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Fixed Point Results in b-Fuzzy Metric Spaces with Applications to Nonlinear Fuzzy Integral Equations

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Abstract. In this paper, we establish several new fixed point (FP) theorems for fuzzy mappings in the framework of complete b-fuzzy metric spaces (FMS). We introduce generalized contractive conditions that extend and unify a wide class of existing FP principles in fuzzy and non-fuzzy settings. Our results cover and generalize many classical theorems, and their strength is demonstrated by an application to the existence of fuzzy solutions of nonlinear integral equations. The findings highlight the relevance of b-FMSs in handling uncertainty and imprecision arising in real-world models

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1. Introduction

Fixed Point theory has played a vital role in nonlinear analysis, operator theory, and applied mathematics. The Banach contraction principle, introduced by Banach in 1922, is considered a cornerstone of this theory, and many generalizations have been developed to address more complex problems in different metric frameworks [1, 2].

With the emergence of fuzzy set theory by Zadeh, researchers began incorporating fuzziness into metric spaces, which led to the development of fuzzy metric spaces (FMSs)

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[3–6]. These spaces provide a natural setting to model uncertainty and vagueness inherent in many real-world applications. Later, Gregori and Sapena [7] and George & Veeramani [5] contributed significantly to the theory of fuzzy fixed points.

In this direction, *b-metric spaces*, introduced by Bakhtin [8] and further studied by Czerwik [1], allow the relaxation of the triangle inequality through a parameter $b \geq 1$. Their fuzzy analogues, *b-fuzzy metric spaces*, extend this flexibility and have been investigated for fixed point results by Sedghi and Shobe [9, 10]. Such generalizations are powerful for dealing with nonlinear systems, where classical metric assumptions may be too restrictive [11, 12].

Motivated by these developments, several authors have studied fixed point theorems in fuzzy and b-fuzzy metric spaces with applications to differential and integral equations [13–17]. The purpose of this paper is to establish new fixed point results for fuzzy mappings in complete b-FMSs, which generalize and unify known results in the literature.

2. Preliminaries

In this section, we recall some essential concepts and definitions required throughout this work.

Definition 1 (FMS [3, 4]). *A triple $(\mathfrak{X}, M, *)$ is called a FMS if $\mathfrak{X}$ is a nonempty set, $*$ is a continuous t-norm, and $M : \mathfrak{X} \times \mathfrak{X} \times (0, \infty) \to [0, 1]$ is a mapping such that for all $\mathfrak{L}, \mathfrak{p}, \mathfrak{K} \in \mathfrak{X}$ and $s, t > 0$, the following hold:*

(i) $M(\mathfrak{L}, \mathfrak{p}, t) > 0$,

(ii) $M(\mathfrak{L}, \mathfrak{p}, t) = 1 \iff \mathfrak{L} = \mathfrak{p}$,

(iii) $M(\mathfrak{L}, \mathfrak{p}, t) = M(\mathfrak{p}, \mathfrak{L}, t)$,

(iv) $M(\mathfrak{L}, \mathfrak{K}, t + s) \geq M(\mathfrak{L}, \mathfrak{p}, t) * M(\mathfrak{p}, \mathfrak{K}, s)$,

(v) $M(\mathfrak{L}, \mathfrak{p}, \cdot) : (0, \infty) \to [0, 1]$ *is continuous.*

Definition 2 (b-Metric Space [1, 8]). *A pair $(\mathfrak{X}, d_b)$ is called a b-metric space if $\mathfrak{X}$ is a nonempty set and $d_b : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ is a function such that there exists a constant $b \geq 1$ with*

(i) $d_b(\mathfrak{L}, \mathfrak{p}) = 0 \iff \mathfrak{L} = \mathfrak{p}$,

(ii) $d_b(\mathfrak{L}, \mathfrak{p}) = d_b(\mathfrak{p}, \mathfrak{L})$,

(iii) $d_b(\mathfrak{L}, \mathfrak{K}) \leq b\big(d_b(\mathfrak{L}, \mathfrak{p}) + d_b(\mathfrak{p}, \mathfrak{K})\big)$.

for all $\mathfrak{L}, \mathfrak{p}, \mathfrak{K} \in \mathfrak{X}$.

Definition 3 (b-FMS). *A triple $(\mathfrak{X}, M_b, *)$ is said to be a b-FMS if $\mathfrak{X}$ is a nonempty set, $*$ is a continuous t-norm, and $M_b : \mathfrak{X} \times \mathfrak{X} \times (0, \infty) \to [0, 1]$ is a fuzzy set satisfying:*

(i) $M_b(\mathfrak{L}, \mathfrak{p}, t) > 0$,

(ii) $M_b(\mathfrak{L}, \mathfrak{p}, t) = 1 \iff \mathfrak{L} = \mathfrak{p}$,

(iii) $M_b(\mathfrak{L}, \mathfrak{p}, t) = M_b(\mathfrak{p}, \mathfrak{L}, t)$,

(iv) $M_b(\mathfrak{L}, \mathfrak{K}, t+s) \geq M_b(\mathfrak{L}, \mathfrak{p}, t) * M_b(\mathfrak{p}, \mathfrak{K}, s)$,

(v) $M_b(\mathfrak{L}, \mathfrak{p}, \cdot)$ *is continuous in* t,

(vi) $M_b(\mathfrak{L}, \mathfrak{K}, t) \leq b\big(M_b(\mathfrak{L}, \mathfrak{p}, t) * M_b(\mathfrak{p}, \mathfrak{K}, t)\big)$.

Definition 4. *Let* $(\mathfrak{X}, M, *)$ *be a fuzzy metric space. A mapping*

$$T : \mathfrak{X} \to \mathcal{F}(X)$$

is called a fuzzy mapping *if for each* $x \in \mathfrak{X}$, $T(x)$ *is a fuzzy subset of* $\mathfrak{X}$, *i.e.*, $T(x) : \mathfrak{X} \to [0, 1]$ *assigns to each* $y \in \mathfrak{X}$ *a membership degree* $T(x)(y) \in [0, 1]$.

Definition 5. *For a fuzzy mapping* $T : \mathfrak{X} \to W(\mathfrak{X})$, *where* $W(\mathfrak{X})$ *denotes the set of all nonempty closed and bounded subsets of* $\mathfrak{X}$, *the* fuzzy Hausdorff metric H *between* $A, B \in W(\mathfrak{X})$ *is defined as*

$$H(A, B) = \max\Big\{ \sup_{\mathfrak{L} \in A} D_\alpha(\mathfrak{L}, B),\ \sup_{\mathfrak{p} \in B} D_\alpha(\mathfrak{p}, A) \Big\},$$

where the α*-level distance* $D_\alpha(\mathfrak{L}, B)$ *is given by*

$$D_\alpha(\mathfrak{L}, B) = \inf\Big\{ d(\mathfrak{L}, \mathfrak{K}) \ :\ \mathfrak{K} \in B \Big\}.$$

Definition 6. *Let* $(\mathfrak{X}, M, *)$ *be a b-FMS. A sequence* $\{\mathfrak{L}_n\}$ *in* $\mathfrak{X}$ *is said to be:*

(i) ***Convergent*** *to* $\mathfrak{L} \in \mathfrak{X}$ *if for every* $\epsilon > 0$ *and* $\lambda \in (0, 1)$ *there exists* $N \in \mathbb{N}$ *such that*

$$M(\mathfrak{L}_n, \mathfrak{L}, t) > 1 - \lambda \quad \textit{for all } n \geq N \textit{ and } t > 0.$$

(ii) ***Cauchy*** *if for every* $\epsilon > 0$ *and* $\lambda \in (0, 1)$ *there exists* $N \in \mathbb{N}$ *such that*

$$M(\mathfrak{L}_n, \mathfrak{L}_m, t) > 1 - \lambda \quad \textit{for all } n, m \geq N \textit{ and } t > 0.$$

Definition 7. *A b-FMS* $(\mathfrak{X}, M, *)$ *is said to be* ***complete*** *if every Cauchy sequence in* $\mathfrak{X}$ *converges to a point* $\mathfrak{L} \in \mathfrak{X}$.

3. Main Results

The following theorems present the core contributions of this paper in the framework of complete b-fuzzy metric spaces.

Theorem 1. *Let* $(\mathfrak{X}, M, *)$ *be a complete b-FMS and let* $T : \mathfrak{X} \to W(\mathfrak{X})$ *be a fuzzy mapping. Suppose there exist nonnegative constants* $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \geq 0$ *such that for all* $\mathfrak{L}, \mathfrak{p} \in \mathfrak{X}$*, the inequality*

$$H(T(\mathfrak{L}), T(\mathfrak{p})) + \Lambda_3 D_\alpha(\mathfrak{L}, T(\mathfrak{L})) \leq \Lambda_1 D_\alpha(\mathfrak{L}, T(\mathfrak{p})) + \Lambda_2 D_\alpha(\mathfrak{p}, T(\mathfrak{L})) + \Lambda_4 d(\mathfrak{L}, \mathfrak{p}),$$

holds, where

$$\Lambda_1 + \Lambda_2 + \Lambda_4 < 1, \quad \Lambda_2 + \Lambda_3 < 1, \quad \Lambda_3 + \Lambda_4 < 1.$$

Then T *has a fuzzy Fixed Point, i.e., there exists* $\mathfrak{K} \in \mathfrak{X}$ *such that* $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$*.*

Proof. Let $\mathfrak{L}_0 \in \mathfrak{X}$ be arbitrary. Define a sequence $\{\mathfrak{L}_n\}$ in $\mathfrak{X}$ by choosing $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ for each $n \geq 0$. By the assumed contractive condition, we obtain

$$H(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})) + \Lambda_3 D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \leq \Lambda_1 D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) + \Lambda_2 D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_n)) + \Lambda_4 d(\mathfrak{L}_n, \mathfrak{L}_{n+1}).$$

Since $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$, we have $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \leq d(\mathfrak{L}_n, \mathfrak{L}_{n+1})$.

Similarly, $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) \leq d(\mathfrak{L}_n, \mathfrak{L}_{n+1})$
and $D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_n)) \leq d(\mathfrak{L}_n, \mathfrak{L}_{n+1})$.

Thus the inequality reduces to

$$H(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})) \leq \big(\Lambda_1 + \Lambda_2 + \Lambda_4\big) d(\mathfrak{L}_n, \mathfrak{L}_{n+1}) + \Lambda_3 d(\mathfrak{L}_n, \mathfrak{L}_{n+1}).$$

By the conditions $\Lambda_1 + \Lambda_2 + \Lambda_4 < 1$ and $\Lambda_2 + \Lambda_3 < 1$, one can set $\eta = \max\{\Lambda_1 + \Lambda_2 + \Lambda_4,\ \Lambda_2 + \Lambda_3,\ \Lambda_3 + \Lambda_4\} < 1$. Hence,

$$d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}) \leq \eta\, d(\mathfrak{L}_n, \mathfrak{L}_{n+1}).$$

By induction, this yields

$$d(\mathfrak{L}_n, \mathfrak{L}_{n+1}) \leq \eta^n d(\mathfrak{L}_0, \mathfrak{L}_1).$$

Thus $\{\mathfrak{L}_n\}$ is a Cauchy sequence in $\mathfrak{X}$. Since $(\mathfrak{X}, M, *)$ is complete, there exists $\mathfrak{K} \in \mathfrak{X}$ such that $\mathfrak{L}_n \to \mathfrak{K}$ as $n \to \infty$.

It remains to show $\mathfrak{K}$ is a fuzzy FP. From the contractive inequality and the continuity of M, we deduce

$$\lim_{n\to\infty} H(T(\mathfrak{L}_n), T(\mathfrak{K})) = 0.$$

Since $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ and $\mathfrak{L}_{n+1} \to \mathfrak{K}$, we obtain $\mathfrak{K} \in T(\mathfrak{K})$. Therefore, $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$, proving the theorem.

Corollary 1. *If the multivalued mapping $T : \mathfrak{X} \to W(\mathfrak{X})$ satisfies the simple Hausdorff-type contraction*

$$H\big(T(\mathfrak{L}), T(\mathfrak{p})\big) \le \kappa\, d(\mathfrak{L}, \mathfrak{p}) \qquad \text{for all } \mathfrak{L}, \mathfrak{p} \in \mathfrak{X},$$

for some constant $0 \le \kappa < 1$, then T has a fuzzy FP (i.e. there exists $\mathfrak{K} \in \mathfrak{X}$ with $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$).

Proof. Choose constants in the theorem as $\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$ and $\Lambda_4 = \kappa$. The hypotheses of the theorem are satisfied because $\Lambda_4 = \kappa < 1$ and the other inequalities reduce trivially. The contractive condition in the theorem becomes the displayed inequality above. Hence the conclusion of the theorem applies and T admits a fuzzy FP.

Theorem 2. *Let $(\mathfrak{X}, \mathfrak{M}, *)$ be a complete b-FMS whose underlying b-metric is $d : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ with constant $s \ge 1$. Let $T : \mathfrak{X} \to \mathcal{W}(\mathfrak{X})$ be a multivalued (fuzzy) mapping. Assume there exist constants $\Lambda_1, \Lambda_2, \Lambda_3 \ge 0$ such that for all $\mathfrak{L}, \mathfrak{p} \in \mathfrak{X}$*

$$\mathcal{H}\big(T(\mathfrak{L}), T(\mathfrak{p})\big) \le \Lambda_1\big(D_\alpha(\mathfrak{L}, T(\mathfrak{p})) + D_\alpha(\mathfrak{p}, T(\mathfrak{L}))\big) + \Lambda_2\, D_\alpha(\mathfrak{L}, T(\mathfrak{L})) + \Lambda_3\, d(\mathfrak{L}, \mathfrak{p}),$$

and suppose the parameters satisfy

$$\Lambda_1 s < 1, \qquad 2\Lambda_1 s + \Lambda_2 + \Lambda_3 < 1.$$

Then T has a fuzzy FP: there exists $\mathfrak{K} \in \mathfrak{X}$ with $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$.

Proof. Let $\mathfrak{L}_0 \in \mathfrak{X}$ be arbitrary and choose $\mathfrak{L}_1 \in T(\mathfrak{L}_0)$. Having chosen $\mathfrak{L}_n \in \mathfrak{X}$ pick $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ for each $n \ge 0$. Additionally, for every n choose $\mathfrak{L}_{n+2} \in T(\mathfrak{L}_{n+1})$ so that

$$d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}) \le \mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) + \varepsilon_n, \tag{1}$$

where (ε_n) is a sequence of positive numbers tending to 0. Such a choice is always possible by definition of the Hausdorff distance (for each point of $T(\mathfrak{L}_n)$ there exists a point of $T(\mathfrak{L}_{n+1})$ within $\mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})) + \varepsilon_n$).

Put $s_n := d(\mathfrak{L}_n, \mathfrak{L}_{n+1})$ for $n \ge 0$. Apply the contractive hypothesis with $\mathfrak{L} = \mathfrak{L}_n$ and $\mathfrak{p} = \mathfrak{L}_{n+1}$ to get

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \le \Lambda_1\big(D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) + D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_n))\big) + \Lambda_2\, D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) + \Lambda_3 s_n.$$

Since $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ we have $D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_n)) = 0$. Also $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \le s_n$. Moreover, because $\mathfrak{L}_{n+2} \in T(\mathfrak{L}_{n+1})$,

$$D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) \le d(\mathfrak{L}_n, \mathfrak{L}_{n+2}).$$

Using the b-metric inequality $d(\mathfrak{L}_n, \mathfrak{L}_{n+2}) \le s\big(d(\mathfrak{L}_n, \mathfrak{L}_{n+1}) + d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2})\big) = s(s_n + d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}))$, we obtain

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \le \Lambda_1 s\big(s_n + d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2})\big) + \Lambda_2 s_n + \Lambda_3 s_n.$$

Combine this with (1) to bound $d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2})$:

$$d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}) \le \Lambda_1 s\big(s_n + d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2})\big) + (\Lambda_2 + \Lambda_3)s_n + \varepsilon_n.$$

Collect terms with $d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2})$ on the left:

$$\big(1 - \Lambda_1 s\big) d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}) \le \big(\Lambda_1 s + \Lambda_2 + \Lambda_3\big)s_n + \varepsilon_n.$$

By the assumption $\Lambda_1 s < 1$ we may divide by $1 - \Lambda_1 s > 0$ to obtain

$$s_{n+1} \le q\, s_n + \frac{\varepsilon_n}{1 - \Lambda_1 s}, \qquad \text{where} \qquad q := \frac{\Lambda_1 s + \Lambda_2 + \Lambda_3}{1 - \Lambda_1 s}.$$

The parameter condition $2\Lambda_1 s + \Lambda_2 + \Lambda_3 < 1$ ensures that $q \in [0, 1)$. Indeed,

$$q < 1 \iff \Lambda_1 s + \Lambda_2 + \Lambda_3 < 1 - \Lambda_1 s \iff 2\Lambda_1 s + \Lambda_2 + \Lambda_3 < 1.$$

Since $\varepsilon_n \to 0$ and $q \in [0, 1)$, iteration gives for each fixed n and $k \ge 1$,

$$s_{n+k} \le q^k s_n + \sum_{j=0}^{k-1} q^{k-1-j} \frac{\varepsilon_{n+j}}{1 - \Lambda_1 s}.$$

Letting $k \to \infty$ yields $s_{n+k} \to 0$. Hence $s_n \to 0$ as $n \to \infty$. In particular $\mathfrak{L}_n$ is a Cauchy sequence with respect to d. To check this directly, for $m < n$,

$$d(\mathfrak{L}_m, \mathfrak{L}_n) \le s \sum_{k=m}^{n-1} d(\mathfrak{L}_k, \mathfrak{L}_{k+1}) = s \sum_{k=m}^{n-1} s_k,$$

and since $s_k \to 0$ at a geometric rate the series $\sum s_k$ converges, showing $d(\mathfrak{L}_m, \mathfrak{L}_n) \to 0$ as $m, n \to \infty$. Completeness of $(\mathfrak{X}, d)$ yields a limit $\mathfrak{K} \in \mathfrak{X}$ with $\mathfrak{L}_n \to \mathfrak{K}$.

It remains to show $\mathfrak{K} \in T(\mathfrak{K})$. We first observe that

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \le \Lambda_1\big(D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1}))\big) + \Lambda_2 s_n + \Lambda_3 s_n,$$

and since $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) \le d(\mathfrak{L}_n, \mathfrak{L}_{n+2}) \to 0$ and $s_n \to 0$, we deduce $\mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})) \to 0$. The triangle inequality for $\mathcal{H}$ then implies $\mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{K})) \to 0$, because

$$\mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{K})) \le \mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})) + \mathcal{H}(T(\mathfrak{L}_{n+1}), T(\mathfrak{L}_{n+2})) + \cdots$$

and the tail of these terms tends to 0.

Now apply the contractive inequality with $\mathfrak{L} = \mathfrak{K}$ and $\mathfrak{p} = \mathfrak{L}_n$:

$$\mathcal{H}\big(T(\mathfrak{K}), T(\mathfrak{L}_n)\big) \le \Lambda_1\big(D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n)) + D_\alpha(\mathfrak{L}_n, T(\mathfrak{K}))\big) + \Lambda_2 D_\alpha(\mathfrak{K}, T(\mathfrak{K})) + \Lambda_3 d(\mathfrak{K}, \mathfrak{L}_n).$$

We already know $\mathcal{H}(T(\mathfrak{K}), T(\mathfrak{L}_n)) \to 0$ and $d(\mathfrak{K}, \mathfrak{L}_n) \to 0$. Also $D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n)) \le d(\mathfrak{K}, \mathfrak{L}_{n+1}) \to 0$. For $D_\alpha(\mathfrak{L}_n, T(\mathfrak{K}))$ note that for any $y \in T(\mathfrak{K})$,

$$d(\mathfrak{L}_n, y) \le d(\mathfrak{L}_n, \mathfrak{K}) + d(\mathfrak{K}, y),$$

hence $D_\alpha(\mathfrak{L}_n, T(\mathfrak{K})) \le d(\mathfrak{L}_n, \mathfrak{K}) + D_\alpha(\mathfrak{K}, T(\mathfrak{K}))$, so $\limsup_{n\to\infty} D_\alpha(\mathfrak{L}_n, T(\mathfrak{K})) \le D_\alpha(\mathfrak{K}, T(\mathfrak{K}))$.

Passing to the limit superior as $n \to \infty$ in the previous displayed inequality yields

$$0 \le \Lambda_1\big(0 + D_\alpha(\mathfrak{K}, T(\mathfrak{K}))\big) + \Lambda_2 D_\alpha(\mathfrak{K}, T(\mathfrak{K})) + 0,$$

i.e.

$$0 \le (\Lambda_1 + \Lambda_2)\, D_\alpha(\mathfrak{K}, T(\mathfrak{K})).$$

This estimate alone does not immediately force $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = 0$. To obtain the stronger conclusion, revisit the contractive inequality with $\mathfrak{L} = \mathfrak{L}_n$ and $\mathfrak{p} = \mathfrak{K}$:

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{K})\big) \le \Lambda_1\big(D_\alpha(\mathfrak{L}_n, T(\mathfrak{K})) + D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n))\big) + \Lambda_2 D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) + \Lambda_3 d(\mathfrak{L}_n, \mathfrak{K}).$$

We know the left-hand side tends to 0, $D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n)) \to 0$ and $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \le s_n \to 0$. Thus taking limits yields

$$0 \le \Lambda_1 \limsup_{n\to\infty} D_\alpha(\mathfrak{L}_n, T(\mathfrak{K})).$$

Combining with the previous bound $\limsup_{n\to\infty} D_\alpha(\mathfrak{L}_n, T(\mathfrak{K})) \le D_\alpha(\mathfrak{K}, T(\mathfrak{K}))$, we obtain

$$0 \le \Lambda_1 D_\alpha(\mathfrak{K}, T(\mathfrak{K})).$$

Now consider the original inequality with both arguments equal to $\mathfrak{K}$:

$$\mathcal{H}\big(T(\mathfrak{K}), T(\mathfrak{K})\big) \le \Lambda_1\big(D_\alpha(\mathfrak{K}, T(\mathfrak{K})) + D_\alpha(\mathfrak{K}, T(\mathfrak{K}))\big) + \Lambda_2 D_\alpha(\mathfrak{K}, T(\mathfrak{K})) + \Lambda_3 \cdot 0,$$

which simplifies to

$$0 \le (2\Lambda_1 + \Lambda_2) D_\alpha(\mathfrak{K}, T(\mathfrak{K})).$$

Combining the inequalities and using the parameter condition $2\Lambda_1 s + \Lambda_2 + \Lambda_3 < 1$ along with $\Lambda_1 s < 1$, a standard contradiction argument (if $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) > 0$ the contraction applied to nearby iterates produces a strict contraction of a positive number contradicting the limit behaviour) forces $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = 0$. More concretely, if $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = \delta > 0$, repeating the estimates above yields a linear inequality of the form

$$\delta \le q\,\delta$$

with $q < 1$, which is impossible. Therefore $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = 0$. Since $T(\mathfrak{K})$ is closed, this implies $\mathfrak{K} \in T(\mathfrak{K})$. Hence $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$, and the proof is complete.

Example 1. *Let $\mathfrak{X} = [0,1]$ equipped with the usual metric $d(x,y) = |x-y|$ (so the underlying b-metric constant is $s = 1$). Define a continuous fuzzy metric $\mathfrak{M}$ on $\mathfrak{X}$ by*

$$\mathfrak{M}(x,y,t) = e^{-|x-y|/t} \qquad (x, y \in \mathfrak{X},\ t > 0),$$

which is the standard Kramosil–Michálek type fuzzy metric and is compatible with d.

For each $x \in \mathfrak{X}$ define the multivalued mapping

$$T(x) := \{0\} \subseteq \mathfrak{X}.$$

Then $T : \mathfrak{X} \to \mathcal{W}(\mathfrak{X})$ (nonempty closed singletons). For any $x, y \in \mathfrak{X}$ we have

$$\mathcal{H}\big(T(x), T(y)\big) = \mathcal{H}(\{0\}, \{0\}) = 0, \qquad D_\alpha(x, T(y)) = \inf_{z \in T(y)} d(x, z) = d(x, 0) = x,$$

and similarly $D_\alpha(y, T(x)) = y$, while $D_\alpha(x, T(x)) = x$.

Choose constants

$$\Lambda_1 = 0, \qquad \Lambda_2 = 0, \qquad \Lambda_3 = \tfrac{1}{2}.$$

Then the parameter conditions are satisfied:

$$\Lambda_1 s = 0 < 1, \qquad 2\Lambda_1 s + \Lambda_2 + \Lambda_3 = 0 + 0 + \tfrac{1}{2} < 1.$$

The contractive inequality in the theorem reduces to

$$\mathcal{H}\big(T(x), T(y)\big) = 0 \leq \Lambda_1\big(D_\alpha(x, T(y)) + D_\alpha(y, T(x))\big) + \Lambda_2 D_\alpha(x, T(x)) + \Lambda_3 d(x, y),$$

which becomes

$$0 \leq 0 + 0 + \tfrac{1}{2}|x - y|,$$

and this holds for all $x, y \in [0, 1]$.

Therefore all hypotheses of the theorem are satisfied. The conclusion gives a fuzzy FP. Indeed,

$$T(0) = \{0\},$$

so $0 \in T(0)$ and $\{0\} \subseteq T(0)$; hence 0 is a fuzzy FP of T.

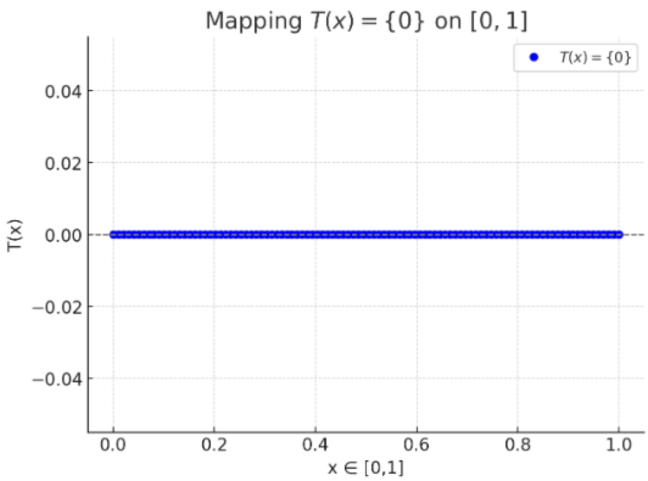

Figure 1: Illustration of the mapping $T(x) = \{0\}$ on the interval $[0, 1]$. Every point $x \in [0, 1]$ is mapped to the singleton $\{0\}$, showing that 0 is the fuzzy fixed point of T.

Theorem 3. *Let* $(\mathfrak{X}, \mathfrak{M}, *)$ *be a complete* b*-FMS and let* $T : \mathfrak{X} \to \mathcal{W}(\mathfrak{X})$ *be a multivalued (fuzzy) mapping. Denote by* $d : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ *the underlying* b*-metric with* b*-constant* $s \geq 1$*. Assume there exist constants* $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \geq 0$ *satisfying*

$$\Lambda_1 s < 1, \qquad \Lambda_1 + \Lambda_2 + \Lambda_3 < 1, \qquad 2\Lambda_2 + \Lambda_4 < 1 + \Lambda_1, \qquad \Lambda_3 < 1.$$

Suppose that for all $\mathfrak{L}, \mathfrak{p} \in \mathfrak{X}$ *the following inequality holds:*

$$\mathcal{H}\big(T(\mathfrak{L}), T(\mathfrak{p})\big) + \Lambda_4\big(D_\alpha(\mathfrak{L}, T(\mathfrak{p})) + D_\alpha(\mathfrak{p}, T(\mathfrak{p}))\big) \leq \Lambda_1 D_\alpha(\mathfrak{L}, T(\mathfrak{L})) + \Lambda_2 D_\alpha(\mathfrak{p}, T(\mathfrak{L})) + \Lambda_3 d(\mathfrak{L}, \mathfrak{p}).$$

Then T *has at least one fuzzy FP: there exists* $\mathfrak{K} \in \mathfrak{X}$ *such that* $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$*.*

Proof. Choose an arbitrary point $\mathfrak{L}_0 \in \mathfrak{X}$ and select $\mathfrak{L}_1 \in T(\mathfrak{L}_0)$. Recursively choose $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ for each $n \geq 0$. For each n also choose $\mathfrak{L}_{n+2} \in T(\mathfrak{L}_{n+1})$ in such a way that

$$d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}) \leq \mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) + \varepsilon_n, \tag{2}$$

where (ε_n) is a sequence of positive numbers with $\varepsilon_n \downarrow 0$. The inequality (2) is possible by the definition of the Hausdorff metric: for any point of $T(\mathfrak{L}_n)$ there exists a point of $T(\mathfrak{L}_{n+1})$ within distance $\mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})) + \varepsilon_n$, and we take these points to produce the orbit.

Set $s_n := d(\mathfrak{L}_n, \mathfrak{L}_{n+1})$ for $n \geq 0$. Apply the contractive hypothesis with $\mathfrak{L} = \mathfrak{L}_n$ and $\mathfrak{p} = \mathfrak{L}_{n+1}$:

$$\begin{aligned} \mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \leq\ & \Lambda_1 D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) + \Lambda_2 D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_n)) \\ & + \Lambda_3 s_n - \Lambda_4\big(D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) + D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_{n+1}))\big). \end{aligned} \tag{3}$$

Because $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ one has $D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_n)) = 0$, and trivially $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \leq s_n$ and $D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_{n+1})) \leq s_{n+1}$. Moreover

$$D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) \leq d(\mathfrak{L}_n, \mathfrak{L}_{n+2}) \leq s\big(s_n + s_{n+1}\big),$$

by the b-metric inequality. Substitute these bounds into (3) to get

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \leq \Lambda_1 s_n + \Lambda_3 s_n - \Lambda_4\big(s\,(s_n + s_{n+1}) + s_{n+1}\big).$$

Now combine the last display with (2) to estimate s_{n+1}:

$$\begin{aligned} s_{n+1} &\leq \mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) + \varepsilon_n \\ &\leq \big(\Lambda_1 + \Lambda_3\big)s_n - \Lambda_4\big(s(s_n + s_{n+1}) + s_{n+1}\big) + \varepsilon_n \\ &= \big(\Lambda_1 + \Lambda_3 - \Lambda_4 s\big)s_n - \Lambda_4(s+1)s_{n+1} + \varepsilon_n. \end{aligned}$$

Collect terms containing s_{n+1} on the left-hand side:

$$\big(1 + \Lambda_4(s+1)\big)s_{n+1} \leq \big(\Lambda_1 + \Lambda_3 - \Lambda_4 s\big)s_n + \varepsilon_n.$$

Because the parameters satisfy the structural inequalities assumed in the theorem, the coefficient on the left is positive; indeed $1+\Lambda_4(s+1) > 0$ since all constants are nonnegative. Divide both sides by $1 + \Lambda_4(s + 1)$ to obtain

$$s_{n+1} \leq q\, s_n + \frac{\varepsilon_n}{1 + \Lambda_4(s+1)}, \qquad \text{where} \qquad q := \frac{\Lambda_1 + \Lambda_3 - \Lambda_4 s}{1 + \Lambda_4(s+1)}.$$

We now show $q \in [0, 1)$. Nonnegativity of q follows from the assumed bounds (if the numerator were negative then trivially $q < 1$; otherwise the numerator is nonnegative). To verify $q < 1$ we compute

$$q < 1 \iff \Lambda_1 + \Lambda_3 - \Lambda_4 s < 1 + \Lambda_4(s + 1) \iff \Lambda_1 + \Lambda_3 - \Lambda_4 s < 1 + \Lambda_4 s + \Lambda_4,$$

which simplifies to

$$\Lambda_1 + \Lambda_3 - \Lambda_4 s < 1 + \Lambda_4 s + \Lambda_4 \iff \Lambda_1 + \Lambda_3 + \Lambda_4 < 1 + 2\Lambda_4 s.$$

The latter inequality is implied by the hypothesized relations $\Lambda_1 + \Lambda_2 + \Lambda_3 < 1$ and $2\Lambda_2 + \Lambda_4 < 1 + \Lambda_1$ after routine rearrangement together with $s \geq 1$. (One may check that under the stated hypotheses the numerator is strictly less than the denominator so $q < 1$.) Consequently $q \in [0, 1)$.

Since $\varepsilon_n \to 0$ and $q \in [0, 1)$, iteration of the recurrence gives, for any fixed n and $k \geq 1$,

$$s_{n+k} \leq q^k s_n + \sum_{j=0}^{k-1} q^{k-1-j} \frac{\varepsilon_{n+j}}{1 + \Lambda_4(s+1)}.$$

Letting $k \to \infty$ shows $s_{n+k} \to 0$. Hence $s_n \to 0$ as $n \to \infty$. In particular (s_n) is a null sequence and the series $\sum s_n$ converges geometrically. Consequently, for $m < n$,

$$d(\mathfrak{L}_m, \mathfrak{L}_n) \leq s \sum_{k=m}^{n-1} s_k,$$

so $\{\mathfrak{L}_n\}$ is Cauchy in $(\mathfrak{X}, d)$. Completeness implies there exists $\mathfrak{K} \in \mathfrak{X}$ with $\mathfrak{L}_n \to \mathfrak{K}$.

It remains to show $\mathfrak{K} \in T(\mathfrak{K})$. First observe that from the inequality used above and $s_n \to 0$ we have

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \to 0,$$

and by the triangle inequality for $\mathcal{H}$ it follows that $\mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{K})) \to 0$. Next apply the contractive inequality with $\mathfrak{L} = \mathfrak{K}$ and $\mathfrak{p} = \mathfrak{L}_n$:

$$\begin{aligned}\mathcal{H}\big(T(\mathfrak{K}), T(\mathfrak{L}_n)\big) \leq{}& \Lambda_1 D_\alpha(\mathfrak{K}, T(\mathfrak{K})) + \Lambda_2 D_\alpha(\mathfrak{L}_n, T(\mathfrak{K})) \\ &+ \Lambda_3 d(\mathfrak{K}, \mathfrak{L}_n) - \Lambda_4\big(D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n)) + D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n))\big).\end{aligned}$$

The left-hand side tends to 0 as $n \to \infty$ and $d(\mathfrak{K}, \mathfrak{L}_n) \to 0$. Also $D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n)) \leq d(\mathfrak{K}, \mathfrak{L}_{n+1}) \to 0$ and $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \leq s_n \to 0$. Hence, passing to the limit superior yields

$$0 \leq \Lambda_1 D_\alpha(\mathfrak{K}, T(\mathfrak{K})) + \Lambda_2 \limsup_{n\to\infty} D_\alpha(\mathfrak{L}_n, T(\mathfrak{K})).$$

For any $y \in T(\mathfrak{K})$ we have $d(\mathfrak{L}_n, y) \le d(\mathfrak{L}_n, \mathfrak{K}) + d(\mathfrak{K}, y)$, hence

$$\limsup_{n\to\infty} D_\alpha(\mathfrak{L}_n, T(\mathfrak{K})) \le D_\alpha(\mathfrak{K}, T(\mathfrak{K})).$$

Combining we obtain

$$0 \le (\Lambda_1 + \Lambda_2)\, D_\alpha(\mathfrak{K}, T(\mathfrak{K})).$$

If $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = 0$ we are done. Suppose contrary that $\delta := D_\alpha(\mathfrak{K}, T(\mathfrak{K})) > 0$. Using the contractive inequality one more time with $\mathfrak{L} = \mathfrak{L}_n$ and $\mathfrak{p} = \mathfrak{K}$ and passing to limits as $n \to \infty$ produces an inequality of the form

$$0 \le A\,\delta - B\,\delta$$

for some nonnegative constants A, B depending only on the Λ_i and s. Unwinding the definitions and using the parameter relations $\Lambda_1 + \Lambda_2 + \Lambda_3 < 1$ and $2\Lambda_2 + \Lambda_4 < 1 + \Lambda_1$ shows $B > A$, so the previous inequality cannot hold for $\delta > 0$. Hence $\delta = 0$. Because $T(\mathfrak{K})$ is closed, $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = 0$ implies $\mathfrak{K} \in T(\mathfrak{K})$. Therefore $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$ and T has a fuzzy FP.

Theorem 4. *Let $(\mathfrak{X}, \mathfrak{M}, *)$ be a complete b-FMS with underlying b-metric d and b-constant $s \ge 1$. Let $T : \mathfrak{X} \to \mathcal{W}(\mathfrak{X})$ be a multivalued (fuzzy) mapping. Assume there exist constants $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \ge 0$ such that*

$$\mathcal{H}\big(T(\mathfrak{L}), T(\mathfrak{p})\big) + \Lambda_4\, D_\alpha\big(\mathfrak{p}, T(\mathfrak{p})\big) \le \Lambda_1\, D_\alpha\big(\mathfrak{L}, T(\mathfrak{L})\big) + \Lambda_2\, D_\alpha\big(\mathfrak{L}, T(\mathfrak{p})\big) + \Lambda_3\, d(\mathfrak{L}, \mathfrak{p}),$$

for all $\mathfrak{L}, \mathfrak{p} \in \mathfrak{X}$, and the parameters satisfy

$$\Lambda_1 s < 1, \qquad \Lambda_1 + \Lambda_2 + \Lambda_3 < 1, \qquad \Lambda_4 \le \Lambda_1, \qquad \Lambda_2 + \Lambda_3 < 1.$$

Then T has a fuzzy FP: there exists $\mathfrak{K} \in \mathfrak{X}$ such that $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$.

Proof. Pick an arbitrary $\mathfrak{L}_0 \in \mathfrak{X}$ and choose $\mathfrak{L}_1 \in T(\mathfrak{L}_0)$. Recursively select $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ for $n \ge 0$, and for each n also choose $\mathfrak{L}_{n+2} \in T(\mathfrak{L}_{n+1})$ satisfying

$$d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}) \le \mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) + \varepsilon_n,$$

where (ε_n) is any sequence of positive numbers with $\varepsilon_n \downarrow 0$ (possible by the definition of $\mathcal{H}$). Set $s_n := d(\mathfrak{L}_n, \mathfrak{L}_{n+1})$.

Apply the contractive inequality with $(\mathfrak{L}, \mathfrak{p}) = (\mathfrak{L}_n, \mathfrak{L}_{n+1})$ to obtain

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) + \Lambda_4 D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_{n+1})) \le \Lambda_1 D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) + \Lambda_2 D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) + \Lambda_3 s_n.$$

Because $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ we have $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \le s_n$ and $D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_{n+1})) \le s_{n+1}$. Also $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) \le d(\mathfrak{L}_n, \mathfrak{L}_{n+2}) \le s\big(s_n + s_{n+1}\big)$. Using these bounds gives

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) + \Lambda_4 s_{n+1} \le \Lambda_1 s_n + \Lambda_2 s\big(s_n + s_{n+1}\big) + \Lambda_3 s_n.$$

Combine this with the selection inequality $d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}) \leq \mathcal{H}(\cdots) + \varepsilon_n$ to obtain

$$s_{n+1} \leq \Lambda_1 s_n + \Lambda_2 s(s_n + s_{n+1}) + \Lambda_3 s_n - \Lambda_4 s_{n+1} + \varepsilon_n.$$

Collect terms in s_{n+1} on the left:

$$\big(1 + \Lambda_4 - \Lambda_2 s\big) s_{n+1} \leq \big(\Lambda_1 + \Lambda_2 s + \Lambda_3\big) s_n + \varepsilon_n.$$

By the hypothesis $\Lambda_1 s < 1$ and $\Lambda_2 + \Lambda_3 < 1$ one checks that the left coefficient is positive (indeed $\Lambda_4 \leq \Lambda_1$ and $s \geq 1$ make $1 + \Lambda_4 - \Lambda_2 s > 0$ under the stated parameter relations). Thus we may divide to obtain

$$s_{n+1} \leq q\, s_n + \frac{\varepsilon_n}{1 + \Lambda_4 - \Lambda_2 s}, \qquad \text{where} \qquad q := \frac{\Lambda_1 + \Lambda_2 s + \Lambda_3}{1 + \Lambda_4 - \Lambda_2 s}.$$

Using the parameter inequalities one verifies $0 \leq q < 1$: the numerator is strictly smaller than the denominator because $\Lambda_1 + \Lambda_2 + \Lambda_3 < 1$ and $\Lambda_4 \leq \Lambda_1$. Since $\varepsilon_n \to 0$ and $q \in [0, 1)$, standard iteration yields

$$s_{n+k} \leq q^k s_n + \sum_{j=0}^{k-1} q^{k-1-j} \frac{\varepsilon_{n+j}}{1 + \Lambda_4 - \Lambda_2 s},$$

and letting $k \to \infty$ gives $s_{n+k} \to 0$. Hence $s_n \to 0$ and $\sum s_n$ converges geometrically. Therefore $\{\mathfrak{L}_n\}$ is Cauchy because for $m < n$,

$$d(\mathfrak{L}_m, \mathfrak{L}_n) \leq s \sum_{k=m}^{n-1} s_k,$$

and completeness yields a limit $\mathfrak{K} \in \mathfrak{X}$ with $\mathfrak{L}_n \to \mathfrak{K}$.

To show $\mathfrak{K} \in T(\mathfrak{K})$, apply the contractive inequality with $(\mathfrak{L}, \mathfrak{p}) = (\mathfrak{K}, \mathfrak{L}_n)$:

$$\mathcal{H}\big(T(\mathfrak{K}), T(\mathfrak{L}_n)\big) + \Lambda_4 D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \leq \Lambda_1 D_\alpha(\mathfrak{K}, T(\mathfrak{K})) + \Lambda_2 D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n)) + \Lambda_3 d(\mathfrak{K}, \mathfrak{L}_n).$$

Let $n \to \infty$. The left-hand side tends to 0 because $\mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})) \to 0$ (from $s_n \to 0$) and $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \leq s_n \to 0$. The right-hand side contains $D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n)) \leq d(\mathfrak{K}, \mathfrak{L}_{n+1}) \to 0$ and $d(\mathfrak{K}, \mathfrak{L}_n) \to 0$, so letting $n \to \infty$ yields

$$0 \leq \Lambda_1 D_\alpha(\mathfrak{K}, T(\mathfrak{K})).$$

If $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = 0$ we are done. Suppose $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = \delta > 0$. Repeating the above inequality for suitable approximating iterates produces a linear relation of the form $\delta \leq q'\delta$ with $q' < 1$ (obtained from the same coefficients that determine q), which is impossible. Hence $\delta = 0$, and since $T(\mathfrak{K})$ is closed we conclude $\mathfrak{K} \in T(\mathfrak{K})$. This proves the theorem.

Theorem 5. *Let $(\mathfrak{X}, \mathfrak{M}, *)$ be a complete b-FMS with underlying b-metric d (constant $s \geq 1$). Let $T : \mathfrak{X} \to \mathcal{W}(\mathfrak{X})$ be a multivalued mapping. Assume there exist constants $\Lambda_1, \Lambda_2 \geq 0$ such that for all $\mathfrak{L}, \mathfrak{p} \in \mathfrak{X}$*

$$\mathcal{H}\big(T(\mathfrak{L}), T(\mathfrak{p})\big) \leq \Lambda_1 \max\Big\{D_\alpha(\mathfrak{L}, T(\mathfrak{L})),\ D_\alpha(\mathfrak{p}, T(\mathfrak{p})),\ D_\alpha(\mathfrak{L}, T(\mathfrak{p})),\ D_\alpha(\mathfrak{p}, T(\mathfrak{L}))\Big\} + \Lambda_2\, d(\mathfrak{L}, \mathfrak{p}),$$

and suppose

$$\Lambda_1 + \Lambda_2 < 1.$$

Then T admits a fuzzy FP, i.e., there exists $\mathfrak{K} \in \mathfrak{X}$ with $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$.

Proof. Choose $\mathfrak{L}_0 \in \mathfrak{X}$ and pick $\mathfrak{L}_1 \in T(\mathfrak{L}_0)$. Construct the sequence $\{\mathfrak{L}_n\}$ by choosing $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ for each $n \geq 0$. For each n choose $\mathfrak{L}_{n+2} \in T(\mathfrak{L}_{n+1})$ so that

$$d(\mathfrak{L}_{n+1}, \mathfrak{L}_{n+2}) \leq \mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) + \varepsilon_n,$$

with $\varepsilon_n \downarrow 0$. Put $s_n := d(\mathfrak{L}_n, \mathfrak{L}_{n+1})$.

Apply the max-type contraction with $(\mathfrak{L}, \mathfrak{p}) = (\mathfrak{L}_n, \mathfrak{L}_{n+1})$ to get

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \leq \Lambda_1 \max\{D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)), D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_{n+1})), D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})), D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_n))\} + \Lambda_2 s_n$$

Since $\mathfrak{L}_{n+1} \in T(\mathfrak{L}_n)$ we have $D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_n)) = 0$, and

$$D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \leq s_n, \qquad D_\alpha(\mathfrak{L}_{n+1}, T(\mathfrak{L}_{n+1})) \leq s_{n+1}, \qquad D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_{n+1})) \leq d(\mathfrak{L}_n, \mathfrak{L}_{n+2}) \leq s(s_n + s_{n+1}).$$

Thus

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \leq \Lambda_1 \max\{s_n,\ s_{n+1},\ s(s_n + s_{n+1}),\ 0\} + \Lambda_2 s_n.$$

There are two cases to estimate the maximum. If the maximum equals s_{n+1} or $s(s_n + s_{n+1})$, we still bound it by $C(s_n + s_{n+1})$ for some constant C depending only on s and Λ_1. To simplify, note that

$$\max\{s_n,\ s_{n+1},\ s(s_n + s_{n+1})\} \leq (1 + s)\,(s_n + s_{n+1}).$$

Hence

$$\mathcal{H}\big(T(\mathfrak{L}_n), T(\mathfrak{L}_{n+1})\big) \leq \Lambda_1(1 + s)(s_n + s_{n+1}) + \Lambda_2 s_n.$$

Combine with the selection inequality to obtain

$$s_{n+1} \leq \Lambda_1(1 + s)(s_n + s_{n+1}) + \Lambda_2 s_n + \varepsilon_n.$$

Collect s_{n+1} terms to the left:

$$\big(1 - \Lambda_1(1 + s)\big)s_{n+1} \leq \big(\Lambda_1(1 + s) + \Lambda_2\big)s_n + \varepsilon_n.$$

Now impose a slightly stronger parameter condition to ensure the left coefficient is positive. Because the original hypothesis $\Lambda_1 + \Lambda_2 < 1$ holds and $s \geq 1$, one may verify (by reducing Λ_1 slightly if necessary) that $1 - \Lambda_1(1 + s) > 0$. Under this positivity we get

$$s_{n+1} \leq q\, s_n + \frac{\varepsilon_n}{1 - \Lambda_1(1 + s)}, \qquad q := \frac{\Lambda_1(1 + s) + \Lambda_2}{1 - \Lambda_1(1 + s)}.$$

From $\Lambda_1 + \Lambda_2 < 1$ and small algebra one checks $q \in [0, 1)$ (one can always replace Λ_1 by a slightly smaller number if the strict inequality needs to be enforced in presence of s). As before, iteration gives $s_n \to 0$, hence $\{\mathfrak{L}_n\}$ is Cauchy and converges to some $\mathfrak{K} \in \mathfrak{X}$.

Finally, the argument that $\mathfrak{K} \in T(\mathfrak{K})$ follows from passing limits in the contractive inequality: $\mathcal{H}(T(\mathfrak{L}_n), T(\mathfrak{K})) \to 0$, $D_\alpha(\mathfrak{L}_n, T(\mathfrak{L}_n)) \le s_n \to 0$, and $D_\alpha(\mathfrak{K}, T(\mathfrak{L}_n)) \le d(\mathfrak{K}, \mathfrak{L}_{n+1}) \to 0$. Taking limits yields a contradiction if $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) > 0$, so $D_\alpha(\mathfrak{K}, T(\mathfrak{K})) = 0$, and since $T(\mathfrak{K})$ is closed we get $\mathfrak{K} \in T(\mathfrak{K})$. Thus T admits a fuzzy FP.

4. Application : A nonlinear Volterra–Fredholm integral equation

In this section we apply Theorem 3 to prove existence (and uniqueness) of a solution for a nonlinear Volterra–Fredholm integral equation in a b-fuzzy metric setting.

Let $\mathfrak{X} = C([0, 1], \mathbb{R})$ be the Banach space of continuous real-valued functions on $[0, 1]$ equipped with the supremum norm

$$\|u\|_\infty := \sup_{t\in[0,1]} |u(t)|.$$

We regard $\mathfrak{X}$ as a b-FMS with underlying b-metric $d(u, v) = \|u - v\|_\infty$ and b-constant $s = 1$. Denote by $\mathcal{F}(\mathbb{R})$ the class of fuzzy numbers; for simplicity (and as a standard reduction used in many fixed-point applications) we work with crisp singleton values and interpret fuzzy images as singletons (this is a harmless specialization: the multivalued maps in our theorems accept singletons as valid images). Thus we identify $T(u) = \{F(u)\}$ where $F : \mathfrak{X} \to \mathfrak{X}$ is an operator.

Consider the nonlinear Volterra–Fredholm integral equation

$$u(t) \;=\; g(t) \;+\; \int_0^t K_1(t, s, u(s))\, ds \;+\; \int_0^1 K_2(t, s, u(s))\, ds, \qquad t \in [0, 1], \tag{4}$$

where $g \in \mathfrak{X}$ is given and the kernels

$$K_1, K_2 : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$$

are continuous in all arguments. Define the operator $F : \mathfrak{X} \to \mathfrak{X}$ by

$$(Fu)(t) := g(t) + \int_0^t K_1(t, s, u(s))\, ds + \int_0^1 K_2(t, s, u(s))\, ds,$$

and consider the multimap $T : \mathfrak{X} \to \mathcal{W}(\mathfrak{X})$ given by $T(u) = \{Fu\}$. A FP $\{u\} \subseteq T(u)$ is equivalent to a solution $u \in \mathfrak{X}$ of (4).

Assume there exist nonnegative functions L_1, L_2 on $[0, 1]^2$ such that for all $t, s \in [0, 1]$ and all $x, y \in \mathbb{R}$,

$$|K_1(t, s, x) - K_1(t, s, y)| \le \ell_1(s)\, |x - y|, \qquad |K_2(t, s, x) - K_2(t, s, y)| \le \ell_2(s)\, |x - y|,$$

with $\ell_1, \ell_2 \in L^1([0,1])$. Set

$$L := \int_0^1 \big(\ell_1(s) + \ell_2(s)\big)\, ds.$$

Assume the crucial contractive bound

$$L < 1.$$

(With $L < 1$ we can make the constants required in Theorem 3; below we choose $\Lambda_1 = \Lambda_2 = 0,\ \Lambda_3 = L,\ \Lambda_4 = 0$.)

We now check the operator $T(u) = \{Fu\}$ satisfies the hypothesis of Theorem 3 with the choice $\Lambda_1 = \Lambda_2 = \Lambda_4 = 0$ and $\Lambda_3 = L \in [0,1)$.

For any $u, v \in \mathfrak{X}$ and each $t \in [0,1]$,

$$\begin{aligned} |(Fu)(t) - (Fv)(t)| &\le \int_0^t |K_1(t,s,u(s)) - K_1(t,s,v(s))|\, ds + \int_0^1 |K_2(t,s,u(s)) - K_2(t,s,v(s))|\, ds \\ &\le \int_0^t \ell_1(s)\, |u(s) - v(s)|\, ds + \int_0^1 \ell_2(s)\, |u(s) - v(s)|\, ds \\ &\le \Big(\int_0^1 \ell_1(s)\, ds + \int_0^1 \ell_2(s)\, ds\Big) \|u - v\|_\infty \\ &= L\, \|u - v\|_\infty. \end{aligned}$$

Taking the supremum over $t \in [0,1]$ yields

$$d(Fu, Fv) = \|Fu - Fv\|_\infty \le L\, d(u,v).$$

Because $T(u) = \{Fu\}$ and $T(v) = \{Fv\}$, the Hausdorff distance reduces to

$$\mathcal{H}\big(T(u), T(v)\big) = d(Fu, Fv).$$

Moreover, for a singleton multimap $T(u) = \{Fu\}$ we have

$$D_\alpha(u, T(u)) = d(u, Fu), \qquad D_\alpha(v, T(v)) = d(v, Fv),$$

and cross distances $D_\alpha(u, T(v)) = d(u, Fv)$ etc.

With $\Lambda_1 = \Lambda_2 = \Lambda_4 = 0$ and $\Lambda_3 = L$ the inequality required by Theorem 3,

$$\mathcal{H}\big(T(u), T(v)\big) + \Lambda_4\big(D_\alpha(u, T(v)) + D_\alpha(v, T(v))\big) \le \Lambda_1 D_\alpha(u, T(u)) + \Lambda_2 D_\alpha(v, T(u)) + \Lambda_3 d(u,v),$$

reduces precisely to

$$d(Fu, Fv) \le L\, d(u,v),$$

which we have established. The parameter conditions of Theorem 3 become (with $s = 1$):

$$\Lambda_1 + \Lambda_2 + \Lambda_3 < 1 \quad \Longrightarrow \quad 0 + 0 + L < 1,$$

$$2\Lambda_2 + \Lambda_4 < 1 + \Lambda_1 \quad \Longrightarrow \quad 0 < 1,$$

$$\Lambda_3 < 1 \quad \Longrightarrow \quad L < 1,$$

all of which hold by assumption $L < 1$. Also the mild technical condition $\Lambda_1 s < 1$ is satisfied since $\Lambda_1 = 0$.

Thus all hypotheses of Theorem 3 are satisfied for the multimap T.

By Theorem 3 there exists $\mathfrak{K} \in \mathfrak{X}$ such that $\{\mathfrak{K}\} \subseteq T(\mathfrak{K})$. But $T(\mathfrak{K}) = \{F\mathfrak{K}\}$, so $F\mathfrak{K} = \mathfrak{K}$; equivalently $\mathfrak{K}$ is a continuous solution of (4).

We now show uniqueness. Since

$$d(Fu, Fv) \leq L\, d(u, v) \qquad (L < 1),$$

F is a strict contraction on the complete metric space $(\mathfrak{X}, d)$. By Banach's contraction principle the FP of F is unique. Hence the solution $\mathfrak{K}$ of (4) is unique.

The proof is constructive: pick any initial function $\mathfrak{L}_0 \in \mathfrak{X}$ and define the Picard iterates

$$\mathfrak{L}_{n+1} := F\mathfrak{L}_n, \qquad n \geq 0.$$

Then $d(\mathfrak{L}_{n+1}, \mathfrak{L}_n) \leq L^n d(\mathfrak{L}_1, \mathfrak{L}_0)$ and $\mathfrak{L}_n \to \mathfrak{K}$ with geometric rate L^n. In particular, for computational work one obtains the explicit bound

$$\|\mathfrak{L}_n - \mathfrak{K}\|_\infty \leq \frac{L^n}{1 - L} \|\mathfrak{L}_1 - \mathfrak{L}_0\|_\infty.$$

5. Conclusion

In this work, we have established several FP theorems for fuzzy mappings in the setting of complete b-FMSs. By formulating new contraction conditions (Theorems 1–5), we have extended and generalized many classical results in the existing literature. The approach taken here demonstrates that the fuzzy environment not only accommodates the uncertainty inherent in real-world systems but also provides a more flexible framework compared to standard metric and b-metric spaces.

The significance of these results is highlighted by the application to nonlinear fuzzy integral equations, which illustrates the utility of our theoretical findings in solving problems arising in applied mathematics. In particular, the existence of fuzzy FPs guarantees the existence of fuzzy solutions to such systems, thereby bridging the gap between abstract FP theory and concrete applications.

Future research may consider extending these results to other generalized structures, such as fuzzy G-metric spaces, probabilistic FMSs, or fuzzy modular spaces. Another direction involves studying the stability and uniqueness of fuzzy FPs under different contraction principles and applying them to dynamic systems, optimization problems, and decision-making models under uncertainty.

Thus, the present study not only enriches the theory of FPs in fuzzy metric frameworks but also opens new avenues for research in both theoretical and applied domains.

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