\aleph \times \aleph$. A binary relation Γ on $\aleph$ is a subclass Γ of $\aleph \times \aleph$. To indicate that (b, n) is an element of Γ, we write $b\Gamma n$. If a relation Γ on $\aleph$ is reflexive (i.e., $b\Gamma b$ for all $b \in \aleph$), symmetric ($b\Gamma n \Leftrightarrow n\Gamma b$), and transitive (i.e.,$b\Gamma m$ when $b\Gamma n$ and $n\Gamma m$), we call it an equivalence. Furthermore, a relation is referred to as comparable if it fulfills $b\Gamma n$ or $n\Gamma b$ for any $b, n \in \aleph$.*

Definition 2. *[1] Let Γ represent an eqr on $\aleph$, and $\Delta \sqsubseteq \aleph$ so, the Lw and Up-Ap_s of Δ will be represented as:*

$$\underline{\Gamma}(\Delta) = \sqcup\{D \in \tfrac{\aleph}{\Gamma} : D \sqsubseteq \Delta\}.$$

$$\overline{\Gamma}(\Delta) = \sqcup\{D \in \tfrac{\aleph}{\Gamma} : D \sqcap \Delta \neq \phi\}$$

Notation $\frac{\aleph}{\Gamma}$ represents the family that includes all equivalency classes brought to by the relation Γ.
Henceforth referred to as an Ap space is the pair $(\aleph, \Gamma)$. It is described as follows if $\overline{\Gamma}(\Delta)$ and $\underline{\Gamma}(\Delta)$ are not equal, Δ is said to be rough. On the other hand, A subset Δ is said to be exact if $\underline{\Gamma}(\Delta) = \overline{\Gamma}(\Delta)$.

The following proposition lists the main characteristics of the conventional *Rs* model.

Proposition 1. *[1, 2]*
Examine the definition of an eqr Γ on $\aleph$. The following properties are true for sets D and Δ:

(Ł1) $\underline{\Gamma}(D) \sqsubseteq D$.

(Ł2) $\underline{\Gamma}(\aleph) = \aleph$.

(Ł3) $\underline{\Gamma}(\phi) = \phi$.

(Ł4) $\underline{\Gamma}(D) \sqsubseteq \underline{\Gamma}(\Delta)$ *whenever* $D \sqsubseteq \Delta$.

(Ł5) $\underline{\Gamma}((D) \sqcap \Delta) = \underline{\Gamma}(D) \sqcap \underline{\Gamma}(\Delta)$.

(Ł6) $\underline{\Gamma}(D) \sqcup \underline{\Gamma}(\Delta) \sqsubseteq \underline{\Gamma}(D \sqcup \Delta)$.

(Ł7) $\underline{\Gamma}(D^c) = (\overline{\Gamma}(D))^c$.

(Ł8) $\underline{\Gamma}(\underline{\Gamma}(D)) = \underline{\Gamma}(D)$.

(Ł9) $\underline{\Gamma}((\underline{\Gamma}(D))^c) = (\overline{\Gamma}(D))^c$.

(Ł10) $\underline{\Gamma}(\Delta) = \Delta, \forall \Delta \in \frac{\aleph}{\Gamma}$.

$(U1)$ $D \sqsubseteq \overline{\Gamma}(D)$.

$(U2)$ $\overline{\Gamma}(\aleph) = \aleph$.

$(U3)$ $\overline{\Gamma}(\phi) = \phi$.

$(U4)$ *If* $D \sqsubseteq \Delta$, *then* $\overline{\Gamma}(D) \sqsubseteq \overline{\Gamma}(\Delta)$.

$(U5)$ $\overline{\Gamma}(D \sqcap \Delta) \sqsubseteq \overline{\Gamma}(D) \sqcap \overline{AP}(\Delta)$.

$(U6)$ $\overline{\Gamma}(D \sqcup \Delta) = \overline{\Gamma}(D) \sqcup \overline{\Gamma}(\Delta)$.

$(U7)$ $\overline{\Gamma}(D^c) = (\underline{\Gamma}(D))^c$.

$(U8)$ $\overline{\Gamma}(\overline{\Gamma}(D)) = \overline{\Gamma}(D)$.

$(U9)$ $\overline{\Gamma}((\overline{\Gamma}(D))^c) = (\underline{\Gamma}(D))^c$.

$(U10)$ $\overline{\Gamma}(\Delta) = \Delta, \forall \Delta \in \frac{\aleph}{\Gamma}$.

Numerous methods have been used to expand traditional theory [26], and the validity of these traits has been well validated. Regretfully, it has been discovered that some qualities are questionable. However, it is seen to be beneficial in these approaches to acquire as many of these traits as is practical.

Definition 3. *[1, 2] The* Σ*-accuracy and* $\Re$*-roughness criteria of* Δ *are ascertained by taking an eqr* Γ *on* $\aleph$ *into consideration:*

$$\Sigma(\Delta) = \frac{|\underline{\Gamma}(\Delta)|}{|\overline{\Gamma}(\Delta)|},\ |\overline{\Gamma}(\Delta)| \neq 0.$$

$$\Re(\Delta) = 1 - \Sigma(\Delta).$$

The equivalency relations are not achievable in many situations. Consequently, weaker relationships than full equivalency have been used to augment the traditional approach.

2.2. ϱ-Neighborhood space types

Definition 4. *[13, 15, 43, 44] In an arbitrary relation Γ on $\aleph$. where $\varrho \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$, the ϱ-nbd$_s$ of an $e \in \aleph$ (denoted by $M_\varrho(e)$) are defined as follows:*

(1) $M_r(e) = \{\pi \in \aleph : e\Gamma\pi\}$.

(2) $M_l(e) = \{\pi \in \aleph : \pi\Gamma e\}$.

(3)
$$M_{\langle r\rangle}(e) = \begin{cases} \sqcap_{e\in M_r(\pi)} M_r(\pi) & : \ \exists\ M_r(\pi) \text{ containing } e, \\ \phi & : \ \text{otherwise.} \end{cases}$$

(4)
$$M_{\langle l\rangle}(e) = \begin{cases} \sqcap_{e\in M_l(\pi)} M_l(\pi) & : \ \exists\ M_l(\pi) \text{ containing } e, \\ \phi & : \ \text{otherwise.} \end{cases}$$
.

(5) $M_i(e) = M_r(e) \sqcap M_l(e)$.

(6) $M_u(e) = M_r(e) \sqcup M_l(e)$.

(7) $M_{\langle i\rangle}(e) = M_{\langle r\rangle}(e) \sqcap M_{\langle l\rangle}(e)$.

(8) $M_{\langle u\rangle}(e) = M_{\langle r\rangle}(e) \sqcup M_{\langle l\rangle}(e)$.

From now on, unless otherwise noted, we shall take into $\varrho = \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$.

Definition 5. *[43] Let us examine a relation Γ on $\aleph$ and demonstrate a mapping ξ_ϱ from $\aleph$ to $2^\aleph$ that connects each member of $\aleph$ to its ϱ-nbd in $2^\aleph$. The triple $(\aleph, \Gamma, \xi_\varrho)$, shortened to ϱ-NS, is hence referred to as a ϱ-nbd space.*

We used the aforementioned *nbd$_s$* to develop new *Lw* and *Up-Ap*, as well as accuracy (roughness) requirements. To improve *Ap* quality and accuracy, we compared several types of *nbd$_s$*.

Definition 6. *[13, 15, 43, 44] The Lw and Up-Ap of each subset Δ in relative to $M_\varrho(e)$-nbd$_s$ are introduced as follows:*

$$\mathcal{M}_{M_\varrho}(\Delta) = \{e \in \aleph : M_\varrho(e) \sqsubseteq \Delta\}.$$

$$\mathcal{M}^{M_\varrho}(\Delta) = \{e \in \aleph : M_\varrho(e) \sqcap \Delta \neq \phi\}.$$

Definition 7. *[12, 13, 15] The Σ_{M_ϱ}-accuracy and $\Re_{M_\varrho}$-roughness criteria of nonempty set Δ in regarding to Γ are computed as:*

$$\Sigma_{M\varrho}(\Delta) = \frac{|\mathcal{M}_{M\varrho}(\Delta) \sqcap \Delta|}{|\mathcal{M}^{M\varrho}(\Delta) \sqcup \Delta|}, \text{ and}$$

$$\Re_{M\varrho}(\Delta) = 1 - \Sigma_{M\varrho}(\Delta).$$

Definition 8. *[1, 2]*

Let Γ_1 and Γ_2 be relations on $\aleph$ such that $\Gamma_1 \sqsubseteq \Gamma_2$. The monotonicity property in accuracy and roughness of any set is demonstrated by Ap obtained from M-nbd$_s$, $\Sigma_{M\varrho_1}(\Delta) \geq \Sigma_{M\varrho_2}(\Delta)$, and $\Re_{M\varrho_1}(\Delta) \leq \Re_{M\varrho_2}(\Delta)$, in that order.

Definition 9. *[16, 17] Let us examine a relation Γ on $\aleph$. The $\mathcal{H}$-nbd$_s$ of an element $e \in \aleph$ for each ϱ are expressed as follows:*

(1) $\mathcal{H}_r(e) = \{\pi \in \aleph : M_r(e) = M_r(\pi)\}$.

(2) $\mathcal{H}_l(e) = \{\pi \in \aleph : M_l(e) = M_l(\pi)\}$.

(3) $\mathcal{H}_i(e) = \mathcal{H}_r(e) \sqcap \mathcal{H}_l(e)$.

(4) $\mathcal{H}_u(e) = \mathcal{H}_r(e) \sqcup \mathcal{H}_l(e)$.

(5) $\mathcal{H}_{\langle r\rangle}(e) = \{\pi \in \aleph : M_{\langle r\rangle}(e) = M_{\langle r\rangle}(\pi)\}$.

(6) $\mathcal{H}_{\langle l\rangle}(e) = \{\pi \in \aleph : M_{\langle l\rangle}(e) = M_{\langle l\rangle}(\pi)\}$.

(7) $\mathcal{H}_{\langle i\rangle}(e) = \mathcal{H}_{\langle r\rangle}(e) \sqcap \mathcal{H}_{\langle l\rangle}(e)$.

(8) $\mathcal{H}_{\langle u\rangle}(e) = \mathcal{H}_{\langle r\rangle}(e) \sqcup \mathcal{H}_{\langle l\rangle}(e)$.

Definition 10. *[29] Let us examine a relation Γ on $\aleph$. The $\Bbbk$-nbd$_s$ of an element $e \in \aleph$ for each ϱ are expressed as follows:*

(1) $\Bbbk_r(e) \sqsubseteq \{\pi \in \aleph : M_r(\pi) \sqsubseteq M_r(e)\}$.

(2) $\Bbbk_l(e) = \{\pi \in \aleph : M_l(\pi) \sqsubseteq M_l(e)\}$.

(3) $\Bbbk_i(e) = \Bbbk_r(e) \sqcap \Bbbk_l(e)$.

(4) $\Bbbk_u(e) = \Bbbk_r(e) \sqcup \Bbbk_l(e)$.

(5) $\Bbbk_{\langle r\rangle}(e) = \{\pi \in \aleph : M_{\langle r\rangle}(\pi) \sqsubseteq M_{\langle r\rangle}(e)\}$.

(6) $\Bbbk_{\langle l\rangle}(e) = \{\pi \in \aleph : M_{\langle l\rangle}(\pi) \sqsubseteq M_{\langle l\rangle}(e)\}$.

(7) $\Bbbk_{\langle i\rangle}(e) = \Bbbk_{\langle r\rangle}(e) \sqcap \Bbbk_{\langle l\rangle}(e)$.

(8) $\Bbbk_{\langle u\rangle}(e) = \Bbbk_{\langle r\rangle}(e) \sqcup \Bbbk_{\langle l\rangle}(e)$.

Definition 11. *[45] Let us examine a relation* Γ *on* $\aleph$*. The* $\nparallel$*-nbd*$_s$ *of an element* $e \in \aleph$ *for each* ϱ *are expressed as follows:*

(1) $\nparallel_r(e) = \{\pi \in \aleph : M_r(e) \sqsubseteq M_r(\pi)\}$.

(2) $\nparallel_l(e) = \{\pi \in \aleph : M_l(e) \sqsubseteq M_l(\pi)\}$.

(3) $\nparallel_i(e) = \nparallel_r(e) \sqcap \nparallel_l(e)$.

(4) $\nparallel_u(e) = \nparallel_r(e) \sqcup \nparallel_l(e)$.

(5) $\nparallel_{\langle r\rangle}(e) = \{\pi \in \aleph : M_{\langle r\rangle}(e) \sqsubseteq M_{\langle r\rangle}(\pi)\}$.

(6) $\nparallel_{\langle l\rangle}(e) = \{\pi \in \aleph : M_{\langle l\rangle}(e) \sqsubseteq M_{\langle l\rangle}(\pi)\}$.

(7) $\nparallel_{\langle i\rangle}(e) = \nparallel_{\langle r\rangle}(e) \sqcap \nparallel_{\langle l\rangle}(e)$.

(8) $\nparallel_{\langle u\rangle}(e) = \nparallel_{\langle r\rangle}(e) \sqcup \nparallel_{\langle l\rangle}(e)$.

2.3. Cardinality ϱ-Neighborhood system

This section is dedicated to introducing the concept of cardinality nbd_s, in accordance with any binary relation. The goal of studying cardinality nbd_s is to address specific situations where the number of members in M_ϱ-nbd_s affects the situation. We'll examine their primary characteristics and identify the circumstances in which some of them are the same. Various examples are offered to bolster the links and outcomes obtained. The cardinality of $M_\varrho(.)$ is indicated by $|M_\varrho(.)|$ for each $\varrho \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$.

Definition 12. *[46] For every* ϱ*, the* ϱ*-cardinality* nbd_s *of* $e \in \aleph$*, denoted by* $\mathcal{C}_\varrho(e)$*, are found using an arbitrary relation* Γ *on* $\aleph$ *as follows:*

(1) $\mathcal{C}_r(e) = \{\pi \in \aleph : |M_r(e)| = |M_r(\pi)|\}$.

(2) $\mathcal{C}_l(e) = \{\pi \in \aleph : M_l(e) = |M_l(\pi)|\}$.

(3) $\mathcal{C}_i(e) = \mathcal{C}_r(e) \sqcap \mathcal{C}_l(e)$.

(4) $\mathcal{C}_u(e) = \nparallel_r(e) \sqcup \mathcal{C}_l(e)$.

(5) $\mathcal{C}_{\langle r\rangle}(e) = \{\pi \in \aleph : |M_{\langle r\rangle}(e)| = |M_{\langle r\rangle}(\pi)|\}$.

(6) $\mathcal{C}_{\langle l\rangle}(e) = \{\pi \in \aleph : |M_{\langle l\rangle}(e)| = |M_{\langle l\rangle}(\pi)|\}$.

(7) $\mathcal{C}_{\langle i\rangle}(e) = \mathcal{C}_{\langle r\rangle}(e) \sqcap \mathcal{C}_{\langle l\rangle}(e)$.

(8) $\mathcal{C}_{\langle u\rangle}(e) = \mathcal{C}_{\langle r\rangle}(e) \sqcup \mathcal{C}_{\langle l\rangle}(e)$.

Proposition 2. *[46]*

(1) $\mathcal{C}_i \sqsubseteq \mathcal{C}_r \sqcap \mathcal{C}_l \sqsubseteq \mathcal{C}_r \sqcup \mathcal{C}_l \sqsubseteq \mathcal{C}_u$, *and* $\mathcal{C}_{\langle i\rangle} \sqsubseteq \mathcal{C}_{\langle r\rangle} \sqcap \mathcal{C}_{\langle l\rangle} \sqsubseteq \mathcal{C}_{\langle r\rangle} \sqcup \mathcal{C}_{\langle l\rangle} \sqsubseteq \mathcal{C}_{\langle u\rangle}$.

(2) All $\mathcal{C}_\varrho$ *are equal if* Γ *is a symmetric relation.*

Proposition 3. *[46]*
(1) $e \in \mathcal{C}_i(b)$ *iff* $|M_r(e)| = |M_r(b)|$ *and* $|M_l(e)| = |M_l(b)|$.

(2) $e \in \mathcal{C}_u(b)$ *iff* $|M_r(e)| = |M_r(b)|$ *or* $|M_l(e)| = |M_l(b)|$.

(3) $e \in \mathcal{C}_{\langle i\rangle}(b)$ *iff* $|M_{\langle r\rangle}(e)| = |M_{\langle r\rangle}(b)|$ *and* $|M_{\langle l\rangle}(e)| = |M_{\langle l\rangle}(b)|$.

(4) $e \in \mathcal{C}_{\langle u\rangle}(b)$ *iff* $|M_{\langle r\rangle}(e)| = |M_{\langle r\rangle}(b)|$ *or* $|M_{\langle l\rangle}(e)| = |M_{\langle l\rangle}(b)|$.

Corollary 1. *[46] If the relation* Γ *is symmetric, then:*

(1) $\mathcal{C}_i(b) = \{\pi \in \aleph : |M_i(b)| = |M_r(\pi)|\}$.

(2) $\mathcal{C}_{\langle i\rangle}(b) = \{\pi \in \aleph : |M_{\langle i\rangle}(b)| = |M_{\langle i\rangle}(\pi)|\}$.

(3) $\mathcal{C}_u(b) = \{\pi \in \aleph : |M_u(b)| = |M_u(\pi)|\}$.

(4) $\mathcal{C}_{\langle u\rangle}(b) = \{\pi \in \aleph : |M_{\langle u\rangle}(b)| = |M_{\langle u\rangle}(\pi)|\}$.

Proposition 4. *[46] Let* $(\aleph, \Gamma, \xi_\varrho)$ *be a* ϱ*-NS. If* $e \in \aleph$, *then* $\mathcal{C}_\varrho(e) \neq \phi \forall \varrho$.

Proposition 5. *[46] Let* $(\aleph, \Gamma, \xi_\varrho)$ *be a* ϱ*-NS and* $e \in \aleph$. *Then,* $e \in \mathcal{C}_\varrho(b)$ *iff* $b \in \mathcal{C}_\varrho(e) \forall \varrho$.

Proposition 6. *[46] Let* $(\aleph, \Gamma, \xi_\varrho)$ *be a* ϱ*-NS. If* $e \in \mathcal{C}_\varrho(b)$, $b \in \mathcal{C}_\varrho(n)$, *then* $e \in \mathcal{C}_\varrho(n)$, *in the cases of* $\varrho \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.

Corollary 2. *[46]*
Let $(\aleph, \Gamma, \xi_\varrho)$ *be a* ϱ*-NS and* $e \in \aleph$. *Then,* $e \in \mathcal{C}_\varrho(b)$ *iff* $\mathcal{C}_\varrho(e) = \mathcal{C}_\varrho(b)$, *in the cases of* $\varrho \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.

Corollary 3. *[46] In the instances of* $\varrho \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$, *the relation* Γ *where* $b\Gamma e \Leftrightarrow b \in \mathcal{C}_\varrho(e)$ *is an equivalence.*

Corollary 4. *[46] Under a symmetric relation, the cardinality nbd_s create a partition for $\aleph$ for each ϱ.*

Proposition 7. *[46] $\mathcal{C}_\varrho = \mathcal{C}_{\langle\varrho\rangle}$ for ϱ belongs to $\{r, l, i, u\}$, if Γ is a preorder relation on $\aleph$.*

Proposition 8. *[46] Let $(\aleph, \Gamma, \xi_\varrho)$ be a ϱ-NS. If $e \in \aleph$, then $\nparallel_\varrho(e) \sqsubseteq \mathcal{C}_\varrho(e) \forall \varrho$.*

Definition 13. *[46] Let $(\aleph, \Gamma, \xi_\varrho)$ be a ϱ-NS. Based on cardinality nbd_s, the $\mathcal{C}_\varrho$-Lw-Ap $\mathcal{M}_{\mathcal{C}_\varrho}(\Delta)$, and $\mathcal{C}_\varrho$-Up-Ap $\mathcal{M}^{\mathcal{C}_\varrho}(\Delta)$ of a set Δ, assigned as:*

$$\mathcal{M}_{\mathcal{C}_\varrho}(\Delta) = \{b \in \aleph : \mathcal{C}_\varrho(b) \sqsubseteq \Delta\}.$$

$$\mathcal{M}^{\mathcal{C}_\varrho}(\Delta) = \{b \in \aleph : \mathcal{C}_\varrho(b) \sqcap \Delta \neq \phi\}.$$

Definition 14. *[46] The $\mathcal{C}_\varrho$-boundary, $\mathcal{C}_\varrho$-positive, $\mathcal{C}_\varrho$-negative regions of a subset Δ within a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ are identified respectively as:*

$$\mathcal{B}_{\mathcal{C}_\varrho}(\Delta) = \mathcal{M}^{\mathcal{C}_\varrho}(\Delta) \backslash \mathcal{M}_{\mathcal{C}_\varrho}(\Delta).$$

$$\mathcal{P}^{\mathcal{C}_\varrho}(\Delta) = \mathcal{M}_{\mathcal{C}_\varrho}(\Delta).$$

$$\mathcal{N}^{\mathcal{C}_\varrho}(\Delta) = \aleph \backslash \mathcal{M}^{\mathcal{C}_\varrho}(\Delta).$$

Definition 15. *[46] The $\mathcal{C}_\varrho$-accuracy and $\mathcal{C}_\varrho$-roughness criteria of $\Delta \neq \phi$ of a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ are identified respectively as:*

$$\Sigma_{\mathcal{C}_\varrho}(\Delta) = \tfrac{|\mathcal{M}_{\mathcal{C}_\varrho}(\Delta)|}{|\mathcal{M}^{\mathcal{C}_\varrho}(\Delta)|}, |\mathcal{M}^{\mathcal{C}_\varrho}(\Delta)| \neq 0.$$

$$\Re_{\mathcal{C}_\varrho}(\Delta) = 1 - \Sigma_{\mathcal{C}_\varrho}(\Delta).$$

Theorem 1. *[46] Let $(\aleph, \Gamma, \xi_\varrho)$ be a ϱ-NS. Based on cardinality nbd_s, the family $\tau_{\mathcal{C}_\varrho} = \{\Delta \sqsubseteq \aleph : \forall e \in \Delta, \mathcal{C}_\varrho(e) \sqsubseteq \Delta\}$ is a topology on $\aleph$, for each ϱ.*

Lemma 1. *[46] Let $(\aleph, \Gamma, \xi_\varrho)$ be a ϱ-NS and $e \in \aleph$.*
If $\varrho \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$, then $\mathcal{C}_\varrho(e)$ is $\tau_{\mathcal{C}_\varrho}$-open set.

Definition 16. *[42] A nonempty subcollection $\mathcal{G} \sqsubseteq 2^{\aleph}$ is defined as a grill on $\aleph$ if the following are satisfied:*
$\phi \notin \mathcal{G}$
$H \in \mathcal{G}$, $H \sqsubseteq J \sqsubseteq \aleph$ leads to $J \in \mathcal{G}$,
if $H \sqcup J \in \mathcal{G}$ for $H, J \sqsubseteq \aleph$, then $H \in \mathcal{G}$ or $J \in \mathcal{G}$.

3. Grills and cardinality create new rough-set paradigms in neighborhoods

This section aims to delineate and examine novel *R-Ap* spaces generated by cardinality nbd_s and grills, with an emphasis on establishing new areas and standards for accuracy and roughness.

3.1. The initial class of Rs paradigms

This section is devoted to presenting novel Rs paradigms that are influenced by the concepts of grills and cardinality nbd_s. We demonstrate that, in contrast to the earlier models of Rs_s, In these paradigms, the *Lw-Ap* is amplified and the *Up-Ap* is minified. On the other hand, we talk about the current models' flaws.

Definition 17. *Let $(\aleph, \Gamma, \xi_\varrho)$ be a ϱ-NS and $\mathcal{G}$ be a grill on $\aleph$. The pair $(\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta), \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta))$ represents Lw, and Up-Ap of a subset Δ with respect to grills and cardinality nbd_s, and they are calculated by:*

$$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta) = \{b \in \aleph : \mathcal{C}_\varrho(b) \sqcap \Delta^c \notin \mathcal{G}\},$$

$$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta) = \{b \in \aleph : \mathcal{C}_\varrho(b) \sqcap \Delta \in \mathcal{G}\}$$

Remark 1. *If $\mathcal{G} = \aleph$ in Definition 3.1 then, $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta) = \phi$.*

Theorem 2. *Let $(\aleph, \Gamma, \xi_\varrho)$ be a ϱ-NS and $\mathcal{G}$ be a grill on $\aleph$. If $\Delta, D \sqsubseteq \aleph$, then $\forall \varrho$ the next statement hold true.*

(1) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\aleph) = \aleph$ and $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\phi) = \phi$.

(2) If $D \sqsubseteq \Delta$, then $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$ and $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$.

(3) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D \sqcap \Delta) = \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$ and $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D \sqcup \Delta) = \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D) \sqcup \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$.

(4) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta^c) = (\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta))^c$ and $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta^c) = (\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta))^c$.

(5) If $\Delta^c \in \mathcal{G}$, then $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta) = \aleph$ and $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta^c) = \phi$.

(6) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)) \sqsupseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$ and $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$, $\forall \varrho \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.

(7) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(n)) \sqsupseteq \mathcal{C}_\varrho(n)$
$\forall \varrho \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.

Proof. (1) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\aleph) = \{b \in \aleph : \mathcal{C}_\varrho(b) \setminus \aleph = \phi \notin \mathcal{G}\} = \aleph$ and
$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\phi) = \{b \in \aleph : \mathcal{C}_\varrho(b) \sqcap \phi \notin \mathcal{G}\} = \phi$.

(2) Clear.

(3) From (ii) we notice that $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D \sqcap \Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$. In contrast, suppose $b \in \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$. Then $b \in \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D)$ and $b \in \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$ it indicates that $\mathcal{G}_{\mathcal{C}_\varrho(b)} \backslash D \in \mathcal{G}$ and $\mathcal{G}_{\mathcal{C}_\varrho(b)} \backslash \Delta \in \mathcal{G}$. So, $\mathcal{G}_{\mathcal{C}_\varrho(b)} \backslash (D \sqcap \Delta) \in \mathcal{G}$. Thus, $b \in \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D \sqcap \Delta)$. Hence, $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D \sqcap \Delta)$. Similarly, it may be demonstrated that

$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D \sqcup \Delta) = \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D) \sqcup \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$.

(4) $b \in \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta^c) \Leftrightarrow \mathcal{C}_\varrho(b) \backslash \Delta^c \in \mathcal{G}$
$\Leftrightarrow \mathcal{C}_\varrho(b) \sqcap \Delta \in \mathcal{G}$
$\Leftrightarrow b \neq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$
$\Leftrightarrow b \in (\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta))^c$.
Likewise, it can be demonstrated $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta^c) = (\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta))^c$.
Let $b \in \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta^c) \Rightarrow \mathcal{C}_\varrho(b) \backslash \Delta^c \notin \mathcal{G}$
$\Leftrightarrow \mathcal{C}_\varrho(b) \sqcap \Delta \notin \mathcal{G}$
$\Leftrightarrow b \in (\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta))^c$.

(5) Let $\Delta^c \in \mathcal{G}$. For any $b \in \aleph$, $\mathcal{C}_\varrho(b) \backslash \Delta = \mathcal{C}_\varrho(b) \sqcap \Delta^c \in \mathcal{G}$. Hence, $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta) = \aleph$. From (4), $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta^c) = \phi$.

(6) Suppose $\varrho \in \{r, \langle r \rangle, l, \langle l \rangle, i, \langle i \rangle\}$. We will just demonstrate $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$. Let $b \in \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta))$, then $\mathcal{C}_\varrho(b) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta) \in \mathcal{G}$. Hence, $\mathcal{C}_\varrho(b) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta) \neq \phi$ i.e. there exists $m \in \aleph$ s.t. $m \in \mathcal{C}_\varrho(b)$, and $m \in \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$. This results in that $\mathcal{C}_\varrho(m) \sqcap \Delta \in \mathcal{G}$. Considering Corollary 2.20, $\mathcal{C}_\varrho(m) = \mathcal{C}_\varrho(b)$. Consequently, $\mathcal{C}_\varrho(b) \sqcap \Delta \in \mathcal{G}$ and so $b \in \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$.

(7) Suppose $\varrho \in \{r, \langle r \rangle, l, \langle l \rangle, i, \langle i \rangle\}$. Let $b \in \mathcal{C}_\varrho(n)$. In line with Corollary 2.20, $\mathcal{C}_\varrho(n) = \mathcal{C}_\varrho(b)$. Then, $\mathcal{C}_\varrho(b) \backslash \mathcal{C}_\varrho(n) \neq \phi \in \mathcal{G}$, and so $b \in \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(n))$, and $\mathcal{C}_\varrho(n) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(n))$.

It is clear that the following implication follows from points (2) of Theorem 3.3.

Corollary 5. *A grill on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ is denoted by $\mathcal{G}$. For any ϱ, the following ideas are true if $D, \Delta \sqsubseteq \aleph$:*

(1) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \sqcup \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D \sqcup \aleph)$.

(2) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D \sqcap \Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$.

Proposition 9. *Let $\mathcal{G}$ be a grill on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$. If $D, \Delta \sqsubseteq \aleph$, then*

(1) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_u}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(\Delta) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_l}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(\Delta) \sqcup \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_l}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_i}}(\Delta)$.

(2) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_i}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_r}}(\Delta) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_l}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_r}}(\Delta) \sqcup \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_l}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_u}}(\Delta)$.

(3) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle u \rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle r \rangle}}}(\Delta) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle l \rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle r \rangle}}}(\Delta) \sqcup \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle l \rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle i \rangle}}}(\Delta)$.

(4) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle i \rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle r \rangle}}}(\Delta) \sqcap \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle l \rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle r \rangle}}}(\Delta) \sqcup \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle l \rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle u \rangle}}}(\Delta)$.

Proof. This derives from Proposition 2.14 (1).

Proposition 10. *On a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$, let $\mathcal{G}$ be a grill, and let Γ be a symmetric relation. Then, $\forall \Delta \sqsubseteq \aleph$, all $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)(\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta))$ are equal.*

Proof. This derives from Proposition 2.14(2).

To demonstrate this, we provide the following example:

(1) In general, the opposite of items (2), (6), and (7) in Theorem 3.3 is not successful,

(2) Corollary 3.4's converse isn't always true,

(3) The subsets $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(b))$ and $\mathcal{C}_\varrho(b)$ are independent of one another in the case of $\varrho \in \{u, \langle i \rangle\}$,

(4) Proposition 3.5's opposite need not be true, and

(5) The current approach violates a few of Pawlak's paradigm's properties.

Example 1. *Consider the binary relation $\Gamma = \{(b,n),(n,n),((n,m),(m,e)\}$ on $\aleph = \{b,e,n,m\}$. The cardinality nbds for each element of $\aleph$ in Table 1 are then calculated.*

Table 1: $\mathcal{C}_\varrho$-nbd_s for members of $\aleph$.

	b	n	m	e
$\mathcal{C}_r$	$\{b,m\}$	$\{n\}$	$\{b,m\}$	$\{e\}$
$\mathcal{C}_l$	$\{b\}$	$\{n\}$	$\{em\}$	$\{e,m\}$
$\mathcal{C}_i$	$\{b\}$	$\{n\}$	$\{m\}$	$\{e\}$
$\mathcal{C}_u$	$\{b,m\}$	$\{n\}$	$\{b,e,m\}$	$\{e,m\}$
$\mathcal{C}_{\langle r \rangle}$	$\{b\}$	$\{n,e\}$	$\{m\}$	$\{n,e\}$
$\mathcal{C}_{\langle l \rangle}$	$\{b\}$	$\{n,m\}$	$\{n,m\}$	$\{e\}$
$\mathcal{C}_{\langle i \rangle}$	$\{b\}$	$\{n\}$	$\{m\}$	$\{e\}$
$\mathcal{C}_{\langle u \rangle}$	$\{b\}$	$\{e,n,m\}$	$\{n,m\}$	$\{n,e\}$

If $\mathcal{G} = 2^{\aleph} \backslash \phi$, then for each ϱ, the $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$, $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$ are computed in Table 2 and Table 3. It can now be seen as follows:

Currently, the following is visible:

(1) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\{D\}) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\{\Delta\})$ for each ϱ, and $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\{D\}) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\{\Delta\})$ whereas $\{D\} \sqsubseteq \{\Delta\}$.

(2) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\{n\} \sqcup \{e\}) = \{n,e\} \not\sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\{n\}) \sqcup \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\{e\}) = \phi$.

(3) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle r \rangle}}}\{(n)\} \sqcap \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle r \rangle}}}\{(e)\} = \{e\} \not\sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle r \rangle}}}\{(n) \sqcap \{e\}\} = \phi$.

Table 2: The Ap. for $\{r, l, i, u\}$.

Δ	$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_r}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_l}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_l}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_i}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_i}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_u}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_u}}(\Delta)$
$\{b\}$	ϕ	$\{b,n\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	ϕ	$\{b,m\}$
$\{n\}$	ϕ	$\{n\}$	$\{n\}$	$\{n\}$	$\{n\}$	$\{n\}$	$\{n\}$	$\{n\}$
$\{m\}$	ϕ	$\{b,m\}$	ϕ	$\{m,e\}$	$\{m\}$	$\{m\}$	ϕ	$\{b,m,e\}$
$\{e\}$	$\{e\}$	$\{e\}$	ϕ	$\{m,e\}$	$\{e\}$	$\{e\}$	ϕ	$\{m,e\}$
$\{b,n\}$	$\{n\}$	$\{b,n,m\}$	$\{b,n\}$	$\{b,n\}$	$\{b,n\}$	$\{b,n\}$	$\{n\}$	$\{b,n,m\}$
$\{b,m\}$	$\{b,m\}$	$\{b,m\}$	$\{b\}$	$\{b,m,e\}$	$\{b,m\}$	$\{b,m\}$	$\{b\}$	$\{b,m,e\}$
$\{b,e\}$	$\{e\}$	$\{b,m,e\}$	$\{b\}$	$\{b,e,m\}$	$\{b,e\}$	$\{b,e\}$	ϕ	$\{b,m,e\}$
$\{n,m\}$	$\{n\}$	$\{b,n,m\}$	$\{n\}$	$\{n,m,e\}$	$\{n,m\}$	$\{n,m\}$	$\{n\}$	$\aleph$
$\{n,e\}$	$\{n,e\}$	$\{n,e\}$	$\{n\}$	$\{n,m,e\}$	$\{n,e\}$	$\{n,e\}$	$\{n,e\}$	$\{n,e\}$
$\{m,e\}$	$\{e\}$	$\{b,m,e\}$	$\{m,e\}$	$\{m,e\}$	$\{m,e\}$	$\{m,e\}$	$\{e\}$	$\{b,m,e\}$
$\{b,n,m\}$	$\{b,n,m\}$	$\{b,n,m\}$	$\{b,n\}$	$\aleph$	$\{b,n,m\}$	$\{b,n,m\}$	$\{b,n\}$	$\aleph$
$\{b,n,e\}$	$\{n,e\}$	$\aleph$	$\{b,n\}$	$\aleph$	$\{b,n,e\}$	$\{b,n,e\}$	$\{n\}$	$\aleph$
$\{b,m,e\}$	$\{b,m,e\}$	$\{b,m,e\}$	$\{b,m,e\}$	$\{b,m,e\}$	$\{b,m,e\}$	$\{b,m,e\}$	$\{b,m,e\}$	$\{b,m,e\}$
$\{n,m,e\}$	$\{n,e\}$	$\aleph$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n\}$	$\aleph$
$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

The primary benefit of the current models over the earlier models presented in [24] is their ability to minimize the *Br* by enlarging the *Lw-Ap* and downsizing the *Lw-Ap* of subsets. This is demonstrated by the following outcome.

Theorem 3. *Let $\mathcal{G}$ be a grill on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ and let $D \sqsubseteq \aleph$. We have the subsequent relations for each ϱ.*

(1) $\mathcal{M}_{\mathcal{C}_\varrho}(D) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D)$,

(2) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\{D\}) \sqsubseteq \mathcal{M}^{\mathcal{C}_\varrho}(D)$.

Proof. Let $\pi \in \mathcal{M}_{\mathcal{C}_\varrho}(D)$. Then, $\mathcal{C}_\varrho(\pi) \sqsubseteq D$, and $\mathcal{C}_\varrho(\pi) \sqcap D^c \notin \mathcal{G}$. Currently, we have $\pi \in \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D)$. Hence, $\mathcal{M}_{\mathcal{C}_\varrho}(D) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D)$. A similar argument can be used to prove the second statement.

To demonstrate why the reverse of the preceding theorem fails in the first sentence, we shall provide an example.

Example 2. *Let $D = \{m\}$. Then, $\mathcal{M}_{\mathcal{C}_\varrho}(D) = \phi$, whereas $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) = \{m\}$.*

In the following, we show some of the models' weaknesses.

Remark 2. *Let $\mathcal{G} \neq 2^\aleph \backslash \phi$ be a grill on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ and let $D, \Delta \sqsubseteq \aleph$. Some shortcomings of the present preliminary set models are illustrated in the following assertions.*

(1) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \not\sqsubseteq D \not\sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\{D\})$.

Table 3: The Ap. for $\{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

Δ	$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle r\rangle}}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle r\rangle}}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle l\rangle}}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle l\rangle}}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle i\rangle}}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle i\rangle}}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle u\rangle}}}(\Delta)$	$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_{\langle u\rangle}}}(\Delta)$
$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{n\}$	ϕ	$\{n,e\}$	ϕ	$\{n,m\}$	$\{n\}$	$\{n\}$	ϕ	$\{n,m,e\}$
$\{m\}$	$\{m\}$	$\{n,m\}$	ϕ	$\{n,m\}$	$\{m\}$	$\{m\}$	ϕ	$\{n,m\}$
$\{e\}$	ϕ	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	ϕ	$\{n,e\}$
$\{b,n\}$	$\{b\}$	$\{b,n,e\}$	$\{b\}$	$\{b,n,m\}$	$\{b,n\}$	$\{b,n\}$	$\{b\}$	$\aleph$
$\{b,m\}$	$\{b,m\}$	$\{b,n,m\}$	$\{b\}$	$\{b,n,m\}$	$\{b,m\}$	$\{b,m\}$	$\{b\}$	$\{b,n,m\}$
$\{b,e\}$	$\{b\}$	$\{b,e\}$	$\{b,e\}$	$\{b,e\}$	$\{b,e\}$	$\{b,e\}$	$\{b\}$	$\{b,n,e\}$
$\{n,m\}$	$\{n,m\}$	$\{n,m,e\}$	$\{n,m\}$	$\{n,m\}$	$\{n,m\}$	$\{n,m\}$	$\{m\}$	$\{n,m,e\}$
$\{n,e\}$	$\{n,e\}$	$\{n,e\}$	$\{e\}$	$\{n,m,e\}$	$\{n,e\}$	$\{n,e\}$	$\{e\}$	$\{n,m,e\}$
$\{m,e\}$	$\{m,e\}$	$\{n,m,e\}$	$\{m,e\}$	$\{n,m,e\}$	$\{m,e\}$	$\{m,e\}$	$\{e\}$	$\{n,m,e\}$
$\{b,n,m\}$	$\{b,n,m\}$	$\aleph$	$\{b,n,m\}$	$\{b,n,m\}$	$\{b,n,m\}$	$\{b,n,m\}$	$\{b,m\}$	$\aleph$
$\{b,n,e\}$	$\{b,e\}$	$\{b,n,e\}$	$\{b,e\}$	$\aleph$	$\{b,n,e\}$	$\{b,n,e\}$	$\{b,e\}$	$\{b,n,e\}$
$\{b,m,e\}$	$\{b,m\}$	$\aleph$	$\{b,e\}$	$\aleph$	$\{b,m,e\}$	$\{b,m,e\}$	$\{b\}$	$\aleph$
$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$	$\{n,m,e\}$
$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$	$\aleph$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

(2) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\phi) \neq \phi$.

(3) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\aleph) \neq \aleph$.

(4) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)) \neq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta)$ *for* $\varrho \in \{\langle r\rangle, \langle u\rangle\}$.

(5) Let $\pi \in \aleph$. *Then,* $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(\pi)) \nsubseteq \mathcal{C}_\varrho(\pi) \forall \varrho$.

Example 3.11 illustrates property $(1 \to 5)$ of Remark 3.10

Example 3. *In Example 3.7. Let* $\mathcal{G} = \{\{b\}, \{b,e\}, \{b,n\}, \{b,m\}, \{b,e,n\}, \{b,e,m\}, \{b,n,m\}, \aleph\}$ *be a grill on a* ϱ*-NS* $(\aleph, \Gamma, \xi_\varrho)$

(1) If $\Delta = \{b,n,m\}$, *then* $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(\Delta) = \aleph \nsubseteq \Delta$,

(2) If $\Delta = \{b,e\}$, *then* $\Delta \nsubseteq \{b,m\} = \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\{\Delta\})$.

(3) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(\phi) = \{n,e\} \neq \phi$.

(4) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_r}}(\aleph) = \{b,m\} \neq \aleph$.

(5) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(\{n\}) = \{n,e\} \nsubseteq \{n\}, \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_i}}(\{n\}) = \{n,m,e\} \nsubseteq \{n\}, \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_u}}(\{e\}) = \{n,e\} \nsubseteq \{e\}$.

Hence, $\forall \pi \in \aleph, \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\{\pi\}) \nsubseteq \mathcal{C}_\varrho(\{\pi\})$

(6) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle r\rangle}}}(\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle r\rangle}}}(\{e\})) = \{n,e\} \neq \{n,m,e\} = \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_{\langle r\rangle}}}(\{e\})$.

Proposition 11. *Let $\mathcal{G}, \mathcal{J}$ be grills on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$. If $D, \Delta \sqsubseteq \aleph$, $\mathcal{G} \sqsubseteq \mathcal{J}$ then, the following claims are true for all ϱ.*

(1) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \sqsubseteq \mathcal{J}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D)$.

(2) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D)$.

Proof. This is evident.

Remark 3. *Let $\mathcal{G} = \{\{b\}, \{b,e\}, \{b,n\}, \{b,m\}, \{b,e,n\}, \{b,e,m\}, \{b,n,m\}, \aleph\}$, and $\mathcal{J} = 2^{\aleph} \backslash \phi$ be grills on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$.*

(1) If $D = \{b\}$, then $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(D) = \aleph \nsubseteq \phi = \mathcal{J}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(D)$.

(2) If $D = \{b,n\}$, then $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_r}}(D) = \{b,n,m\} \nsubseteq \{b,m\} = \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_r}}(D)$.

3.2. The second class of Rs paradigms

Some drawbacks and unacceptable characteristics of the first category of *Rs* paradigms include: The property that $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) \subseteq D \subseteq \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\{D\})$ does not apply to all subsets, consequently, particularly when $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\phi) \neq \phi$ and $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\aleph) \neq \aleph$, results in irrational descriptions of those basic set models or mistrust of the information gleaned from them. Since the original accuracy measure formula yields values greater than one or undefined cases for certain subsets, we are unable to apply it, i.e. in Example 3.11, we have

$\frac{|\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(\{b\})|}{|\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_r}}(\{b\})|} = \frac{4}{2} > 1$, and $\frac{|\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_r}}(\phi)|}{|\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_r}}(\phi)|} = \frac{2}{0}$.

These situations have little practical significance and are worthless. This section aims to improve the initial *Rs* paradigm by enhancing *Lw-Ap* and *Up-Ap*, while maintaining its advantages.

Definition 18. *Let $\mathcal{G}$ be a grill on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$. Based on grills and cardinality nbd_s, the $\mathcal{G}_{\mathcal{C}_\varrho} Lw - Ap$ ($\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}()$), and $\mathcal{G}_{\mathcal{C}_\varrho}$Up-Ap $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}()$ of $\Delta \sqsubseteq \aleph$ are respectively computed by:*

$$\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta) = \{\delta \in \Delta : \mathcal{C}_\varrho(\delta) \sqcap \Delta^c \notin \mathcal{G}\}$$

$$\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta) = \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(\Delta) \sqcup \Delta$$

Definition 19. *The $\mathcal{G}_{\mathcal{C}_\varrho}$-boundary, $\mathcal{G}_{\mathcal{C}_\varrho}$-positive, and $\mathcal{G}_{\mathcal{C}_\varrho}$-negative regions of a subset Δ within a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ with grill $\mathcal{G}$ on $\aleph$ are respectively computed by:*

$$\mathcal{G}_{\mathcal{B}_{\mathcal{C}_\varrho}}(\Delta) = \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta) \setminus \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$$

$$\mathcal{G}_{\mathcal{P}_{\mathcal{C}_\varrho}}(\Delta) = \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$$

$$\mathcal{G}_{\mathcal{N}_{\mathcal{C}_\varrho}}(\Delta) = \aleph \backslash \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)$$

The following measurements can be used to numerically characterize Rs_s in relation to $\mathcal{C}_\varrho$-nbd_s and grills.

Definition 20. *The $\mathcal{G}_{\mathcal{C}_\varrho}$-accuracy, and $\mathcal{G}_{\mathcal{C}_\varrho}$-roughness criteria of $\Delta \neq$ of a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ with grill $\mathcal{G}$ on $\aleph$ are provided, respectively, by:*

$$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta) = \frac{|\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)|}{|\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)|}, \ |\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)| \neq 0.$$

$$\mathcal{G}_{\mathcal{R}_{\mathcal{C}_\varrho}}(\Delta) = 1 - \mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta).$$

The statement of Pawlak's properties using the $\mathcal{G}_{\mathcal{C}_\varrho}$-$Lw$ and $\mathcal{G}_{\mathcal{C}^\varrho}$-$Up$-$Ap$ will be looked at in the following theorem.

Theorem 4. *For a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ and $D, \Delta \sqsubseteq \aleph$, let $\mathcal{G} = 2^{\aleph} \backslash \phi$ be a grill. Next, we have the properties listed below.*

(1) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta) \sqsubseteq \Delta \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)$.

(2) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\phi) = \phi$, *and* $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\phi) = \phi$.

(3) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\aleph) = \aleph$, *and* $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\aleph) = \aleph$.

(4) If $D \sqsubseteq \Delta$, *then* $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$, *and* $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(D) \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)$.

(5) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)) = \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$ *for* $\varrho \in \{r, l, i, \langle l \rangle, \langle i \rangle\}$.

(6) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$ *for* $\varrho \in \{u, \langle u \rangle, \langle r \rangle\}$.

(7) Let $\pi \in \aleph$. *Then* $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(\pi)) = \mathcal{C}_\varrho(\pi)$ *for* $\varrho \in \{r, l, i, \langle l \rangle, \langle i \rangle\}$.

(8) Let $\pi \in \aleph$. *Then* $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(\pi)) \sqsubseteq \mathcal{C}_\varrho(\pi)$ *for* $\varrho \in \{u, \langle u \rangle, \langle r \rangle\}$.

(9) $\mathcal{M}^{\mathcal{C}_\varrho}(\mathcal{M}^{\mathcal{C}_\varrho}(\Delta)) = \mathcal{M}^{\mathcal{C}_\varrho}(\Delta)$ *for* $\varrho \in \{r, l, i, \langle l\rangle, \langle i\rangle\}$.

(10) $\mathcal{M}^{\mathcal{C}_\varrho}(\mathcal{M}^{\mathcal{C}_\varrho}(\Delta)) \sqsupseteq \mathcal{M}^{\mathcal{C}_\varrho}(\Delta)$ *for* $\varrho \in \{u, \langle u\rangle, \langle r\rangle\}$.

(11) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D) \sqcap \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta) = \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D \sqcap \Delta)$ *for* ϱ.

(12) $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(D) \sqcup \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta) = \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(D \sqcup \Delta)$ *for* ϱ.

Proof. Validating (1), (2), (3), (4), (11), and (12) is simple in accordance with Definition 3.14.

(5) Let ϱ be a member of $\{r, l, i, \langle l\rangle, \langle i\rangle\}$. Using the current theorem's characteristics (1) and (4), $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$.

The $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)) = \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$ is the result of utilizing (6) of Theorem 3.3 to show the other direction.

(6) Let $\varrho \in \{u, \langle u\rangle, \langle r\rangle\}$. Using the current theorem's characteristics (1) and (4), $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$.

(7) Let $\varrho \in \{r, l, i, \langle l\rangle, \langle i\rangle\}$. Applying the current theorem's property (1), we have $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(\pi)) \sqsubseteq \mathcal{C}_\varrho(\pi)$, $\forall \pi \in \aleph$. Theorem 2's (7) is used to demonstrate the opposite direction. Thus, $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(\pi)) = \mathcal{C}_\varrho(\pi)$.

(8) Assume that $\varrho \in \{u, \langle u\rangle, \langle r\rangle\}$. By property (1) of the current theorem, we, have $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\mathcal{C}_\varrho(\pi)) \sqsubseteq \mathcal{C}_\varrho(\pi) \forall \pi \in \aleph$.

(9) Identical to evidence of property (5).

(10) Identical to evidence of property (5).

Proposition 12. *For a* ϱ*-NS* $(\aleph, \Gamma, \xi_\varrho)$ *and* $\Delta \sqsubseteq \aleph$*, let* $\mathcal{G}$ *be a grill, then*

(1) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_u}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_r}}(\Delta) \sqcap \mathcal{G}_{\mathcal{M}_{\mathcal{C}_l}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_r}}(\Delta) \sqcup \mathcal{G}_{\mathcal{M}_{\mathcal{C}_l}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_i}}(\Delta)$.

(2) $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_i}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_r}}(\Delta) \sqcap \mathcal{G}_{\mathcal{M}^{\mathcal{C}_l}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_r}}(\Delta) \sqcup \mathcal{G}_{\mathcal{M}^{\mathcal{C}_l}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_u}}(\Delta)$.

(3) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_{\langle u\rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_{\langle r\rangle}}}(\Delta) \sqcap \mathcal{G}_{\mathcal{M}_{\mathcal{C}_{\langle l\rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_{\langle r\rangle}}}(\Delta) \sqcup \mathcal{G}_{\mathcal{M}_{\mathcal{C}_{\langle l\rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_{\langle i\rangle}}}(\Delta)$.

(4) $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_{\langle i\rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_{\langle r\rangle}}}(\Delta) \sqcap \mathcal{G}_{\mathcal{M}^{\mathcal{C}_{\langle l\rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_{\langle r\rangle}}}(\Delta) \sqcup \mathcal{G}_{\mathcal{M}^{\mathcal{C}_{\langle l\rangle}}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_{\langle u\rangle}}}(\Delta)$.

Proof. Derived from Definition 3.14 and Proposition 3.5.

Corollary 6. *For a* ϱ*-NS* $(\aleph, \Gamma, \xi_\varrho)$ *and* $\Delta \sqsubseteq \aleph$*, let* $\mathcal{G}$ *be a grill, then*

(1) $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_u}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_r}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_i}}(\Delta)$.

(2) $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_u}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_l}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_i}}(\Delta)$.

(3) $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle u \rangle}}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle r \rangle}}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle i \rangle}}}(\Delta)$.

(4) $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle u \rangle}}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle l \rangle}}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle i \rangle}}}(\Delta)$.

Proposition 13. *If Δ is a nonempty subset of $\aleph$, then for any ϱ, $0 \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta) \leq 1$.*

Proof. Derives from the reality that $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta) \sqsubseteq \Delta \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)$.

Definition 21. *We refer to a subset Δ $\mathcal{G}_{\mathcal{C}_\varrho}$-exact if $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta) = 1$. Otherwise it's known $\mathcal{G}_{\mathcal{C}_\varrho}$-rough.*

The opposite side of Corollary 3.19 generally fails, as the following example shows.

Example 4. *Extended in Example 3.11. Tables 4 and 5 calculate the accuracy criteria $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta)$ for each ϱ.*

Table 4: $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta)$ for $\varrho \in \{r, l, i, u\}$.

Δ	$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_r}}(\Delta)$	$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_l}}(\Delta)$	$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_i}}(\Delta)$	$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_u}}(\Delta)$
$\{b\}$	1	1	1	1
$\{n\}$	1	1	1	1
$\{m\}$	1	1	1	0
$\{e\}$	1	1	1	1
$\{b, n\}$	1	1	1	1
$\{b, m\}$	1	1	1	1
$\{b, e\}$	1	1	1	1
$\{n, m\}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
$\{n, e\}$	1	1	1	1
$\{m, e\}$	$\frac{1}{2}$	1	1	1
$\{b, n, m\}$	1	1	1	1
$\{b, n, e\}$	1	1	1	1
$\{b, m, e\}$	1	1	1	1
$\{n, m, e\}$	$\frac{2}{3}$	1	1	1
$\aleph$	1	1	1	1

The following theorem describes how the current models outperform the models provided in [24] in terms of *App* operators. They clearly boost the accuracy measures of subsets and make a true shrink (or removal) for the Br_s.

Theorem 5. *Let $\mathcal{G}$ be a grill for a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ and $D \sqsubseteq \aleph$. Consequently, the following assertion is valid for all ϱ.*

Table 5: $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta)$ for $\varrho \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

Δ	$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle r\rangle}}}(\Delta)$	$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle l\rangle}}}(\Delta)$	$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle i\rangle}}}(\Delta)$	$\mathcal{G}_{\mathcal{A}_{\mathcal{C}_{\langle u\rangle}}}(\Delta)$
$\{b\}$	1	1	1	1
$\{n\}$	1	1	1	1
$\{m\}$	1	1	1	1
$\{e\}$	1	1	1	1
$\{b,n\}$	1	1	1	1
$\{b,m\}$	1	1	1	1
$\{b,e\}$	1	1	1	1
$\{n,m\}$	1	1	1	1
$\{n,e\}$	1	1	1	1
$\{m,e\}$	1	1	1	1
$\{b,n,m\}$	1	1	1	1
$\{b,n,e\}$	1	1	1	1
$\{b,m,e\}$	1	1	1	1
$\{n,m,e\}$	1	1	1	1
$\aleph$	1	1	1	1

(1) $\mathcal{M}_{\mathcal{C}_\varrho}(D) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D)$.

(2) $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(D) \sqsubseteq \mathcal{M}^{\mathcal{C}_\varrho}(D)$.

Proof. By using Theorem 3.8, then $\mathcal{M}_{\mathcal{C}_\varrho}(D) \sqsubseteq \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D)$. Since $\mathcal{M}_{\mathcal{C}_\varrho}(D) \sqsubseteq D$, then $\mathcal{M}_{\mathcal{C}_\varrho}(D) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D)$. It is possible to prove the second statement by following the same logic.

Corollary 7. *For a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ and $D \sqsubseteq \aleph$, let $\mathcal{G}$ be a grill, then*

$$\mathcal{A}_{\mathcal{C}_\varrho}(D) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta) \forall \varrho.$$

To demonstrate that the inverse of the aforementioned theorem and corollary fails, use the following example.

Example 5. *Extending from Example 3.11. If $\Delta = \{b,n\}$, then $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta), \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)$, and $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta)$ are computed for $\varrho \in \{r,u\}$ as follows:*

(1) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta) = \{b,n\}$, $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta) = \{b,n\}$, *and* $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta) = 1$.

(2) $\mathcal{M}_{\mathcal{C}_\varrho}(\Delta) = \{n\}$, $\mathcal{M}^{\mathcal{C}_\varrho}(\Delta) = \{b,n,m\}$, *and* $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta) = \frac{1}{3}$.

Proposition 14. *For any ϱ, suppose that $\mathcal{G}$ is a grill on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$. If $\Delta \sqsubseteq \aleph$, then*

(1) $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\nparallel_\varrho}}(\Delta)$.

(2) $\mathcal{G}_{\mathcal{M}^{\nparallel_\varrho}}(\Delta) \sqsubseteq \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)$.

(3) $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta) \leq \mathcal{G}_{\mathcal{A}_{\nparallel_\varrho}}(\Delta)$.

Proof. (1) Let $\delta \in \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(\Delta)$, then $\mathcal{C}_\varrho(\delta) \sqcap \Delta^c \notin \mathcal{G}$ (by Definition 4.1), $\nparallel_\varrho(\delta) \sqcap \Delta^c \notin \mathcal{G}$. This is show that $\delta \in \mathcal{G}_{\mathcal{M}_{\nparallel_\varrho}}(\Delta)$.

(2) Let $\delta \in \mathcal{G}_{\mathcal{M}^{\nparallel_\varrho}}(\Delta)$, i.e., $\nparallel_\varrho \sqcap \Delta \in \mathcal{G}$, Hence, $\mathcal{C}_\varrho \sqcap \Delta \in \mathcal{G}$, $\delta \in \mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(\Delta)$. So, $\delta \in \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta)$,(by Definition 3.14).

(3) The evidence is obvious.

Remark 4. *Let $\mathcal{G}_1$, and $\mathcal{G}_2$ be grills on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$, $\Delta \sqsubseteq \aleph$. If $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$ then $\{\mathcal{G}_1\}_{\mathcal{A}_{\mathcal{C}_\varrho}}(\Delta) \geq \{\mathcal{G}_2\}_{\mathcal{A}_{\mathcal{C}_\varrho}}$, for each ϱ.*

4. Various topologies created using grills and cardinality nbd_s

In this section, for any given relation, we use grills and cardinal nbd_s to build a variety of topologies that are finer than those previously produced by cardinal nbd_s as detailed in [46].

Theorem 6. *For a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$, let $\mathcal{G}$ be a grill. The family*

$$\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}} = \{\Delta \sqsubseteq \aleph : \forall \delta \in \Delta, \mathcal{C}_\varrho(\delta) \sqcap \Delta^c \notin \mathcal{G}\} \text{ a topology on } \aleph.$$

Proof. First, for any ϱ, it is obvious that $\aleph, \phi \in \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.
Second, define Δ_1, Δ_2 as elements of $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$, and $\delta \in \Delta_1 \sqcap \Delta_2$. Then $\mathcal{C}_\varrho(\delta) \sqcap \Delta_1^c \notin \mathcal{G}$, and $\mathcal{C}_\varrho(\delta) \sqcap \Delta_2^c \notin \mathcal{G}$. Hence, $\mathcal{C}_\varrho(\delta) \sqcap (\Delta_1 \sqcap \Delta_2)^c \notin \mathcal{G}$. This implies that $\Delta_1 \sqcap \Delta_2 \in \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.
Lastly, for each $i \in I$, assume that Δ_i belonged to $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$. Let $\delta \in \sqcup_{i \in I} \Delta_i$, then there is $i_0 \in I$ s.t. $\delta \in \Delta_{i_0}$ and $\mathcal{C}_\varrho(\delta) \sqcap \Delta_{i_0}^c \notin \mathcal{G}$. Since $\Delta_{i_0} \sqsubseteq \sqcup_{i \in I} \Delta_i$. Then, $\mathcal{C}_\varrho(\delta) \sqcap (\sqcup_{i \in I} \Delta_i)^c \notin \mathcal{G}$, and $\sqcup_{i \in I} \Delta_i \in \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.

The $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$-open sets are the members of $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$, and the $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$-closed sets are its complement.

Proposition 15. *For any ϱ, suppose that $\mathcal{G}$ is a grill on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$. Then*

(1) For each ϱ, $\Psi_{\mathcal{C}_\varrho} \sqsubseteq \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.

(2) For each ϱ, $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}} \sqsubseteq \mathcal{G}_{\Psi_{\nparallel_\varrho}}$.

Proof. (1) $\mathcal{C}_\varrho(e) \sqcap D^c \notin \mathcal{G}$ $\forall e \in D$ is implied by the fact that $\mathcal{C}_\varrho(e) \sqsubseteq D$ $\forall e \in D$.
(2) According to Proposition 2.24, for any ϱ, we have $\|_\varrho(e) \sqsubseteq \mathcal{C}_\varrho(e)$. Therefore, using the grill property, we determine that $\mathcal{C}_\varrho(e) \sqcap D^c \notin \mathcal{G}$.

Example 6. *From Example 3.11, we have:*

$\Psi_{\mathcal{C}_r} = \{\{n\}, \{e\}, \{n, e\}, \{b, m\}, \{b, m, n\}, \{b, m, e\}, \aleph, \phi\}$.

$\mathcal{G}_{\Psi_{\mathcal{C}_r}} = \{\{n\}, \{b\}, \{e\}, \{b, n\}, \{b, m\}, \{b, e\}, \{n, e\}, \{b, m, n\}, \{b, n, e\}, \{b, m, e\}, \aleph, \phi\}$.

$\Psi_{\mathcal{C}_l} = \{\{n\}, \{b\}, \{m, e\}, \{b, n\}, \{b, m, e\}, \{n, m, e\}, \aleph, \phi\}$.

$\mathcal{G}_{\Psi_{\mathcal{C}_l}} = 2^{\aleph}$.

$\Psi_{\mathcal{C}_i} = 2^{\aleph}$.

$\mathcal{G}_{\Psi_{\mathcal{C}_i}} = 2^{\aleph}$.

$\Psi_{\mathcal{C}_u} = \{\{n\}, \{b, m, e\}, \aleph, \phi\}$.

$\mathcal{G}_{\Psi_{\mathcal{C}_u}} = \{\{n\}, \{b\}, \{e\}, \{b, n\}, \{b, m\}, \{b, e\}, \{n, e\}, \{b, m, n\}, \{b, n, e\}, \{b, m, e\}, \aleph, \phi\}$.

$\Psi_{\mathcal{C}_{\langle r\rangle}} = \{\{b\}, \{m\}, \{n, e\}, \{b, m\}, \{b, n, e\}, \{n, m, e\}, \aleph, \phi\}$.

$\mathcal{G}_{\Psi_{\mathcal{C}_{\langle r\rangle}}} = 2^{\aleph}$.

$\Psi_{\mathcal{C}_{\langle l\rangle}} = \{\{b\}, \{e\}, \{n, m\}, \{b, e\}, \{n, m, e\}, \{n, m, b\}, \aleph, \phi\}$.

$\mathcal{G}_{\Psi_{\mathcal{C}_{\langle l\rangle}}} = 2^{\aleph}$.

$\Psi_{\mathcal{C}_{\langle i\rangle}} = 2^{\aleph}$.

$\mathcal{G}_{\Psi_{\mathcal{C}_{\langle i\rangle}}} = 2^{\aleph}$.

$\Psi_{\mathcal{C}_{\langle u\rangle}} = \{\{b\}, \{n, m, e\}, \phi, \aleph\}$.

$\mathcal{G}_{\Psi_{\mathcal{C}_{\langle u\rangle}}} = 2^{\aleph}$.

Lemma 2. *Let $\mathcal{G}, \mathcal{J}$ be grills on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$ such that $\mathcal{G} \sqsubseteq \mathcal{J}$ for any ϱ. $\mathcal{J}_{\Psi_{\mathcal{C}_\varrho}} \sqsubseteq \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$ follows.*

Proof. Straight to prove.

The following example shows that Lemma 4.4's inverse implication is not always true.

Example 7. *Keeping with Example 3.11.*
Let $\mathcal{G} = \{\{b\}, \{b, e\}, \{b, n\}, \{b, m\}, \{b, e, n\}, \{b, e, m\}, \{b, n, m\}, \aleph\}$, and $\mathcal{J} = 2^{\aleph} \backslash \phi$, and $\varrho = r$.

Then, $\mathcal{G}_{\Psi_{\mathcal{C}_r}} = \{\{n\},\{b\},\{e\},\{n,b\},\{b,m\},\{b,e\},\{n,e\},\{b,n,m\},\{b,n,e\},\{b,m,e\},\aleph,\phi\} \not\sqsubseteq \{\{n\},\{e\},\{b,m\},\{n,e\},\{b,n,m\},\{b,m,e\},\aleph,\phi\} = \mathcal{J}_{\Psi_{\mathcal{C}_r}}$.

Theorem 7. *Topologies satisfy the following relations:*

(1) $\mathcal{G}_{\Psi_{\mathcal{C}_u}} \sqsubseteq \mathcal{G}_{\Psi_{\mathcal{C}_r}} \sqcap \mathcal{G}_{\Psi_{\mathcal{C}_l}} \sqsubseteq \mathcal{G}_{\Psi_{\mathcal{C}_r}} \sqcup \mathcal{G}_{\Psi_{\mathcal{C}_l}} \sqsubseteq \mathcal{G}_{\Psi_{\mathcal{C}_i}}$.

(2) $\mathcal{G}_{\Psi_{\mathcal{C}_{\langle u\rangle}}} \sqsubseteq \mathcal{G}_{\Psi_{\mathcal{C}_{\langle r\rangle}}} \sqcap \mathcal{G}_{\Psi_{\mathcal{C}_{\langle l\rangle}}} \sqsubseteq \mathcal{G}_{\Psi_{\mathcal{C}_{\langle r\rangle}}} \sqcup \mathcal{G}_{\Psi_{\mathcal{C}_{\langle l\rangle}}} \sqsubseteq \mathcal{G}_{\Psi_{\mathcal{C}_{\langle i\rangle}}}$.

Proof. Proposition 2.14's first item justifies these relationships.

Example 4.3 displays that $\mathcal{G}_{\Psi_{\mathcal{C}_r}} \neq \mathcal{G}_{\Psi_{\mathcal{C}_l}}$, $\mathcal{G}_{\Psi_{\mathcal{C}_r}} \neq \mathcal{G}_{\Psi_{\mathcal{C}_i}}$, $\mathcal{G}_{\Psi_{\mathcal{C}_u}} \neq \mathcal{G}_{\Psi_{\mathcal{C}_l}}$, $\mathcal{G}_{\Psi_{\mathcal{C}_u}} \neq \mathcal{G}_{\Psi_{\mathcal{C}_i}}$, $\mathcal{G}_{\Psi_{\mathcal{C}_r}} = \mathcal{G}_{\Psi_{\mathcal{C}_u}}$, $\mathcal{G}_{\Psi_{\mathcal{C}_i}} = \mathcal{G}_{\Psi_{\mathcal{C}_l}}$, and $\mathcal{G}_{\Psi_{\mathcal{C}_{\langle u\rangle}}} = \mathcal{G}_{\Psi_{\mathcal{C}_{\langle r\rangle}}} = \mathcal{G}_{\Psi_{\mathcal{C}_{\langle l\rangle}}} = \mathcal{G}_{\Psi_{\mathcal{C}_{\langle i\rangle}}}$.

Next, crude Ap_s will be created using topologies based on grills and cardinal nbd_s. Additionally, the features of these crude Ap_s will be examined.

Definition 22. *Let* $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$ *represent a topology induced by grills and cardinality* nbd_s*. Then, for each* ϱ*, Lw and Up-Ap of a st* $\Delta \sqsubseteq \aleph$ *are respectively given by:*
$\underline{\mathcal{G}_{\xi_\varrho}}(\Delta) = \mathcal{G}_{int_{\mathcal{C}_\varrho}}(\Delta)$, $\overline{\mathcal{G}_{\xi_\varrho}}(\Delta) = \mathcal{G}_{cl_{\mathcal{C}_\varrho}}(\Delta)$*, where* $\mathcal{G}_{int_{\mathcal{C}_\varrho}}(\Delta)$, $\mathcal{G}_{cl_{\mathcal{C}_\varrho}}(\Delta)$ *respectively represent the interior and closure of a set* Δ *with respect the topology* $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$*. Furthermore,* Δ*'s accuracy criterion is set as follows:* $\mathcal{G}_{\lambda_{\xi_\varrho}}(\Delta) = \frac{|\underline{\mathcal{G}_{\xi_\varrho}}(\Delta)|}{|\overline{\mathcal{G}_{\xi_\varrho}}(\Delta)|}$, $|\overline{\mathcal{G}_{\xi_\varrho}}(\Delta)| \neq 0$.

$0 \leq \mathcal{G}_{\lambda_{\xi_\varrho}} \leq 1$ is clearly. Δ is called a $\mathcal{G}_{\mathcal{C}_\varrho}$-exact set if $\mathcal{G}_{\lambda_{\xi_\varrho}}(\Delta) = 1$. On the other hand, Δ is referred to as a $\mathcal{G}_{\mathcal{C}_\varrho}$-*Rs*.
Concerning to Definition 4.7, the following results can be proven using the topological characteristics of interior and closure operators. It is noteworthy that certain properties absent in the $\mathcal{G}_{\tilde{\mathcal{M}}\mathcal{C}_\varrho}$-,$\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}$-*Ap* are still valid for the $\underline{\mathcal{G}_{\xi_\varrho}}$-, $\overline{\mathcal{G}_{\xi_\varrho}}$-*Ap* that as item (1) of Theorem 3.3.

Theorem 8. *For each* ϱ*, let* $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$ *represent a topology induced by grills and cardinality* nbd_s*, and let* $D, \Delta \sqsubseteq \aleph$*. Next are the following properties:*

(1) $\underline{\mathcal{G}_{\xi_\varrho}}(\Delta) \sqsubseteq \Delta$.

(2) $\underline{\mathcal{G}_{\xi_\varrho}}(\phi) = \phi$.

(3) $\underline{\mathcal{G}_{\xi_\varrho}}(\aleph) = \aleph$.

(4) If $D \sqsubseteq \Delta$*, then* $\underline{\mathcal{G}_{\xi_\varrho}}(D) \sqsubseteq \underline{\mathcal{G}_{\xi_\varrho}}(\Delta)$.

(5) $\underline{\mathcal{G}_{\xi_\varrho}}(D \sqcap \Delta) = \underline{\mathcal{G}_{\xi_\varrho}}(D) \sqcap \underline{\mathcal{G}_{\xi_\varrho}}(\Delta)$.

(6) $\underline{\mathcal{G}_{\xi_\varrho}}(\Delta^c) = (\overline{\mathcal{G}_{\xi_\varrho}}(\Delta))^c$.

(7) $\underline{\mathcal{G}_{\xi_\varrho}}(\underline{\mathcal{G}_{\xi_\varrho}}(\Delta)) = \underline{\mathcal{G}_{\xi_\varrho}}(\Delta)\ \forall\varrho$.

Proof. These connections are true because they correspond to interior topology and *Lw-Ap* operators.

Corollary 8. *For each ϱ, let $\mathcal{G}_{\Psi_{C_\varrho}}$ represent a topology induced by grills and cardinality nbd_s. Then, $\underline{\mathcal{G}_{\xi_\varrho}}(D) \sqcup \underline{\mathcal{G}_{\xi_\varrho}}(\Delta) \sqsubseteq \underline{\mathcal{G}_{\xi_\varrho}}(D \sqcup \Delta)$ for any $D, \Delta \sqsubseteq \aleph$.*

Theorem 9. *For each ϱ, let $\mathcal{G}_{\Psi_{C_\varrho}}$ represent a topology induced by grills and cardinality nbd_s, and let $D, \Delta \sqsubseteq \aleph$. Next are the following properties:*

(1) $\Delta \sqsubseteq \overline{\mathcal{G}_{\xi_\varrho}}(\Delta)$.

(2) $\overline{\mathcal{G}_{\xi_\varrho}}(\phi) = \phi$.

(3) $\overline{\mathcal{G}_{\xi_\varrho}}(\aleph) = \aleph$.

(4) If $D \sqsubseteq \aleph$, *then* $\overline{\mathcal{G}_{\xi_\varrho}}(D) \sqsubseteq \overline{\mathcal{G}_{\xi_\varrho}}(\Delta)$.

(5) $\overline{\mathcal{G}_{\xi_\varrho}}(D \sqcup \Delta) = \overline{\mathcal{G}_{\xi_\varrho}}(\Delta) \sqcup \overline{\mathcal{G}_{\xi_\varrho}}(\Delta)$.

(6) $\overline{\mathcal{G}_{\xi_\varrho}}(\Delta^c) = (\underline{\mathcal{G}_{\xi_\varrho}}(\Delta))^c$.

(7) $\overline{\mathcal{G}_{\xi_\varrho}}(\overline{\mathcal{G}_{\xi_\varrho}}(\Delta)) = \overline{\mathcal{G}_{\xi_\varrho}}(\Delta)\ \forall \varrho$.

Proof. These connections are true because they correspond to closure topology and *Up-Ap* operators.

Corollary 9. *For each ϱ, let $\mathcal{G}_{\Psi_{C_\varrho}}$ represent a topology induced by grills and cardinality nbd_s. Then, $\overline{\mathcal{G}_{\xi_\varrho}}(D \sqcap \Delta) \sqsubseteq \overline{\mathcal{G}_{\xi_\varrho}}(D) \sqcap \overline{\mathcal{G}_{\xi_\varrho}}(\Delta)$ for any $D, \Delta \sqsubseteq \aleph$.*

Proposition 16. *For $\phi \neq \Delta \sqsubseteq \aleph$, $0 \leq \mathcal{G}_{\lambda_{\xi_\varrho}}(\Delta) \leq 1$, and $\mathcal{G}_{\lambda_{\xi_\varrho}}(\aleph) = 1$ for each ϱ.*

Proposition 17. *The following inclusion relations are valid for any subset D of a topological space $(\aleph, \mathcal{G}_{\Psi_{C_\varrho}})$:*

(1) $\underline{\mathcal{G}_{\xi_u}}(D) \sqsubseteq \underline{\mathcal{G}_{\xi_r}}(D) \sqcap \underline{\mathcal{G}_{\xi_l}}(D) \sqsubseteq \underline{\mathcal{G}_{\xi_r}}(D) \sqcup \underline{\mathcal{G}_{\xi_l}}(D) \sqsubseteq \underline{\mathcal{G}_{\xi_i}}(D)$.

(2) $\overline{\mathcal{G}_{\xi_l}}(D) \sqsubseteq \overline{\mathcal{G}_{\xi_r}}(D) \sqcap \overline{\mathcal{G}_{\xi_i}}(D) \sqsubseteq \overline{\mathcal{G}_{\xi_r}}(D) \sqcup \overline{\mathcal{G}_{\xi_i}}(D) \sqsubseteq \overline{\mathcal{G}_{\xi_u}}(D)$.

(3) $\underline{\mathcal{G}_{\xi_{\langle u \rangle}}}(D) = \underline{\mathcal{G}_{\xi_{\langle r \rangle}}}(D) = \underline{\mathcal{G}_{\xi_{\langle l \rangle}}}(D) = \underline{\mathcal{G}_{\xi_{\langle i \rangle}}}(D)$.

(4) $\overline{\mathcal{G}_{\xi_{\langle u \rangle}}}(D) = \overline{\mathcal{G}_{\xi_{\langle r \rangle}}}(D) = \overline{\mathcal{G}_{\xi_{\langle l \rangle}}}(D) = \overline{\mathcal{G}_{\xi_{\langle i \rangle}}}(D)$.

Corollary 10. *The following inequalities are valid for any nonempty subset D of a topological space $(\aleph, \mathcal{G}_{\Psi_{C_\varrho}})$:*

(1) $\mathcal{G}_{\lambda_{\xi_u}}(D) \leq \mathcal{G}_{\lambda_{\xi_r}}(D) \leq \mathcal{G}_{\lambda_{\xi_i}}(D)$.

(2) $\mathcal{G}_{\lambda_{\xi_u}}(D) \leq \mathcal{G}_{\lambda_{\xi_l}}(D) \leq \mathcal{G}_{\lambda_{\xi_i}}(D)$.

(3) $\mathcal{G}_{\lambda_{\xi_{\langle u \rangle}}}(D) = \mathcal{G}_{\lambda_{\xi_{\langle r \rangle}}}(D) = \mathcal{G}_{\lambda_{\xi_{\langle i \rangle}}}(D) = \mathcal{G}_{\lambda_{\xi_{\langle l \rangle}}}(D)$.

The approaches covered in the preceding section will now be compared with the Ap and accuracy standards described in this section, which are based on topological spaces.

Proposition 18. *The relations for each ϱ and $D \sqsubseteq \aleph$ are as follows:*

(1) $\underline{\mathcal{G}_{\xi_\varrho}}(D) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D)$.

(2) $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(D) \sqsubseteq \overline{\mathcal{G}_{\xi_\varrho}}(D)$.

Proof. (1) Let $b \in \underline{\mathcal{G}_{\xi_\varrho}}(D)$. Then we find a subset $V \in \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$ with $b \in V \sqsubseteq D$. We derive $\mathcal{C}_\varrho(b) \sqcap V^c \notin \mathcal{G}$ from the topology structuring method. Now, we get $\mathcal{C}_\varrho(b) \sqcap D^c \notin \mathcal{G}$ since $V \sqsubseteq D$. Hence, $b \in \mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D)$. Since $b \in D$, then $b \in \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D)$, and $\underline{\mathcal{G}_{\xi_\varrho}}(D) \sqsubseteq \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D)$. By the same manner, one can prove (2).

The converse of Proposition 4.15 need not be true, refer to Table 2 and Example 4.3. Suppose that $\sigma = r$ and $D = \{b, n\}$. Then $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D) = \aleph$, $\underline{\mathcal{G}_{\xi_\varrho}}(D) = \{b, n\}$. Hence,The opposite is therefore untrue.

Corollary 11. *For any ϱ, suppose that $\mathcal{G}$ is a grill on a ϱ-NS $(\aleph, \Gamma, \xi_\varrho)$. If $D \sqsubseteq \aleph$, then* $\mathcal{G}_{\lambda_{\xi_\varrho}}(D) \leq \mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(D)$.

5. Diagnostic analysis of heart failure as a medical application

In this section, we assess the efficiency of suggested models in managing heart failure information systems for specific patients [47]. We demonstrate how the current approach improves decision-making and how we use a topological strategy to pinpoint the most important symptoms for identifying heart failure disease. The investigation that follows leads us to the conclusion that the suggested Rs paradigms perform better than their counterparts that are based on cardinality nbd_s without grills. We also make reference to the restrictions connected to the approach described in Sec. 3.1. Table 6 shows data for eight patients ($\aleph = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$) and their associated symptoms (conditional attributes): breathlessness (Br), orthopnea (Or), paroxysmal nocturnal dyspnea (Pnd), impaired exercise tolerance (Iet), and ankle swelling (As). Heart failure is regarded as the deciding attribute. We add a value of s^+ or s^- to each conditional property (symptom) to indicate whether the patient has the symptom or not. In a similar manner, the decision attribute is labeled with $++$ or $--$ to indicate a heart failure report that is positive or negative.

Suppose the system's expert proposed the next relation Γ on the set of patients $\aleph$ to describe the links between them based on their symptoms:

Table 6: Information system for heart failure

Patients	Br	Or	Pnd	Iet	As	Decision.
p_1	s^+	s^+	s^+	s^+	s^-	$++$
p_2	s^-	s^-	s^-	s^+	s^+	$--$
p_3	s^+	s^+	s^+	s^+	s^+	$++$
p_4	s^-	s^-	s^-	s^+	s^-	$--$
p_5	s^+	s^-	s^-	s^+	s^+	$--$
p_6	s^-	s^-	s^-	s^+	s^-	$--$
p_7	s^+	s^+	s^+	s^+	s^+	$++$
p_8	s^+	s^+	s^-	s^+	s^+	$++$

$p_r \Gamma p_t \Leftrightarrow$ there are more than two frequent positive symptoms that distinguish p_r from p_t.

Then,

$\Gamma = \{(p_1, p_1), (p_3, p_3), (p_5, p_5), (p_7, p_7), (p_8, p_8), (p_1, p_3), (p_3, p_1), (p_1, p_7), (p_7, p_1),$
$(p_1, p_8), (p_8, p_1), (p_3, p_5), (p_5, p_3), (p_3, p_7), (p_7, p_3), (p_3, p_8), (p_8, p_3), (p_5, p_7), (p_7, p_5), (p_5, p_8),$
$(p_8, p_5), (p_7, p_8), (p_8, p_7)\}$.
Note that Γ is a symmetric relation that is neither transitive nor reflexive. We start by building the $\mathcal{C}_\varrho$-*nbd* systems in order to process the data that is described by the specified relation. As stated in Proposition 2.14, we deduce that all $\mathcal{C}_\varrho$-nbd_s are similar because of the symmetry of the suggested relation. The $\mathcal{C}_\varrho$-*nbd* for every patient is shown in Table 7.

Let $\mathcal{G} = \{\{p_2, p_4, p_5, p_6\}, \{p_1, p_2, p_4, p_5, p_6\}, \{p_2, p_3, p_4, p_5, p_6\}, \{p_2, p_4, p_5, p_6, p_7\},$
$\{p_2, p_4, p_5, p_6, p_8\}, \{p_1, p_2, p_3, p_4, p_5, p_6\}, \{p_1, p_2, p_4, p_5, p_6, p_7\}, \{p_1, p_2, p_4, p_5, p_6, p_8\},$
$\{p_2, p_3, p_4, p_5, p_6, p_7\}, \{p_2, p_3, p_4, p_5, p_6, p_8\}, \{p_2, p_4, p_5, p_6, p_7, p_8\}, \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\},$
$\{p_1, p_2, p_3, p_4, p_5, p_6, p_8\}, \{p_1, p_2, p_4, p_5, p_6, p_7, p_8\}, \{p_2, p_3, p_4, p_5, p_6, p_7, p_8\}, \phi, \aleph\}$.

Using Rs models and [21], we determine approximations Lw, Up, and accuracy for $D = \{p_1, p_3, p_5\}$, a set of patients with a positive report of heart failure:

(1) $\mathcal{M}_{\mathcal{C}_\varrho}(D) = \{p_1, p_5\}$.

(2) $\mathcal{M}^{\mathcal{C}_\varrho}(D) = \{p_1, p_3, p_5, p_7, p_8\}$.

(3) $\mathcal{B}_{\mathcal{C}_\varrho}(D) = \mathcal{M}^{\mathcal{C}_\varrho}(D) \setminus \mathcal{M}_{\mathcal{C}_\varrho}(D) = \{p_3, p_7, p_8\}$.

(4) $\Sigma_{\mathcal{C}_\varrho}(D) = \frac{2}{5}$.

In Sec 3.1, we showed our preliminary set model.

(1) $\mathcal{G}_{\tilde{\mathcal{M}}_{\mathcal{C}_\varrho}}(D) = \aleph$, and

(2) $\mathcal{G}_{\tilde{\mathcal{M}}^{\mathcal{C}_\varrho}}(D) = \phi$.

In Sec. 3.2, we showed our rough set model. $\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D) = D$,

(2) $\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(D) = D$,

(3) $\mathcal{G}_{\mathcal{B}_{\mathcal{C}_\varrho}}(D) = \mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(\Delta) \setminus \mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D) = \phi$, and

(4) $\mathcal{G}_{\mathcal{A}_{\mathcal{C}_\varrho}}(D) = \frac{|\mathcal{G}_{\mathcal{M}_{\mathcal{C}_\varrho}}(D)|}{|\mathcal{G}_{\mathcal{M}^{\mathcal{C}_\varrho}}(D)|} = 1$.

Models of topology developed in Sect.4. In order to use these models, we first set up a topology as shown in Table 7:

Table 7: $\mathcal{C}_\varrho$ for each patient.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
$M_r()$	$\{p_1, p_3, p_7, p_8\}$	ϕ	$\{p_1, p_3, p_5, p_7, p_8\}$	ϕ	$\{p_3, p_5, p_7, p_8\}$	ϕ	$\{p_1, p_3, p_5, p_7, p_8\}$	$\{p_1, p_3, p_5, p_7, p_8\}$
$M_{\langle r\rangle}()$	$\{p_1, p_3, p_7, p_8\}$	ϕ	$\{p_3, p_7, p_8\}$	ϕ	$\{p_3, p_5, p_7, p_8\}$	ϕ	$\{p_3, p_7, p_8\}$	$\{p_3, p_7, p_8\}$
$\mathcal{C}_\varrho()$	$\{p_1, p_5\}$	$\{p_2, p_4, p_6\}$	$\{p_3, p_7, p_8\}$	$\{p_2, p_4, p_6\}$	$\{p_1, p_5\}$	$\{p_2, p_4, p_6\}$	$\{p_3, p_7, p_8\}$	$\{p_3, p_7, p_8\}$

$\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}} = \{\phi, \aleph, \{p_1, p_5\}, \{p_2, p_4, p_6\}, \{p_3, p_7, p_8\}, \{p_1, p_2, p_4, p_5, p_6\}, \{p_1, p_3, p_5, p_7, p_8\}, \{p_2, p_3, p_4, p_6, p_7, p_8\}\}$. Then we calculate, in the following items, the Ap(Lw and Up), and accuracy for D:

$\underline{\mathcal{G}_{\xi_\varrho}}(D) = \mathcal{G}_{int_{\mathcal{C}_\varrho}}(D) = \{p_1, p_5\}$,

$\overline{\mathcal{G}_{\xi_\varrho}}(D) = \mathcal{G}_{cl_{\mathcal{C}_\varrho}}(D) = \{p_1, p_5\}$,

$\mathcal{B}_{\mathcal{C}_\varrho}(D) = \mathcal{G}_{cl_{\mathcal{C}_\varrho}}(D) \backslash \mathcal{G}_{int_{\mathcal{C}_\varrho}}(D) = \phi$,

$\mathcal{G}_{\lambda_{\xi_\varrho}}(\Delta) = \frac{|\underline{\mathcal{G}_{\xi_\varrho}}(\Delta)|}{|\overline{\mathcal{G}_{\xi_\varrho}}(\Delta)|} = 1$.

It is clear from the aforementioned results that these calculations support the conclusions made in Theorem 3.23 and Corollary 3.24. In summary, it can be observed that, in contrast to the Rs models discussed by [46], the models suggested in Sec. 3.2 improve the accuracy measures of subsets by strengthening the Lw, and Up-Ap. Furthermore, the Rs paradigms shown in Sec. 3.2 produce the same Ap spaces as those shown in Sec. 4 due to the topological technique. The computations from the four Rs models discussed above will be compared. We discovered that the greatest estimates (Lw and Up) are produced by the model shown in Sec. 3.2. It is important to note that Model 3.1 has certain shortcomings, including the inability to preserve the essential elements of Ap procedures and excessive accuracy of measurements.

In the remaining portion of this part, we determine the primary symptoms for determining whether a patient has heart failure disease by using the topological spaces that were previously created using grills and cardinality nbd_s. The initial topology is:

$\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}} = \{\phi, \aleph, \{p_1, p_5\}, \{p_2, p_4, p_6\}, \{p_3, p_7, p_8\}, \{p_1, p_2, p_4, p_5, p_6\}, \{p_1, p_3, p_5, p_7, p_8\}, \{p_2, p_3, p_4, p_6, p_7, p_8\}\}$. Next, we will contrast the topologies produced from the same database after eliminating each symptom separately with the initial topology created from the patient information system shown in Table 6. This procedure will be carried out once again for every symptom.

(1) If the symptom "breathlessness" is removed from the input attributes, then $\Gamma_{Br} = \{(p_1, p_1), (p_3, p_3), (p_7, p_7), (p_8, p_8), (p_1, p_3), (p_3, p_1), (p_1, p_7), (p_7, p_1), (p_3, p_7), (p_7, p_3), (p_3, p_8), (p_8, p_3), (p_7, p_8), (p_8, p_7)\}$. It is clear that $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}\text{-}Br \neq \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.

(2) If the symptom "orthopnea" is removed from the input attributes, then $\Gamma_{Or} = \Gamma$. Consequently, $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}\text{-}Br = \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.

(3) If the symptom "paroxysmal nocturnal dyspnea" is removed from the input attributes, then $\Gamma_{Pnd} = \Gamma$. Consequently, $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}\text{-}Pnd = \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.

(4) If the symptom "impaired exercise tolerance" is removed from the input attributes, then $\Gamma_{Iet} \neq \Gamma$. Consequently, $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}\text{-}Pnd \neq \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.

(5) If the symptom "ankle swelling" is removed from the input attributes, then $\Gamma_{As} \neq \Gamma$. Consequently, $\mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}\text{-}Pnd \neq \mathcal{G}_{\Psi_{\mathcal{C}_\varrho}}$.

The computations indicate that the main symptoms are Br, Iet, and As. Stated differently, these symptoms are recognized as the primary markers for identifying if a patient has heart failure disease. On the other hand, eliminating the Or and Br symptom does not change the topology's structure; as a result, it can be skipped during tests.

6. Conclusion and future work

Rs_st, proposed by Pawlak in 1982, is a powerful tool for handling inaccurate and confusing information. The capacity of Rst to represent data using granular structure without having any a priori knowledge outside of the data set itself is one of its main advantages. As is well known, neighborhood systems modeled after arbitrary relations have been used to update the detail that equivalency classes reflect. This helps to eliminate a rigid constraint of an equivalence relation. Recent models have failed to maintain the original paradigm's major elements and have flawed formulas for measuring proven and potential knowledge. In this work, we provide new kinds of generalized Ap spaces that are inspired by the concepts of cardinality nbd_s and grills. The first kind has proven to be effective in extracting as many details as possible from dataset subsets. Nevertheless, it has flaws in that it violates certain of the original model's characteristics. In order to get over this problem, we have introduced the second kind of Ap space, which maintains the features of the original model while appreciably expanding the knowledge that has been gathered. To improve the Ap operators, we inferred the associated positive properties of these models and verified their validity. The suggested crude set model is then represented by a topological frame that we have constructed. We have presented the crude topological model's characteristics and clarified how it differs from its counterpart that is defined without a grill structure. We may conclude that the *Rs* models used here perform better than the current models based on the analysis presented in the medical case of heart failure condition.

Here is our strategy for the future:

† To improve accuracy, the existing models are being expanded using various relations.

‡ In connection with the concepts that were given, the concepts of soft and pythagorean fuzzy soft settings will be discussed.

†‡ Investigate innovative rough set models produced by C_ϱ-neighborhoods and a grill structure created by two grills, $\mathcal{G}_*$ and $\mathcal{G}_{**}$, as outlined below. $\mathcal{G} \doteq \{X \sqcup Y : X \in \mathcal{G}_*, Y \in \mathcal{G}_{**}\}$.

‡‡ In connection with the concepts that were given, the concepts of soft and Picture fuzzy soft settings will be discussed.

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Conflict of interest

Regarding the publishing of this work, the authors affirm that they have no conflicts of interest.

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