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The G-Rank-Mapped Transmuted Fréchet Weibull Distribution under Cubic Model: Mathematical Theory, Computational Statistics, and Sustainability Data Analysis

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Abstract. This work introduces and examines a new probability distribution from the G-rank-mapped transmuted Fréchet-Weibull (G-RTFW) family, using both quadratic and cubic models. The Quadratic Rank Transmuted Fréchet-Weibull (QRTFW) and Cubic Rank Transmuted Fréchet-Weibull (CRTFW) distributions are specifically investigated, with the current study focused solely on the CRTFW model. We calculated and investigated a wide range of statistical and mathematical aspects of the CRTFW distribution, including its moments, moment generating function, characteristic function, quantile function, mode, random variate generation, hazard rate function, entropy, and order statistics. The suggested model is highly adaptable, capable of simulating both unimodal and bimodal behaviors, as well as tolerating symmetric and asymmetric data structures with variable degrees of kurtosis. The maximum likelihood method was used for parameter estimation, and a full Monte Carlo simulation study was carried out to assess the estimators' performance and efficiency under various circumstances. The simulation results showed that the estimators performed effectively, with bias and mean square error reducing as the sample size increased. To demonstrate the CRTFW distribution's practical application, two real-world sustainability-related datasets were examined. The CRTFW model outperformed numerous well-known lifetime and reliability distributions in terms of flexibility and goodness-of-fit, emphasizing its use for both theoretical and applied data modeling.

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1. Introduction

Recent years have seen a rise in interest in studies that aim to either introduce a new distribution or alter the baseline distribution. Because they can be used to add parameters to any distribution, making the final distribution more flexible and able to represent the complexity of highly skewed real-life data sets, these novel distributions are finding widespread usage in many real-life domains [1]. In the field of statistics, it is common practice to generalise probability distributions. Numerous generalised distributions, together with their statistical features, inferential problems, and potential applications, have been introduced in the literature as of late through the use of transmutation methods. A functional composition of the cumulative distribution functions of two probability distributions with the inverse cumulative distribution functions of one and the other was initially defined by [2] as the quadratic rank transmutation map (QRTMp). In order to study the mathematical features of the transmuted Weibull (TW) distribution and the estimates of the model parameters, [3] utilised the QRTMp to create a new version of the Weibull distribution, which they dubbed the transmuted Weibull (TW) distribution. The characteristics and longevity of TW dispersion, as well as its uses in industry and healthcare, were investigated by [4]. The cubic rank transmutation (CRT) map is an updated member of the transmutation map family that was introduced by [5]. They built the cubic transmuted log-logistic (CTLL) and cubic transmuted weibull (CTW) distributions on top of this idea. The cubic transmuted exponential distribution was presented by [6], and its statistical features were investigated through practical applications. The new distribution offers a superior fit, according to research on its statistical features, which was conducted in relation to a cubic transmuted Fréchet distribution suggested by [7]. [8] created the CTW distribution, described its statistical features in detail, and offered practical examples of its use in estimation and real-world scenarios. With the aim of expanding the Burr-XII distribution's practical utility, particularly in the fields of household income, environmental sciences, biology, engineering, and others, [9] presented the cubic transmuted Burr-XII distribution. In their publication, [10] detailed the G-transmuted distribution family and delved into the unique instances of this distribution. They continued by applying the G-transmuted distribution to the Weibull distribution and then watched how the new generalised transmuted distribution behaved and how flexible it was. Moreover, [11] have discussed the methodology of QRTMp and CRT map with some detailed informations on the developments of both the methods that exists in the literature.

The Weibull distribution [12] is a prominent statistical distribution extensively utilised for its efficacy in modelling dependable data. Researchers have documented numerous extensions and altered versions of the Weibull distribution to achieve more adaptable distributions for modelling experimental data [13–17]. They have introduced alternative forms for generalising the Weibull distribution, including the Lindley Weibull distribution, transmuted G-modified Weibull distribution, and Gumbel Weibull distribution. Additionally, [18–23] have contributed towards the extensions of Weibull distribution. The Fréchet distribution [24], recognised as the inverse Weibull distribution, is employed to model extreme value datasets. [25] presented a novel lifetime model known as the gamma extended Fréchet distribution. [26] presented the Beta Fréchet (BF) distribution, a generalisation of the exponentiated Fréchet (EF) and Fréchet distributions, whereas [27] introduced the transmuted Fréchet distribution. The exponentiated generalised Fréchet distribution was developed by [28]. Additionally, [29] introduced the beta generalised exponentiated Fréchet distribution along with its applications. Further [30] proposed a novel extension of Fréchet distribution, called the novel alpha power Fréchet distribution with real life applications. Consequently, multiple academics have advanced various generalisations of the Weibull and Fréchet distributions, which are capable of modelling data across diverse domains, including engineering, medicine, and physics, as well as extreme value phenomena such as earthquakes and floods. Consequently, the relevance of extreme value theory and the capacity to model data from diverse domains using Weibull and Fréchet distributions resulted in the formulation of the Fréchet-Weibull (FW) distribution by [31], which offers enhanced flexibility for engineering, physics, medicine, and environmental datasets. Furthermore, [32] presented the FW distribution and utilised it on two datasets from mechanical engineering, demonstrating its superiority compared to other relevant distributions considered. Recently, [33] extended the FW distribution by combing the transformed transformer method and the new generalized exponentiated method, proposing a new generalized exponentiated Fréchet Weibull distribution. [34] proposed a new distribution called the cubic ranked record-based transmuted Weibull as an alternative to the Weibull distribution and related ones.

This study focuses on the FW distribution and proposes an extension using the transmutation technique. The resulting generalized version is termed the *cubic rank transmuted Fréchet-Weibull (CRTFW) distribution.* The motivation for investigating the CRTFW distribution is multifaceted and can be summarized as follows:

- The Weibull distribution is predominantly used for modeling monotonically increasing or decreasing failure rate functions, whereas the Fréchet distribution is widely applied for modeling extreme value phenomena and rare events.
- The CRT approach allows for greater flexibility in simulating hazard rate functions, enabling them to exhibit various shapes such as increasing, decreasing, constant, bathtub-shaped, or upside-down bathtub-shaped patterns.
- Motivated by this flexibility, we introduce the CRTFW distribution as a novel generalization of the FW distribution, thereby enhancing the baseline distribution's adaptability and reliability in practical applications.
- Additionally, a comparative analysis among the FW, QRTFW, and the proposed CRTFW distributions is conducted to demonstrate the advantages and improved modeling capabilities of the CRTFW distribution.

The rest of the article is organized like this: Sections 2 and 3 present QRTFW and CRTFW distributions. In section 4, we have taken a look at the moment, quantiles, mode, moment generating function, and random number generation as part of the newly suggested CRTFW distribution's statistical features. In Section 5, we discuss the order statistics distribution, CRTFW entropy and reliability analysis. Section 6 explains how the maximal likelihood estimation approach was used to estimate the parameters of the proposed distribution. A simulation study is presented in Section 7 to examine and assess the performance of the model estimator for different sample sizes. Section 8 examines the flexibility of the CRTFW model by analyzing two real-world sustainability datasets. Section 9 concludes with the reporting of results.

2. Mathematical Structure and Visualization of the Quadratic Rank Transmuted Fréchet-Weibull Distribution

The cumulative distribution function (*cdf*) of the quadratic rank transmuted model (QRTM), based on the approach of [5], is defined as

$$F(x) = \lambda G(x) + (1-\lambda)\left[G(x)\right]^2, \quad 0 \le \lambda \le 1, \tag{1}$$

where $G(x)$ is the *cdf* of the baseline distribution. Correspondingly, the probability density function (*pdf*) of the QRTM is given by

$$f(x) = \left[\lambda + 2(1-\lambda)G(x)\right] g(x), \tag{2}$$

where $g(x)$ denotes the *pdf* of the baseline distribution. For the FW distribution, as presented in [31], the *cdf* and *pdf* are given by

$$G(x) = \exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}, \quad x > 0;\ \alpha, \beta, m, k > 0, \tag{3}$$

and

$$g(x) = \alpha k \beta^\alpha m^{\alpha k} x^{-1-\alpha k} \exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}, \tag{4}$$

where α and k are shape parameters, and m and β are scale parameters. Substituting Equation (3) into Equation (1), the *cdf* of the QRTFW distribution becomes

$$F(x) = \lambda \exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\} + (1-\lambda)\left[\exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}\right]^2, \tag{5}$$

with $x > 0$, $\alpha, \beta, m, k > 0$, and $\lambda \in [0, 1]$. The corresponding *pdf* is obtained by differentiating Equation (5) with respect to x, yielding

$$f(x) = \left[\lambda + 2(1-\lambda)\exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}\right] \alpha k \beta^\alpha m^{\alpha k} x^{-1-\alpha k} \exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}. \tag{6}$$

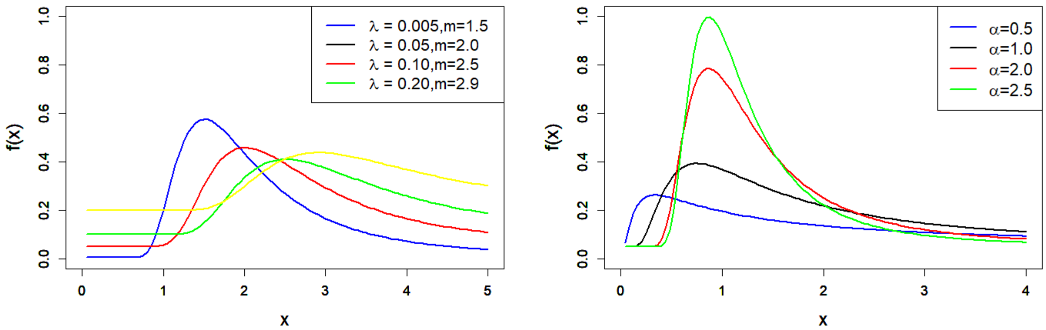

Figure 1: Density function of QRTFW distribution.

Some possible shapes of the *pdf* of the QRTFW distribution for selected values of λ and m, with $\alpha = 0.5$, $\beta = 0.5$, and $k = 5$, are illustrated in Figure 1(Left). Figure 1(Right) shows the *pdf* for selected values of α, setting $\beta = 0.5$, $\lambda = 0.05$, $m = 0.5$, and $k = 1$.

The QRTFW distribution's *pdf* is shown in Figure 1, which highlights a range of potential morphologies. In particular, depending on the parameter values selected, the figure shows density curves with symmetric forms, positively skewed behaviour, and increasing-decreasing patterns. This demonstrates unequivocally the QRTFW distribution's adaptability to modelling a variety of data types.

3. Mathematical Formulation and Graphical Representation of the Cubic Rank Transmuted Fréchet-Weibull Distribution

According to [5], the *cdf* of the cubic rank transmuted model (CRTM) is defined as:

$$F(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1)\left[G(x)\right]^2 + (1 - \lambda_1)\left[G(x)\right]^3, \tag{7}$$

and the corresponding *pdf* is

$$f(x) = \left\{\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1 - \lambda_1)\left[G(x)\right]^2\right\} g(x), \quad 0 \le \lambda_1 \le 1, \quad -1 \le \lambda_2 \le 1,$$

where $G(x)$ and $g(x)$ are the *cdf* and *pdf* of the baseline distribution, respectively. Substituting $G(x)$ from Equation (3) into Equation (7), the *cdf* of the CRTFW distribution is obtained as

$$F(x) = \lambda_1 \exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\} + (\lambda_2 - \lambda_1)\left[G(x)\right]^2 + (1 - \lambda_1)\left[\exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}\right]^3, \tag{8}$$

where $x > 0$, $\alpha, \beta, m, k > 0$, $\lambda_1 \in [0, 1]$, and $\lambda_2 \in [-1, 1]$. Differentiating Equation (8) with respect to x gives the corresponding *pdf* as

$$f(x) \;=\; \left\{\lambda_1 + 2(\lambda_2 - \lambda_1)\exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\} + 3(1 - \lambda_1)\left[\exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}\right]^2\right\}$$

$$\times \alpha k \beta^{\alpha} m^{\alpha k} x^{-1-\alpha k} \exp\left\{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}\right\}, \tag{9}$$

where α and k are shape parameters and m and β are scale parameters. Some sub-models and extensions of the CRTFW distribution include:

- Setting $\lambda_1 = \lambda_2 = 1$ in Equation (8) reduces it to the FW distribution [31] as in Equation (3).
- Setting $\lambda_1 = \lambda_2 = m = k = 1$ in Equation (8) gives the Fréchet distribution [24] with parameters α and β.
- Setting $\beta = 1$ in Equation (3) results in the cubic rank transmuted Fréchet distribution [7].
- Setting $\lambda_2 = 2$ in Equation (3) defines a new quadratic rank transmuted Fréchet-Weibull distribution, introduced in Section 2.

Figure 2(Left) illustrates possible shapes of the *pdf* for selected k values with $\lambda_1 = 0.005, \lambda_2 = -0.05, \alpha = 0.5, \beta = 0.5$, and $m = 1.5$. Figure 2(Right) shows the *pdf* for selected m values with $\lambda_1 = 0.005, \lambda_2 = -0.05, \alpha = 0.5, \beta = 0.5$, and $k = 3$. Similarly, Figures 4 and 5 show the *pdf* and *cdf* of CRTFW distribution for different values of λ_1 and λ_2. Clearly, Figures 2 and 4 demonstrate increasing-decreasing, positively skewed, and symmetric behaviors of the density function, whereas Figures 3 and 5 show monotonically increasing patterns of the distribution function.

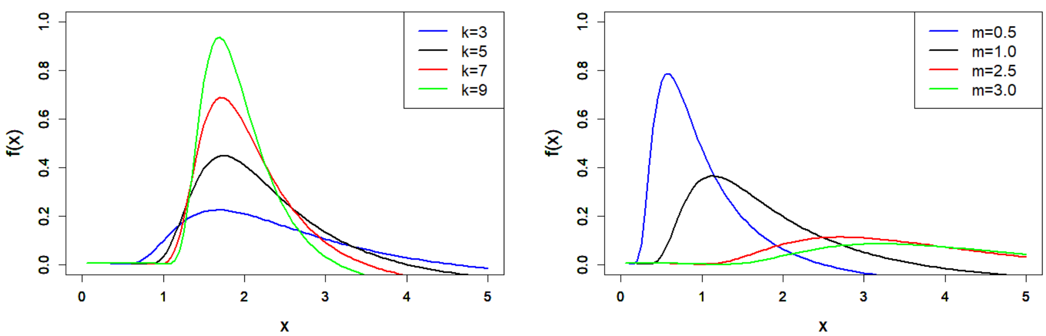

Figure 2: Probability density function of CRTFW distribution for selected values of k and m.

4. Key Statistical Features of the Cubic Rank Transmuted Fréchet-Weibull Distribution

4.1. Quantile Function and Mode

The quantile function, also known as the inverse cumulative distribution function (CDF), provides the value in a probability distribution that corresponds to a given cu-

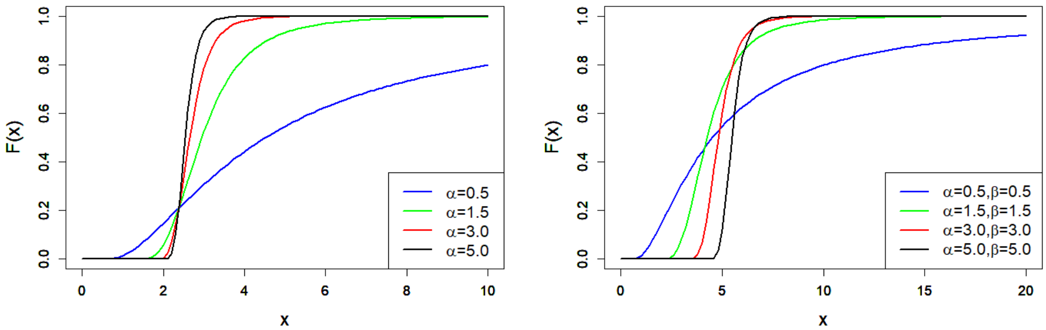

Figure 3: Cumulative distribution function of CRTFW distribution for selected values of α and β.

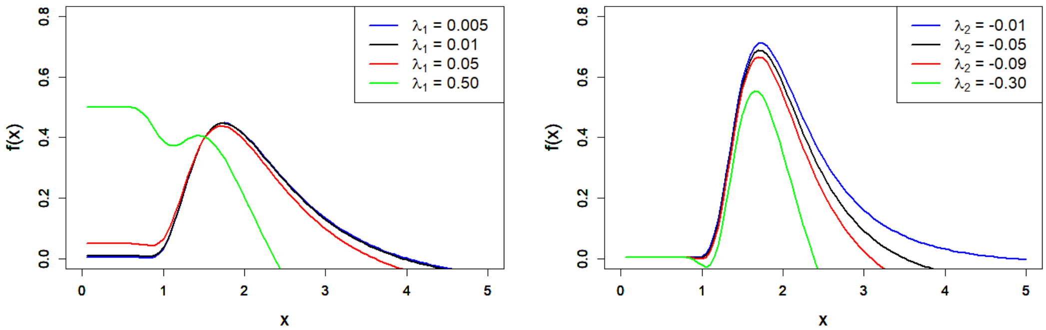

Figure 4: Density function of CRTFW distribution for selected values of λ_1 and λ_2.

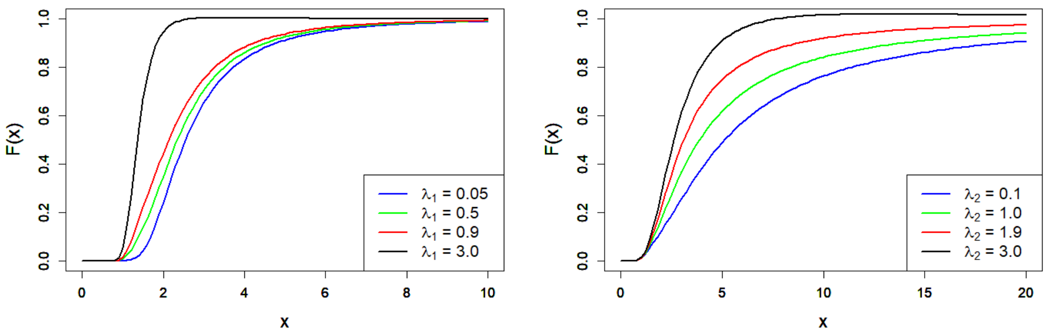

Figure 5: Distribution function of CRTFW distribution for selected values of λ_1 and λ_2.

mulative probability. Meanwhile, mode is a measure of central tendency, highlighting the most typical item by identifying the value that appears most frequently in a data set. The following derivations represent the quantile function and mode of CRTFW distribution, respectively.

The q^{th} quantile x_q of the CRTFW distribution can be obtained by inverting its *cdf* (Equation 8). Let $F(x) = q$, then

$$\lambda_1 G(x) + (\lambda_2 - \lambda_1)\,[G(x)]^2 + (1 - \lambda_1)\,[G(x)]^3 = q,$$

which can be rewritten as

$$\lambda_1 G(x) + (\lambda_2 - \lambda_1)\,[G(x)]^2 + (1 - \lambda_1)\,[G(x)]^3 - q = 0,$$

where $G(x) = \exp\left\{-\beta^\alpha \left(\frac{m}{x}\right)^{\alpha k}\right\}$. Solving this cubic equation for x gives

$$x_q = m\left\{-\frac{1}{\beta^\alpha}\ln y\right\},$$

where

$$\begin{aligned} y &= -\frac{b}{3a} - \frac{\sqrt[3]{2}\xi_1}{3a\left\{\xi_2 + \sqrt{4\xi_1^3 + \xi_2^2}\right\}^{1/3}} + \frac{\left\{\xi_2 + \sqrt{4\xi_1^3 + \xi_2^2}\right\}^{1/3}}{3a\sqrt[3]{2}}, \\ \xi_1 &= -b^2 + 3ac, \quad \xi_2 = -2b^3 + 9abc - 27a^2 d, \\ a &= 1 - \lambda_2, \quad b = \lambda_2 - \lambda_1, \quad c = \lambda_1, \quad d = -q. \end{aligned} \tag{10}$$

By setting $q = 0.25, 0.50$, and 0.75, we obtain the first, second (median), and third quartiles of the CRTFW distribution, respectively. The mode of the distribution is obtained by differentiating the logarithm of the *pdf* and setting it equal to zero:

$$\begin{aligned} \ln f(x) &= \ln\left\{\lambda_1 + 2(\lambda_2 - \lambda_1)\exp\{-\beta^\alpha (m/x)^{\alpha k}\} + 3(1 - \lambda_1)\big[\exp\{-\beta^\alpha (m/x)^{\alpha k}\}\big]^2\right\} \\ &\quad + \ln(\alpha k) + \alpha\ln(\beta) + (\alpha k)\ln(m) - (1 + \alpha k)\ln(x) - \beta^\alpha (m/x)^{\alpha k}, \\ \frac{\partial \ln f(x)}{\partial x} &= 0. \end{aligned}$$

Explicitly, the derivative is

$$\begin{aligned} &\frac{\partial \ln f(x)}{\partial x} \\ = \; &\frac{1}{x}\left[\frac{\alpha k \beta^\alpha m^{\alpha k}\exp\{-\beta^\alpha (m/x)^{\alpha k}\}\left\{2(\lambda_2 - \lambda_1) + 6(1 - \lambda_2)\exp\{-\beta^\alpha (m/x)^{\alpha k}\}\right\}}{\lambda_1 + 2(\lambda_2 - \lambda_1)\exp\{-\beta^\alpha (m/x)^{\alpha k}\} + 3(1 - \lambda_1)\big[\exp\{-\beta^\alpha (m/x)^{\alpha k}\}\big]^2}\right] \\ &+ \frac{\alpha k \beta^\alpha m^{\alpha k}}{x} - (1 + \alpha k). \end{aligned}$$

By applying the Newton-Raphson method, the mode of the CRTFW distribution is found to be

$$x_{\text{mode}} = 1.650405.$$

4.2. Random Number Generation

Random numbers from the cubic rank transmuted Fréchet Weibull (CRTFW) distribution can be generated efficiently using the inverse transform method. A random observation x from the distribution satisfies

$$\lambda_1 e^{-\beta^\alpha (m/x)^{\alpha k}} + (\lambda_2 - \lambda_1)\left[e^{-\beta^\alpha (m/x)^{\alpha k}}\right]^2 + (1-\lambda_1)\left[e^{-\beta^\alpha (m/x)^{\alpha k}}\right]^3 = u,$$

where $u \sim U(0,1)$ is a uniform random variable. Simplifying, the random variable can be expressed as

$$x_q = m\left\{-\frac{1}{\beta^\alpha}\ln y\right\}^{-1/(\alpha k)}, \tag{11}$$

where y is computed using Equation (10) with $d = -u$. Thus, given the parameters $\lambda_1, \lambda_2, \alpha, \beta, k$, and m, we can generate random numbers from the CRTFW distribution directly using Equation (11).

4.3. Moments and Associated Measures

Moments play a fundamental role in characterising the shape and properties of a probability distribution. Specifically, for a given distribution, the first moment represents the expected value, the second central moment corresponds to the variance, and the third and fourth standardized moments describe skewness and kurtosis, respectively. Let X be a random variable. Then, the r^{th} moment about the origin is defined as:

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)\,dx.$$

Using the *pdf* $f(x)$ from Equation (9), the r^{th} moment of the CRTFW distribution can be expressed as

$$\mu_r' = \int_0^\infty x^r \left[\lambda_1 + 2(\lambda_2-\lambda_1)\exp\left\{-\beta^\alpha\left(\tfrac{m}{x}\right)^{\alpha k}\right\} + 3(1-\lambda_2)\left[\exp\left\{-\beta^\alpha\left(\tfrac{m}{x}\right)^{\alpha k}\right\}\right]^2\right]$$
$$\times \alpha k \beta^\alpha m^{\alpha k} x^{-1-\alpha k}\exp\left\{-\beta^\alpha\left(\tfrac{m}{x}\right)^{\alpha k}\right\}dx.$$

After simplification, the r^{th} moment is given by

$$\mu_r' = \frac{m^r \beta^{r/k}\Gamma\left(1-\frac{r}{\alpha k}\right)}{6^{r/(\alpha k)}}\Big[6^{r/(\alpha k)}\lambda_1 + 3^{r/(\alpha k)}(\lambda_2-\lambda_1) + 2^{r/(\alpha k)}(1-\lambda_2)\Big]. \tag{12}$$

Substituting $r = 1$ in Equation (12), the mean of the CRTFW distribution is

$$\mu_1' = \frac{m\beta^{1/k}\Gamma\left(1-\frac{1}{\alpha k}\right)}{6^{1/(\alpha k)}}\Big[6^{1/(\alpha k)}\lambda_1 + 3^{1/(\alpha k)}(\lambda_2-\lambda_1) + 2^{1/(\alpha k)}(1-\lambda_2)\Big].$$

The variance can be calculated using $\text{Var}(X) = \mu_2 = \mu_2' - (\mu_1')^2$, which leads to

$$\sigma^2 = \frac{m^2\beta^{2/k}}{6^{2/(\alpha k)}} \left[\Gamma\left(1 - \tfrac{2}{\alpha k}\right)\left\{6^{2/(\alpha k)}\lambda_1 + 3^{2/(\alpha k)}(\lambda_2 - \lambda_1) + 6^{2/(\alpha k)}(1 - \lambda_2)\right\} - \left(\Gamma\left(1 - \tfrac{1}{\alpha k}\right)\left\{6^{1/(\alpha k)}\lambda_1 + 3^{1/(\alpha k)}(\lambda_2 - \lambda_1) + 6^{1/(\alpha k)}(1 - \lambda_2)\right\}\right)^2\right].$$

Higher-order moments can be obtained by substituting $r > 2$ in Equation (12). Using the valyes of σ^2, μ_3 and μ_4 (see **Appendix**), the coefficients of skewness, kurtosis, and coefficient of variation (CV) can be formulated as:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(c - 3ab + 2a^3)^2}{(b - a^2)^3}, \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{d - 4ac + 6a^2b + 3a^4}{(b - a^2)^2} \text{ and } CV = \frac{\sigma}{\mu} = \frac{\sqrt{b - a^2}}{a}$$

Table 1: Mean, variance, Skewness, Kurtosis, CV and Index of dispersion for the CRTFW distribution.

	Mean	Variance	Skewness	Kurtosis	CV	Index of Dispersion
$k = 0.5$	0.04662	0.00071	0.17344	2.04609	0.57111	0.01520
$k = 1.5$	0.01512	0.00024	0.56704	1.99222	1.02521	0.01589
$k = 1.8$	0.01159	0.00014	0.86991	2.64790	1.02511	0.01219
$k = 2.5$	0.00405	0.00005	1.67504	4.71026	1.71114	0.01185
$m = 0.5$	0.15383	0.00804	0.00378	1.42756	0.58290	0.05227
$m = 1.5$	0.01128	0.00021	0.96897	2.53893	1.27783	0.01842
$m = 2.0$	0.00652	0.00007	0.75778	1.61207	1.25843	0.01033
$m = 2.5$	0.00460	0.00002	0.64562	1.87735	1.02732	0.00485

The mean, variance, skewness, kurtosis, CV and index of dispersion of the CRTFW distribution for different values of k and m are given in Table 1. It can be observed that the mean, variance and index of diversity decreases as the value of k and m increases for fixed values of $\alpha = 0.5$, $\beta = 0.5$, $\lambda_1 = 0.005$, and $\lambda_2 = -0.05$.

4.4. Moment Generating and Characteristic Functions

The moment generating function (MGF) and the characteristic function are important tools in probability theory. They can be used to generate the moments of a distribution and are also valuable in studying various properties of the distribution. The moment generating function of a random variable X is defined as:

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx.$$

Therefore, the MGF of the CRTFW distribution is obtained as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{m^r \beta^{\frac{r}{k}} \Gamma(1 - \frac{r}{\alpha k})}{6^{\frac{r}{\alpha k}}} \left[6^{\frac{r}{\alpha k}}\ \lambda_1 + 3^{\frac{r}{\alpha k}}(\lambda_2 - \lambda_1) + 2^{\frac{r}{\alpha k}}(1 - \lambda_2)\right].$$

The characteristic function of a CRTFW distribution is derived in the following way:

$$\Phi(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{m^r \beta^{\frac{r}{k}} \Gamma(1 - \frac{r}{\alpha k})}{6^{\frac{r}{\alpha k}}} \left[6^{\frac{r}{\alpha k}} \lambda_1 + 3^{\frac{r}{\alpha k}} (\lambda_2 - \lambda_1) + 2^{\frac{r}{\alpha k}} (1 - \lambda_2) \right].$$

5. Reliability Measures, Entropy, and Order Statistics of the Cubic Rank Transmuted Fréchet-Weibull Distribution

5.1. Hazard Rate Function

The hazard rate function (HRF) of a distribution is closely related to its survival function. For the CRTFW distribution, the survival function $S(t)$ is defined as the probability that the lifetime T exceeds a given time t. Thus, the survival function of the CRTFW distribution is given by

$$S(t) = 1 - \left\{ \lambda_1 e^{-\beta^\alpha (m/t)^{\alpha k}} + (\lambda_2 - \lambda_1) \left[e^{-\beta^\alpha (m/t)^{\alpha k}} \right]^2 + (1 - \lambda_1) \left[e^{-\beta^\alpha (m/t)^{\alpha k}} \right]^3 \right\}.$$

Correspondingly, the hazard rate function (HRF) is obtained as

$$h(t) = \frac{\left\{ \lambda_1 + 2(\lambda_2 - \lambda_1) e^{-\beta^\alpha (m/t)^{\alpha k}} + 3(1 - \lambda_1) \left[e^{-\beta^\alpha (m/t)^{\alpha k}} \right]^2 \right\} \alpha k \beta^\alpha m^{\alpha k} t^{-1-\alpha k} e^{-\beta^\alpha (m/t)^{\alpha k}}}{1 - \left\{ \lambda_1 e^{-\beta^\alpha (m/t)^{\alpha k}} + (\lambda_2 - \lambda_1) \left[e^{-\beta^\alpha (m/t)^{\alpha k}} \right]^2 + (1 - \lambda_1) \left[e^{-\beta^\alpha (m/t)^{\alpha k}} \right]^3 \right\}}.$$

Figures 6 and 7 illustrate the possible shapes of the survival function $S(t)$ and the hazard

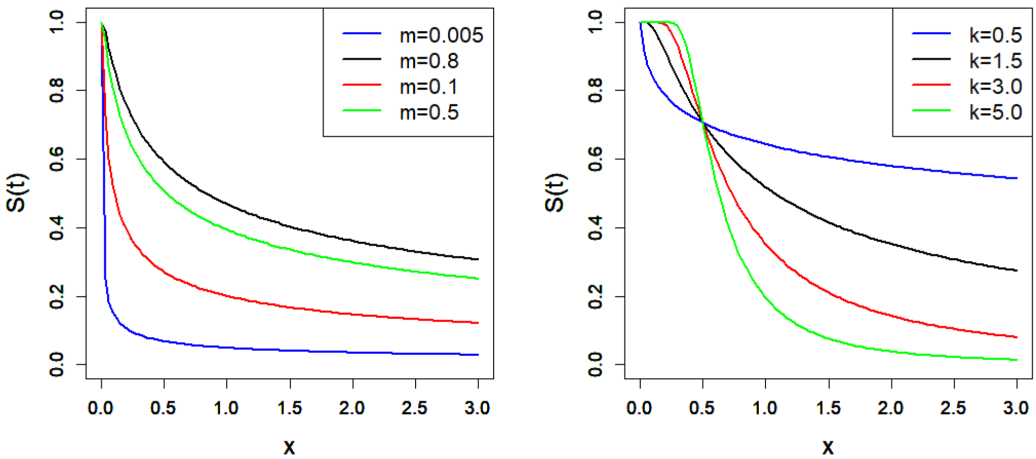

Figure 6: Survival function of the CRTFW distribution for selected values of m and k.

function $h(t)$, respectively. Figure 6(Left) shows the effect of varying the parameter m on

$S(t)$, with $\lambda_1 = 1, \lambda_2 = 1, \alpha = 0.5, \beta = 0.5$, and $k = 1$, while Figure 6(Right) illustrates the influence of the parameter k on $S(t)$, setting $\lambda_1 = 1, \lambda_2 = 1, \alpha = 0.5, \beta = 1.5$, and $m = 0.5$. Figure 7(Left) shows that $h(t)$ is a decreasing function for selected values of m with $\lambda_1 = 0.5, \lambda_2 = 0.5, \alpha = 0.5, \beta = 0.1$, and $k = 0.5$, whereas Figure 7(Right) demonstrates an upside-down bathtub shape for $h(t)$ when $k = 0.8, 0.85, 1.25$, and 1.75 with $\lambda_1 = 0.01, \lambda_2 = 0.5, \alpha = 0.8, \beta = 0.15$, and $m = 0.75$.

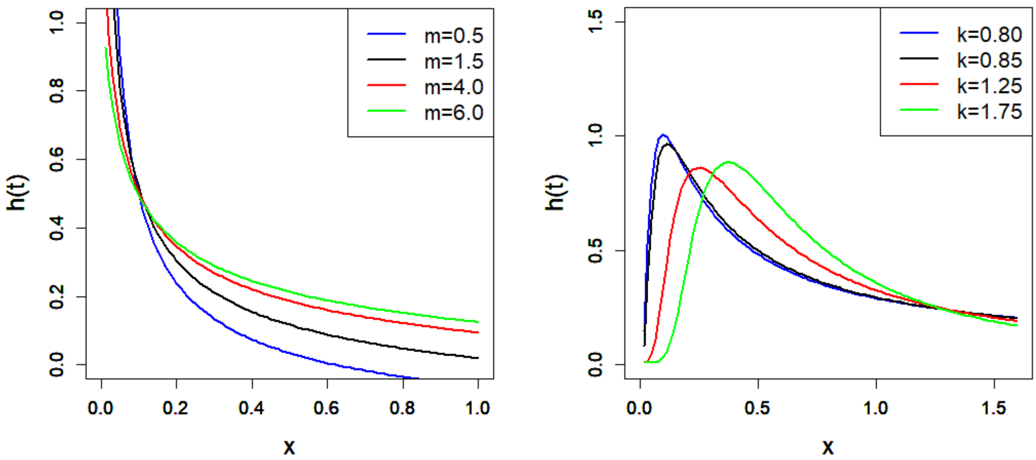

Figure 7: The HRF of the CRTFW distribution for selected values of m and k.

5.2. Entropy

Entropy is a state function that is often misinterpreted as the disorder of a system. More precisely, entropy measures the uncertainty associated with a random variable and plays a key role in thermodynamics and statistical mechanics. In 1948, Claude Shannon introduced the concept of information entropy, also called Shannon entropy [35]. Subsequently, several generalizations of Shannon entropy were developed, such as Rényi and Tsallis entropies [36], which include Shannon entropy as a special case. For the CRTFW distribution, the entropies can be defined as follows. The Rényi entropy for the CRTFW distribution is defined as

$$\begin{aligned} R_r(X) &= \frac{1}{1-r} \log \left[\left(\frac{\alpha k}{m} \right)^{r-1} \beta^{\alpha r} (r\beta\alpha)^{-\left(\frac{r}{\alpha k} + r - \frac{1}{\alpha k}\right)} \Gamma \left(\frac{r}{\alpha k} + r - \frac{1}{\alpha k} \right) \right. \\ & \qquad \left. \times \left\{ \lambda_1^r + 2^{\frac{1-r}{\alpha k}} (\lambda_2 - \lambda_1)^r + 3^{\frac{1-r}{\alpha k}} (1-\lambda_2)^r \right\} \right]. \end{aligned}$$

Similarly, the Tsallis entropy for the CRTFW distribution is defined as

$$T_r(X) = \frac{1}{1-r}\left[\left(\frac{\alpha k}{m}\right)^{r-1}\beta^{\alpha r}(r\beta\alpha)^{-\left(\frac{r}{\alpha k}+r-\frac{1}{\alpha k}\right)}\Gamma\left(\frac{r}{\alpha k}+r-\frac{1}{\alpha k}\right) \times\left\{\lambda_1^r+2^{\frac{1-r}{\alpha k}}(\lambda_2-\lambda_1)^r+3^{\frac{1-r}{\alpha k}}(1-\lambda_2)^r\right\}-1\right].$$

Finally, the Shannon entropy for the CRTFW distribution is given by

$$\begin{aligned} S_H(X) = {} & \log\left(\frac{m\beta^{1/k}}{\alpha k}\right)+\frac{1+\alpha k}{\alpha k}\Big\{\gamma+(\lambda_2-\lambda_1)\log 2+(1-\lambda_2)\log 3\Big\} \\ & +\frac{3\lambda_1+\lambda_2-2}{6}-\int_0^\infty\Big\{\lambda_1+2(\lambda_2-\lambda_1)e^{-\beta^\alpha(m/x)^{\alpha k}} \\ & +3(1-\lambda_1)\left[e^{-\beta^\alpha(m/x)^{\alpha k}}\right]^2\Big\}\alpha k\beta^\alpha m^{\alpha k}x^{-1-\alpha k}e^{-\beta^\alpha(m/x)^{\alpha k}} \\ & \times\log\Big\{\lambda_1+2(\lambda_2-\lambda_1)e^{-\beta^\alpha(m/x)^{\alpha k}}+3(1-\lambda_1)\left[e^{-\beta^\alpha(m/x)^{\alpha k}}\right]^2\Big\}dx, \end{aligned}$$

where $\gamma = -\int_0^\infty e^{-x}\ln(x)\,dx$ denotes the Euler-Mascheroni constant.

5.3. Order Statistics

The study of order statistics deals with the properties and applications of ordered random variables and their functions. Order statistics also play an important role in the estimation of the location and scale parameters of a distribution. Here we have obtained the *pdf* of a single-order statistic and joint-order statistics. According to [37], we have the density function of the r^{th} order statistic as:

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}. \tag{13}$$

The *pdf* of the r^{th} order statistic for the CRTFW distribution is obtained as:

$$\begin{aligned} f_{X_{r:n}}(x) = {} & \frac{n!}{(r-1)!(n-r)!}\left[\lambda_1 e^{-\beta^\alpha\left(\frac{m}{x}\right)^{\alpha k}}+(\lambda_2-\lambda_1)\left\{e^{-\beta^\alpha\left(\frac{m}{x}\right)^{\alpha k}}\right\}^2\right. \\ & \left.+(1-\lambda_1)\left\{e^{-\beta^\alpha\left(\frac{m}{x}\right)^{\alpha k}}\right\}^3\right]^{r-1}\times\left[1-\lambda_1 e^{-\beta^\alpha\left(\frac{m}{x}\right)^{\alpha k}}+(\lambda_2-\lambda_1)\right. \\ & \left.\left\{e^{-\beta^\alpha\left(\frac{m}{x}\right)^{\alpha k}}\right\}^2+(1-\lambda_1)\left\{e^{-\beta^\alpha\left(\frac{m}{x}\right)^{\alpha k}}\right\}^3\right]^{n-r} \\ & \times\left\{\lambda_1+2(\lambda_2-\lambda_1)e^{-\beta^\alpha\left(\frac{m}{x}\right)^{\alpha k}}+3(1-\lambda_1)\left[e^{-\beta^\alpha\left(\frac{m}{x}\right)^{\alpha k}}\right]^2\right\} \end{aligned}$$

$$\times\alpha k\beta^{\alpha}m^{\alpha k}x^{-1-\alpha k}e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}},$$

where $r = 1, 2, \ldots, n$. Therefore, for $r = 1$ and $r = n$, we have the *pdf* of the smallest order statistic and largest order statistic, respectively, as:

$$\begin{aligned} f_{X_{1:n}}(x) &= n\Bigg[1-\lambda_1 e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}+(\lambda_2-\lambda_1)\left\{e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}\right\}^2+(1-\lambda_1) \\ &\left\{e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}\right\}^3\Bigg]^{n-1}\times\Bigg\{\lambda_1+2(\lambda_2-\lambda_1)e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}+3(1-\lambda_1) \\ &\left[e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}\right]^2\Bigg\}\times\alpha k\beta^{\alpha}m^{\alpha k}x^{-1-\alpha k}e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}, \end{aligned}$$

and

$$\begin{aligned} f_{X_{n:n}}(x) &= n\Bigg[\lambda_1 e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}+(\lambda_2-\lambda_1)\left\{e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}\right\}^2+(1-\lambda_1)\left\{e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}\right\}^3\Bigg]^{n-1} \\ &\times\left\{\lambda_1+2(\lambda_2-\lambda_1)e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}+3(1-\lambda_1)\left[e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}\right]^2\right\} \\ &\times\alpha k\beta^{\alpha}m^{\alpha k}x^{-1-\alpha k}e^{-\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}}. \end{aligned}$$

The joint distribution of the r^{th} and s^{th} order statistics from the CRTFW distribution is defined as:

$$f_{r:s:n}(x,y) = C[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y), \tag{14}$$

where

$$C = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

The joint *pdf* of the smallest and largest order statistics for the CRTFW distribution can be easily obtained from Equation (14) by using $r = 1$ and $s = n$. Note that if $\lambda_1 = \lambda_2 = 1$, then we have the *pdf* of the r^{th} order statistic for the Fréchet–Weibull distribution as follows:

$$g_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}\left\{\alpha k\beta^{\alpha}m^{\alpha k}x^{-1-\alpha k}e^{-r\beta^{\alpha}\left(\frac{\lambda}{x}\right)^{\alpha k}}\right\}\left[1-e^{-\beta^{\alpha}\left(\frac{\lambda}{x}\right)^{\alpha k}}\right]^{n-r}.$$

6. Parameter Estimation

This section is devoted to the estimation of the parameters of the proposed CRTFW distribution. For this purpose, we employ the method of maximum likelihood estimation (MLE), which is one of the most widely used and efficient estimation techniques in statistical inference. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n drawn independently from the CRTFW distribution with probability density function $f(x)$. The corresponding log-likelihood function can be expressed as

$$\log L(x) = \left\{ n\log\lambda_1 - 2(\lambda_2-\lambda_1)\beta^{\alpha}\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k} - 6(1-\lambda_1)\beta^{\alpha}\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k} \right\} + n\log(\alpha k) + n\alpha\log(\beta) + n\alpha k\log(m) - (1-\alpha k)\sum_{i=1}^{n}\log(x_i) - \beta^{\alpha}\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k}. \tag{15}$$

The likelihood equations are obtained by differentiating Equation (15) with respect to each of the parameters α, β, λ_1, λ_2, m, and k, and then equating the derivatives to zero. The resulting score functions are given below:

$$\frac{\partial \log L(x)}{\partial\alpha} = \left\{ \frac{n}{\alpha} - (7-2\lambda_1-4\lambda_2)\beta^{\alpha}k\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k} + \log\beta\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k} + n\log(\beta) + nk\log(m) - k\sum_{i=1}^{n}\log(x_i) \right\} = 0,$$

$$\frac{\partial \log L(x)}{\partial\beta} = \left\{ \frac{n\alpha}{\beta} - \alpha\beta^{\alpha-1}(7-2\lambda_1-4\lambda_2)\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k} \right\} = 0,$$

$$\frac{\partial \log L(x)}{\partial\lambda_1} = \left\{ \frac{n}{\lambda_1} - 2\beta^{\alpha}\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k} \right\} = 0,$$

$$\frac{\partial \log L(x)}{\partial\lambda_2} = \left\{ 4\beta^{\alpha}\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k} \right\} = 0,$$

$$\frac{\partial \log L(x)}{\partial m} = \left\{ \frac{n\alpha k}{m} - \beta^{\alpha}(7-2\lambda_1-4\lambda_2)\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k} \alpha k m^{(\alpha k-1)} \right\} = 0,$$

$$\frac{\partial \log L(x)}{\partial k} = \left\{ \frac{n}{k} - \beta^{\alpha}(7-2\lambda_1-4\lambda_2)\alpha\sum_{i=1}^{n}\left(\frac{m}{x_i}\right)^{\alpha k}\log\left(\frac{m}{x_i}\right) + n\alpha\log(m) - \alpha\sum_{i=1}^{n}\log(x_i) \right\} = 0.$$

In principle, the maximum likelihood estimates (MLEs) of the parameters α, β, λ_1, λ_2, m, and k can be derived from the aforementioned system of nonlinear equations. Nonetheless, analytically solving these equations in closed form is exceedingly challenging due to their nonlinear and interrelated nature. Consequently, an iterative numerical technique like the Newton–Raphson algorithm is typically utilized to get approximate solutions. Statistical software packages (e.g., `R`, `MATLAB`, or `Python`) are employed to compute the maximum likelihood estimates efficiently. Section 7 offers a comprehensive examination of the estimating technique, encompassing a simulation study and applications involving real data.

7. Simulation-Based Analysis of Estimator Properties

We performed a simulation research to assess the efficacy of the maximum likelihood estimates (MLEs). In this simulation study, MLEs were computed for random samples from different sizes, which were drawn from the CRTFW distribution. The simulation algorithm is given below:

- Step 1: Generate random samples of sizes $n = 50, 80, 100, 200, 500$, and 600 from the CRTFW distribution by utilizing the parameter values $\alpha = 0.5$, $\beta = 0.5$, $\lambda_1 = 0.01$, $\lambda_2 = 0.1$, $m = 0.5$, and $k = 0.5$.
- Step 2: Evaluate parameter estimates for each sample of a specific size.
- Step 3: Repeat Step 1 and Step 2 for 1,000 times.
- Step 4: To calculate the average bias and mean square error (MSE) of the parameters, i.e., α, β, λ_1, λ_2, m, and k, represented by Q we used the following formulae. Let Q^* be the true values of parameters Q. Then, the bias and MSE from true values of parameters Q^* are defined as:

$$bias(\hat{Q}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{Q} - Q^*), \qquad MSE(\hat{Q}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{Q} - Q^*)^2$$

  Where N is the number of replications and Q represents the parameters i.e., α, β, λ_1, λ_2, m, and k.

For each replication, we calculated the mean, bias, and mean square error (MSE) of the parameter estimates, with the summary results presented in Table 2. The results demonstrate that, with an increase in sample size, the estimated values approach the actual parameter values. Furthermore, both the bias and the mean squared error diminish and converge to zero with increasing sample sizes, so validating the appropriateness and consistency of the maximum likelihood estimation method for the CRTFW distribution.

Table 2: Average estimates of parameters, bias, and MSE for CRTFW distribution.

	size n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{m}$	$\hat{k}$
Estimate	50	0.995089	0.850162	0.059021	0.597612	0.805081	0.852239
	80	0.924812	0.751471	0.086000	0.534561	0.712205	0.828724
	100	0.887845	0.626945	0.057691	0.467345	0.711295	0.766432
	200	0.856031	0.664595	0.040832	0.312545	0.698012	0.735124
	500	0.552406	0.528823	0.03677	0.296969	0.648339	0.564601
	600	0.505899	0.517074	0.014722	0.186874	0.609673	0.508491
Bias	50	0.495089	0.350162	0.049021	0.497612	0.305081	0.352239
	80	0.424812	0.251471	0.076	0.434561	0.212205	0.328724
	100	0.387845	0.126945	0.047691	0.367345	0.211295	0.266432
	200	0.356031	0.164595	0.030832	0.212545	0.198012	0.235124
	500	0.052406	0.028823	0.02677	0.196969	0.148339	0.064601
	600	0.005899	0.017074	0.004722	0.086874	0.109673	0.008491
MSE	50	15.33271	17.12075	0.654881	0.732761	21.96745	2.455233
	80	10.58112	12.59183	0.460856	0.514562	18.87466	1.989642
	100	7.543545	10.73492	0.214024	0.612900	15.71612	0.669543
	200	5.139154	13.54455	0.188822	0.388841	15.58835	0.603745
	500	1.692579	9.84899	0.112303	0.198201	14.71857	0.640984
	600	1.341729	1.240959	0.065385	0.035882	13.62887	0.578568

8. Evaluating Model Fit in Sustainability Data Analysis

This section evaluates the proposed CRTFW distribution using two real lifetime datasets to determine its goodness-of-fit and adaptability. The performance of the CRTFW model is assessed in comparison to several established competing distributions, including the Weibull (W), Fréchet (F), Fréchet–Weibull (FW), and QRTFW distributions. Following this preliminary assessment, we broaden the research to include sustainability-related data to examine the significance of the CRTFW distribution in modeling environmental and sustainability variables. This yields a focused evaluation of model fit in sustainability data analysis, highlighting the efficacy of the suggested distribution in depicting the essential trends in real sustainability measures.

8.1. Dataset I: Water Quality Data

To assess the relevance of the CRTFW distribution in environmental sciences, we analyze a dataset of seasonal estimated boron (B) concentrations (mg/L) in groundwater samples from Rungagora TE Belt, Golaghat district, Assam, India. The measurements were obtained from April 1, 2009, to March 31, 2010. The dataset was utilized with the author's prior consent from the Ph.D. thesis, Department of Chemistry, Gauhati University, Assam. The recorded values are as follows: 0.56, 0.67, 1.09, 1.35, 0.12, 0.27, 0.56, 0.12, 1.09, 0.56, 0.36, 0.29, 0.45, 0.06, 0.11, 0.09, 0.89, 0.38, 0.77, 0.87, 0.09, 0.12, 0.45, 0.23, 1.03, 1.23, 0.98, 0.78, 0.19, 0.11, 0.13, 0.28, 0.56, 0.47, 0.39, 0.41, 0.17, 0.19, 0.11, 0.21, 0.34, 0.67, 0.17, 0.20, 0.09, 0.10, 0.07, 0.09.

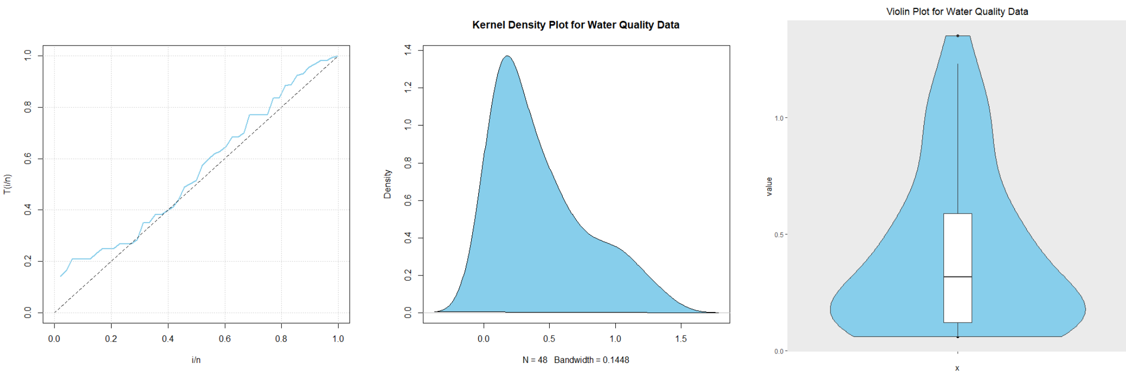

Figure 8: Non parametric plots for water quality data.

Figure 8 represents the non-parametric plots for water quality data. The TTT plot indicates an increasing behavior, kernel density plot indicates a right-skewness and the violin plot shows asymmetry. Thus, the proposed distribution is appropriate for fitting the water quality dataset.

Table 3: Estimated values of AIC, CAIC, BIC, $-\log l$, W, A, KS, and p-value for water quality data.

Distribution	**AIC**	**CAIC**	**BIC**	$-\log l$	**W**	**A**	**KS**	p**-value**
W	14.6514	14.9180	18.3938	5.3257	0.1245	0.8728	0.1079	0.6308
F	15.2230	15.4897	18.9654	14.0748	0.1648	1.0297	0.1140	0.5599
FW	19.2230	20.1532	26.7078	5.6115	0.1648	1.0297	0.1140	0.5600
QRTFW	20.8403	22.2689	30.1963	5.4202	0.1584	0.9966	0.1087	0.6211
CRTFW	**14.6106**	16.6594	25.8378	**1.3053**	**0.0503**	**0.3920**	**0.0777**	**0.9343**

Table 3 presents the goodness-of-fit metrics, encompassing the Akaike Information Criterion (AIC), consistent AIC (CAIC), Bayesian Information Criterion (BIC), negative log-likelihood ($-\log l$), Cramér–von Mises statistic (W), Anderson–Darling statistic (A), Kolmogorov–Smirnov statistic (KS), and the associated p-value. Table 3 clearly demonstrates that the CRTFW distribution produces the lowest values of AIC, $-\log l$, W, A, and KS, along with the highest p-value compared to the other models, thereby indicating the optimal fit to the dataset. The calculations were executed with `R` software. Table 4 displays the maximum likelihood estimates (MLEs) of the parameters together with their corresponding standard errors (SEs) for each fitted distribution. Figures 9 and 10 illustrate the fitted density and cumulative distribution functions, together with the P–P plots. The CRTFW distribution demonstrates the highest concordance with the empirical data.

Table 4: Estimates and SEs of parameters α, β, λ_1, λ_2, m, and k for water quality data.

Distribution	Estimated parameter	Standard error
W	$\hat{\beta} = 2.1660$	0.2630
	$\hat{\lambda} = 1.2582$	0.1412
F	$\hat{\alpha} = 1.2477$	0.1397
	$\hat{\beta} = 0.1913$	0.0234
FW	$\hat{\alpha} = 1.3971$	0.9982
	$\hat{\beta} = 0.3049$	0.7315
	$\hat{\lambda} = 0.7233$	0.2852
	$\hat{k} = 0.8931$	0.2876
QRTFW	$\hat{\alpha} = 1.5224$	0.1934
	$\hat{\beta} = 0.4755$	0.2919
	$\hat{\lambda} = 0.7236$	0.4358
	$\hat{m} = 0.4022$	0.2114
	$\hat{k} = 0.8583$	1.0902
CRTFW	$\hat{\alpha} = 1.4849$	0.3549
	$\hat{\beta} = 0.1565$	0.1967
	$\hat{\lambda}_1 = 0.7035$	0.4026
	$\hat{\lambda}_2 = -0.4309$	0.5999
	$\hat{m} = 0.7995$	0.2570
	$\hat{k} = 1.1687$	0.2794

8.2. Dataset II: Survival Times of Patients

This Section examines a medical dataset comprising the survival durations (measured in days) of 44 patients diagnosed with head and neck cancer. All patients received treatment using a combination of radiotherapy and chemotherapy (RT+CT). The initial dataset was documented by [38], and the recorded survival durations are as follows: 12.20, 23.56, 23.74, 25.87, 31.98, 37.00, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 74.47, 78.26, 81.43, 84.00, 92.00, 94.00, 110.00, 112.00, 119.00, 127.00, 130.00, 133.00, 140.00, 146.00, 155.00, 159.00, 173.00, 179.00, 194.00, 195.00, 209.00, 249.00, 281.00, 319.00, 339.00, 432.00, 469.00, 519.00, 633.00, 725.00, 817.00, 1776.00. To assess the efficacy of the proposed CRTFW distribution in modeling this survival data, we fitted the CRTFW model in conjunction

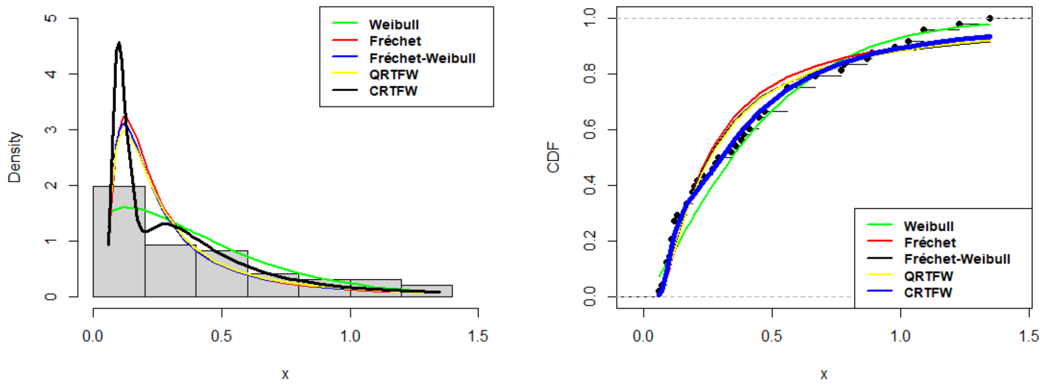

Figure 9: Fitted density and distribution functions of CRTFW model for water quality data.

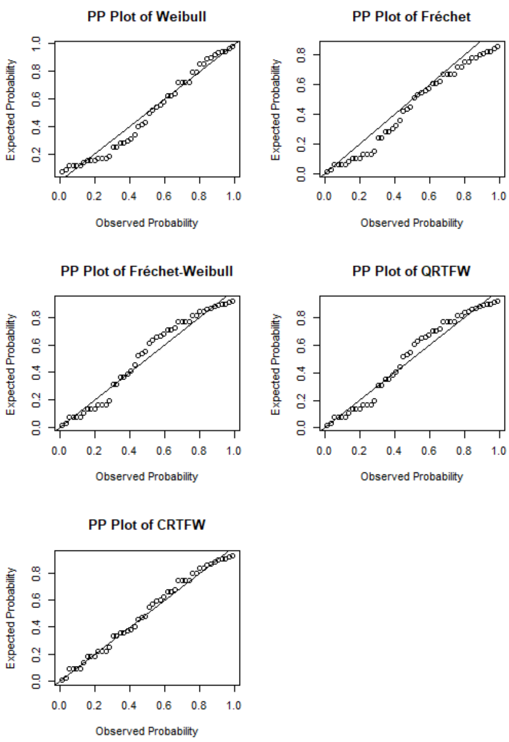

Figure 10: P–P plots of fitted distributions (W, F, FW, QRTFW, CRTFW) for water quality data.

with many rival distributions, including the W, F, FW, and CRTFW distributions. Table 5 provides a comprehensive comparison of the goodness-of-fit metrics, encompassing the AIC, CAIC, BIC, $-\log l$, W, A, KS, and the associated p-values. Table 5 demonstrates that the CRTFW distribution regularly surpasses the other distributions, yielding the lowest values for AIC, $-\log l$, W, A, and KS, while simultaneously obtaining the maximum

p-value. The results demonstrate that the CRTFW distribution offers the optimal fit for the survival data, effectively reflecting the underlying structure of the observed lifetimes compared to alternative models. The maximum likelihood estimates (MLEs) of the model parameters and their corresponding standard errors (SEs) are presented in Table 6.

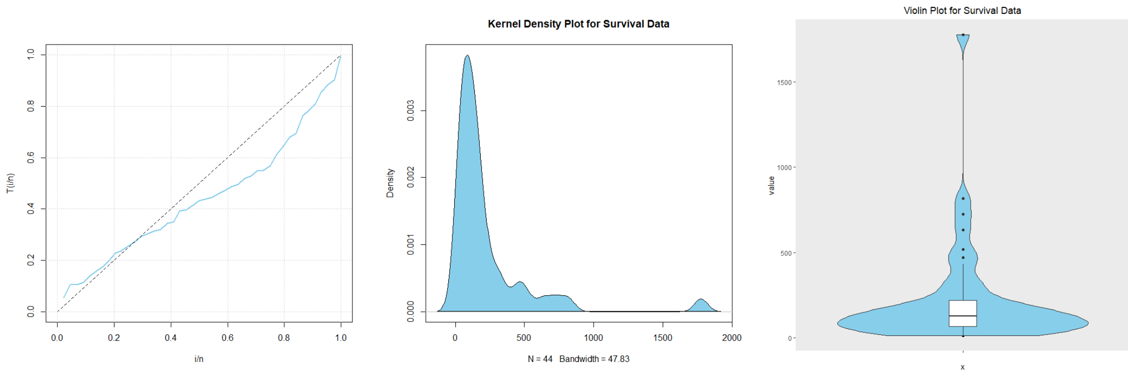

Figure 11: Non parametric plots for survival data.

Figure 11 represents the non-parametric plots for survival times of patients' data. The TTT plot shows an unimodal behavior, kernel density plot indicates a right-skewness and the violin plot indicate a positive skew.

Table 5: Estimated values of AIC, CAIC, BIC, $-\log l$, W, A, KS, and p-value for survival times of patients.

Distribution	**AIC**	**CAIC**	**BIC**	$-\log l$	**W**	**A**	**KS**	p**-value**
W	567.689	567.982	571.258	281.845	0.1410	0.8213	0.1261	0.4499
F	563.141	563.433	566.709	279.570	0.0771	0.4754	0.0938	0.7999
FW	567.140	568.166	574.277	279.570	0.0771	0.4754	0.0927	0.8104
QRTFW	567.646	569.225	576.567	278.823	0.0570	0.3581	0.0829	0.8979
CRTFW	**561.953**	569.224	577.659	**277.477**	**0.0119**	**0.1098**	**0.0465**	**0.9870**

Figures 12 and 13 depict the fitted probability density functions and cumulative distribution functions, respectively, alongside the P–P plots that compare the empirical cumulative distribution with the estimated cumulative distributions derived from the fitted models. The figures unequivocally illustrate that the CRTFW distribution coincides closely with the empirical data, validating its adaptability and suitability in estimating survival durations for head and neck cancer patients.

Table 6: Estimates and SEs of parameters α, β, λ_1, λ_2, m, and k for survival data.

Distribution	Estimated parameter	Standard error
W	$\hat{\beta} = 0.00468$	0.00075
	$\hat{\lambda} = 0.93705$	0.09950
F	$\hat{\alpha} = 1.01393$	0.11180
	$\hat{\beta} = 76.0134$	11.9624
FW	$\hat{\alpha} = 2.17257$	16.5230
	$\hat{\beta} = 9.96635$	358.498
	$\hat{\lambda} = 0.55115$	28.2458
	$\hat{k} = 0.46642$	3.54731
QRTFW	$\hat{\alpha} = 2.41281$	35.9663
	$\hat{\beta} = 2.79247$	50.8027
	$\hat{\lambda} = 0.34095$	0.36846
	$\hat{m} = 5.48477$	191.215
	$\hat{k} = 0.45558$	6.79104
CRTFW	$\hat{\alpha} = 3.93798$	32.6700
	$\hat{\beta} = 2.17735$	39.0578
	$\hat{\lambda}_1 = 0.70344$	0.39757
	$\hat{\lambda}_2 = -0.7230$	0.80218
	$\hat{m} = 4.44806$	193.318
	$\hat{k} = 0.34204$	2.83767

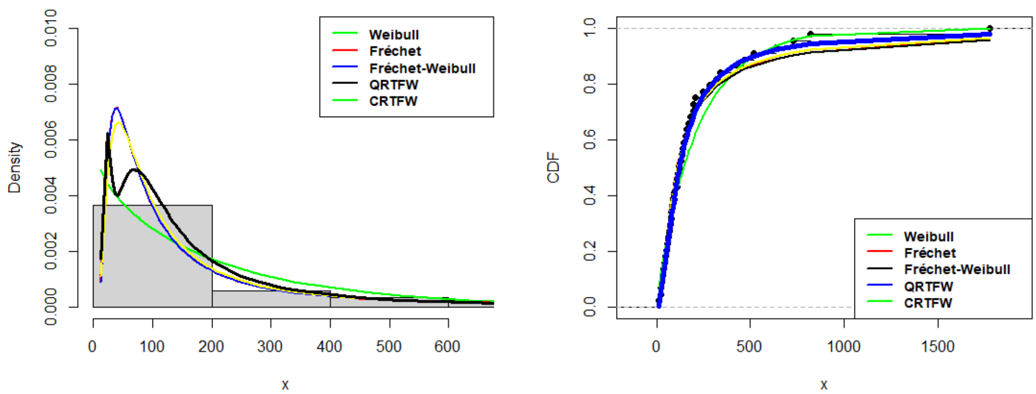

Figure 12: Fitted density and distribution functions of CRTFW model for survival data.

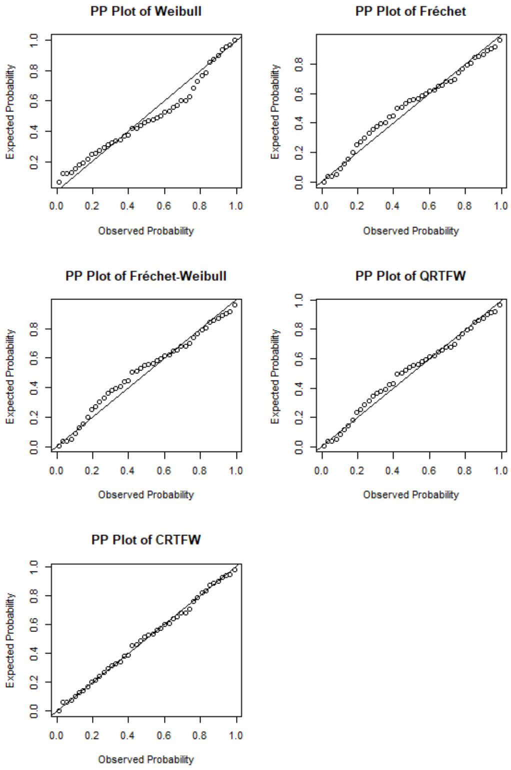

Figure 13: P–P plots of fitted distributions (W, F, FW, QRTFW, CRTFW) for survival data.

9. Conclusions

This study established and investigated a novel cubic rank transmuted Fréchet–Weibull (CRTFW) probability model within the cubic model framework. Several important mathematical and statistical properties of the CRTFW distribution were derived and analyzed, including its moments, moment generating function, characteristic function, quantile function, mode, random variate generation, hazard rate function, entropies, and order statistics. These analyses provided a comprehensive understanding of the distribution's behavior and demonstrated its flexibility in modeling diverse data patterns. The model effectively handled bimodal and unimodal data structures, as well as symmetrical distributions with varying degrees of kurtosis. A thorough Monte Carlo simulation study was run to evaluate the estimators' performance and efficiency under different circumstances, and the maximum likelihood method was used for parameter estimation. As the sample size increased, the simulation results showed that the MLEs worked as expected, with bias and mean square error (MSE) decreasing towards zero, demonstrating the estimate procedure's reliability. We examined two datasets pertaining to sustainability in the real world to show how the CRTFW model works in practice. In comparison to other competing lifespan and reliability distributions, the results showed that the CRTFW distribution captured the underlying data features adequately while also providing higher flexibility and increased goodness-of-fit. We propose expanding the scope of the CRTFW model to include more types of environmental, biological, and engineering datasets in our future studies. To further improve parameter inference, especially in situations with limited samples or censored observations, it could be worth exploring other estimating techniques like Bayesian or robust methods.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

Data Availability

The datasets used and analyzed during the current study are included within this published article.

Appendix

The derivation of the coefficients of skewness, kurtosis, and variation can be obtained by the following: For convenience, let

$$\mu_1' = \mu = \frac{m\beta^{1/k}}{6^{1/(\alpha k)}}a,$$

$$\sigma^2 = \mu_2 = \frac{m^2\beta^{2/k}}{6^{2/(\alpha k)}}(b - a^2),$$

$$\mu_3 = \frac{m^3\beta^{3/k}}{6^{3/(\alpha k)}}(c - 3ab + 2a^3),$$

$$\mu_4 = \frac{m^4\beta^{4/k}}{6^{4/(\alpha k)}}(d - 4ac + 6a^2b + 3a^4),$$

where

$$\begin{aligned}
a &= \Gamma\left(1 - \frac{1}{\alpha k}\right)\left[6^{1/(\alpha k)}\lambda_1 + 3^{1/(\alpha k)}(\lambda_2 - \lambda_1) + 6^{1/(\alpha k)}(1 - \lambda_2)\right],\\
b &= \Gamma\left(1 - \frac{2}{\alpha k}\right)\left[6^{2/(\alpha k)}\lambda_1 + 3^{2/(\alpha k)}(\lambda_2 - \lambda_1) + 6^{2/(\alpha k)}(1 - \lambda_2)\right],\\
c &= \Gamma\left(1 - \frac{3}{\alpha k}\right)\left[6^{3/(\alpha k)}\lambda_1 + 3^{3/(\alpha k)}(\lambda_2 - \lambda_1) + 6^{3/(\alpha k)}(1 - \lambda_2)\right],\\
d &= \Gamma\left(1 - \frac{4}{\alpha k}\right)\left[6^{4/(\alpha k)}\lambda_1 + 3^{4/(\alpha k)}(\lambda_2 - \lambda_1) + 6^{4/(\alpha k)}(1 - \lambda_2)\right].
\end{aligned}$$

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