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On a Subclass of Starlike Functions Related to Pascal and Poisson Distributions

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Abstract. This paper aims to derive coefficient conditions, inclusion relations, and the starlikeness condition for a certain subclass of analytic functions in the open unit disc. Additionally, it establishes the necessary and sufficient conditions for the Pascal and Poisson distributions to belong to this subclass.

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1. Introduction

Assume that $\mathcal{D}$ denotes the family of all analytic functions $\mathcal{F}$ in the open unit disc

$$E = \{\zeta \in \mathbb{C} : |\zeta| < 1\},$$

having the Taylor series expansion

$$\mathcal{F}(\zeta) = \zeta + \sum_{m=2}^{\infty} a_m \zeta^m \quad (a_m \geq 0, m = 2, 3, ...). \tag{1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{D}$ consisting of univalent functions in E. Furthermore, the subclasses $S^*(\gamma)$ and $K(\gamma)$, introduced by Robertson [1], are defined as follows:

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$$S^{*}(\gamma)=\left\{\mathcal{F}\in\mathcal{S}:\ \Re\left(\frac{\zeta\mathcal{F}^{'}(\zeta)}{\mathcal{F}(\zeta)}\right)>\gamma,\ \zeta\in E\right\},0\leq\gamma<1, \tag{2}$$

and

$$K(\gamma)=\left\{\mathcal{F}\in\mathcal{S}:\ \Re\left(1+\frac{\zeta\mathcal{F}^{''}(\zeta)}{\mathcal{F}^{'}(\zeta)}\right)>\gamma,\ \zeta\in E\right\},0\leq\gamma<1. \tag{3}$$

Here, $S^{*}(\gamma)$ and $K(\gamma)$ are, respectively, the well-known subclasses of $\mathcal{S}$ whose members are starlike and convex of order γ. In particular, when $\gamma=0$, these reduce to the subclasses

$$S^{*}(0)=S^{*},\quad K(0)=K,$$

where $S^{*}$and K denote the classical classes of starlike and convex functions in E, respectively.

Definition 1. *Let $\mathcal{H}\left(\alpha\right)$ denote the class of functions $\mathcal{F}\in\mathcal{D}$ that satisfy the following condition*

$$\Re\left\{\alpha\left(1+\frac{\zeta\mathcal{F}^{''}(\zeta)}{\mathcal{F}^{'}(\zeta)}\right)+(1-\alpha)\frac{1}{\mathcal{F}^{'}(\zeta)}\right\}<\frac{2\alpha+1}{2}, \tag{4}$$

where $\alpha>\frac{1}{2}$.

By taking $\alpha=1$ in Definition 1, we obtain the class of analytic functions $\mathcal{H}$ given by

$$\mathcal{H}=\left\{\mathcal{F}\in\mathcal{D}:\Re\left(1+\frac{\zeta\mathcal{F}^{''}(\zeta)}{\mathcal{F}^{'}(\zeta)}\right)<\frac{3}{2}\right\}.$$

The class $\mathcal{H}\left(\alpha\right)$ was introduced by Singh and Singh [2]. They also proved the following results:

1- Every function $\mathcal{F}\in\mathcal{H}\left(\alpha\right)$ is a close-to-convex and bounded in E.

2- Every function $\mathcal{F}\in\mathcal{H}$, belongs to the class S^{*}.

In 1993, Silverman [3] provided characterizations of (Gaussian) hypergeometric functions associated with various subclasses of starlike and convex functions. Building on this approach, Kwon and Cho [4] established the necessary and sufficient conditions for hypergeometric functions to belong to two subclasses of uniformly starlike and uniformly convex functions with negative coefficients. Later, many researchers (see, for example, [5–27]) examined subclasses of $\mathcal{S}$ involving hypergeometric and Bessel functions, as well as the Poisson and Pascal distributions. Furthermore, in the context of quantum calculus, several scholars [28, 29] studied certain subclasses of bi-univalent functions using the q-Pascal and q-Poisson distribution series. These efforts have greatly advanced the development of research in geometric function theory. It is known that a random variable x has the Pascal distribution or negative binomial distribution if it takes the values $0,1,2,3,\ldots$ with probabilities

$$\left(1-p\right)^{n},\frac{pn\left(1-p\right)^{n}}{1!},\frac{p^{2}n\left(n+1\right)\left(1-p\right)^{n}}{2!},\frac{p^{3}n\left(n+1\right)\left(n+2\right)\left(1-p\right)^{n}}{3!},...,$$

respectively, where n denotes the number of successes, p represents the probability of failure, and $1-p$ represents the probability of success in each trial. Hence,

$$P(x=i)=\binom{i+n-1}{n-1}p^{i}(1-p)^{n},\ i=0,1,2,....$$

In recent years, El-Deeb et al. [10] proposed a power series whose coefficients are expressed in terms of the probabilities of the Pascal distribution, defined as follows:

$$\Theta_p^n(\zeta)=\zeta+\sum_{m=2}^{\infty}\binom{m+n-2}{n-1}p^{m-1}(1-p)^{n}\zeta^{m}\qquad(\zeta\in E).\tag{5}$$

Lashin et al. [30, 31] modified (5) to be

$$L_p^n(\zeta)=(1-p)^{n}\zeta+\sum_{m=2}^{\infty}\binom{m+n-2}{n-1}p^{m-1}(1-p)^{n}\zeta^{m}\qquad(\zeta\in E),\tag{6}$$

where $n\in\mathbb{Z}^{+}$ and $0\leq p\leq 1$. They further defined the following series:

$$\Delta_p^n(\zeta)=\frac{L_p^n(\zeta)}{(1-p)^{n}}=\zeta+\sum_{m=2}^{\infty}\binom{m+n-2}{n-1}p^{m-1}\zeta^{m}\qquad(\zeta\in E).\tag{7}$$

Using this operator, Lashin et al. [31] introduced new subclasses of analytic functions and established inclusion relations by applying the subordination technique. For $n\in\mathbb{Z}^{+}$ and $0\leq p\leq 1$, we note that

$$\sum_{m=0}^{\infty}\binom{m+n-1}{n-1}p^{m}=\frac{1}{(1-p)^{n}}.$$

El-Deeb et al. [10] obtained the following relations

$$\sum_{m=2}^{\infty}\binom{m+n-2}{n-1}p^{m-1}=\frac{1}{(1-p)^{n}}-1,\tag{8}$$

$$\sum_{m=2}^{\infty}(m-1)\binom{m+n-2}{n-1}p^{m-1}=\frac{pn}{(1-p)^{n+1}}\tag{9}$$

and

$$\sum_{m=3}^{\infty}(m-1)(m-2)\binom{m+n-2}{n-1}p^{m-1}=\frac{p^{2}n(n+1)}{(1-p)^{n+2}}.\tag{10}$$

On the other hand, a discrete random variable y is said to have the Poisson distribution with expectation k if it takes the values $0,1,2,3,...$ with probabilities

$$e^{-k},\frac{ke^{-k}}{1!},\frac{k^{2}e^{-k}}{2!},\frac{k^{3}e^{-k}}{3!},...,$$

respectively. Thus

$$P(y=j)=\frac{k^{j}e^{-k}}{j!},\quad j=0,1,2,...,\quad k>0.$$

In 2014, Porwal [23] introduced a power series with coefficients derived from the Poisson distribution:

$$N_{k}(\zeta)=\zeta+\sum_{m=2}^{\infty}\frac{k^{m-1}}{(m-1)!}e^{-k}\zeta^{m}\ ,\quad k>0,\ \zeta\in E, \tag{11}$$

and derived the necessary and sufficient conditions for this series to belong to certain subclasses of analytic and univalent functions.

In this paper, the coefficient inequality, inclusion relations, and the starlikeness condition for functions in the class $\mathcal{H}(\alpha)$ are derived. The necessary and sufficient conditions for the Pascal distribution series $\Delta_{p}^{l}(\zeta)$ and the Poisson distribution series $N_{k}(\zeta)$ to belong to this class are also determined. In addition, the necessary and sufficient conditions for certain integral operators associated with the Pascal and Poisson distributions to belong to this class are established.

2. Main results

Throughout this paper, we assume that $\frac{1}{2}<\alpha\leq 1$, $n\in\mathbb{Z}^{+}$, $0\leq p\leq 1$, $k>0$, and $\zeta\in E$. Theorem 1 below states the necessary and sufficient conditions for the function $\mathcal{F}\in\mathcal{D}$ to belong to $\mathcal{H}(\alpha)$.

Theorem 1. *Let $\alpha>\frac{1}{2}$, and let the function $\mathcal{F}$ be given by (1). Then $\mathcal{F}\in\mathcal{H}(\alpha)$ if and only if*

$$\sum_{m=2}^{\infty}m\left[2(m\alpha-1)-(2\alpha-1)\right]a_{m}<2\alpha-1. \tag{12}$$

Equality in (12) is attended for the function

$$f(z)=z+\frac{z^{2}}{2}. \tag{13}$$

Proof. Let inequality (12) hold. Using the same method as Nishiwaki and Owa [32], it suffices to prove that

$$\left|\frac{\alpha\left(1+\frac{\zeta\mathcal{F}''(\zeta)}{\mathcal{F}'(\zeta)}\right)+(1-\alpha)\frac{1}{\mathcal{F}'(\zeta)}-1}{\alpha\left(1+\frac{\zeta\mathcal{F}''(\zeta)}{\mathcal{F}'(\zeta)}\right)+(1-\alpha)\frac{1}{\mathcal{F}'(\zeta)}-[2(\alpha+\frac{1}{2})-1]}\right|<1.$$

We note that

$$\left|\frac{\alpha\left(1+\frac{\zeta\mathcal{F}''(\zeta)}{\mathcal{F}'(\zeta)}\right)+(1-\alpha)\frac{1}{\mathcal{F}'(\zeta)}-1}{\alpha\left(1+\frac{\zeta\mathcal{F}''(\zeta)}{\mathcal{F}'(\zeta)}\right)+(1-\alpha)\frac{1}{\mathcal{F}'(\zeta)}-[2(\alpha+\frac{1}{2})-1]}\right|$$

$$= \left| \frac{\sum_{m=2}^{\infty} m(m\alpha - 1)a_m \zeta^{m-1}}{(2\alpha - 1) - \alpha \sum_{m=2}^{\infty} m(m-2)a_m \zeta^{m-1}} \right|$$

$$\leq \frac{\sum_{m=2}^{\infty} m(m\alpha - 1)a_m \left|\zeta^{m-1}\right|}{(2\alpha - 1) - \sum_{m=2}^{\infty} m(m-2)\alpha a_m \left|\zeta^{m-1}\right|}$$

$$< \frac{\sum_{m=2}^{\infty} m(m\alpha - 1)a_m}{(2\alpha - 1) - \sum_{m=2}^{\infty} m(m-2)\alpha a_m}.$$

The last expression is less than 1 if

$$\sum_{m=2}^{\infty} m[(m\alpha - 1) + (m-2)\alpha]a_m < 2\alpha - 1,$$

which is equivalent to our condition:

$$\sum_{m=2}^{\infty} m\left[2(m\alpha - 1) - (2\alpha - 1)\right]a_m < 2\alpha - 1.$$

Conversely, let the function $\mathcal{F} \in \mathcal{D}$ be in the class $\mathcal{H}(\alpha)$. Then, (4) can be expressed as

$$\Re\left\{\alpha\left(1 + \frac{\zeta \mathcal{F}''(\zeta)}{\mathcal{F}'(\zeta)}\right) + (1-\alpha)\frac{1}{\mathcal{F}'(\zeta)} - 1\right\} < \frac{2\alpha - 1}{2}.$$

or equivalently

$$\Re\left(\frac{\sum_{m=2}^{\infty} m(m\alpha - 1)a_m \zeta^{m-1}}{1 + \sum_{m=2}^{\infty} m a_m \zeta^{m-1}}\right) < \frac{2\alpha - 1}{2}.$$

If we choose ζ on the real axis, then

$$\frac{\sum_{m=2}^{\infty} m(m\alpha - 1)a_m \zeta^{m-1}}{1 + \sum_{m=2}^{\infty} m a_m \zeta^{m-1}},$$

is real. Let $\zeta \to 1^-$ through real values, we obtain

$$\frac{\sum_{m=2}^{\infty} m(m\alpha - 1)a_m}{1 + \sum_{m=2}^{\infty} m a_m} < \frac{2\alpha - 1}{2}.$$

Which is equivalent to (12) and this completes the proof.

Putting $\alpha = 1$ in the above theorem we get the following corollary

Corollary 1. *Let the function $\mathcal{F}$ be defined by (1). Then $\mathcal{F} \in \mathcal{H}$ if and only if*

$$\sum_{m=2}^{\infty} m\,[2m-3]\,a_m < 1. \tag{14}$$

The bounds in (14) is sharp by taking the function

$$f(z) = z + \frac{z^2}{2}.$$

3. Starlikeness for functions in $\mathcal{H}(\alpha)$

In this section, we examine the starlikeness of the class $\mathcal{H}(\alpha)$.

Theorem 2. *Let $\frac{1}{2} < \alpha_1 < \alpha_2$. Then $\mathcal{H}(\alpha_1) \subset \mathcal{H}(\alpha_2)$.*

Proof. Since

$$\begin{aligned} &\frac{m\,[2(m\alpha_1-1)-(2\alpha_1-1)]}{2\alpha_1-1} - \frac{m\,[2(m\alpha_2-1)-(2\alpha_2-1)]}{2\alpha_2-1} \\ = \;&\frac{2m(m-2)(\alpha_2-\alpha_1)}{(2\alpha_2-1)(2\alpha_1-1)} \geq 0, \end{aligned}$$

therefore, by Theorem 1, we have

$$\sum_{m=2}^{\infty} \frac{m\,[2(m\alpha_2-1)-(2\alpha_2-1)]}{2\alpha_2-1} a_m < \sum_{m=2}^{\infty} \frac{m\,[2(m\alpha_1-1)-(2\alpha_1-1)]}{2\alpha_1-1} a_m < 1.$$

That is, if $\mathcal{F} \in \mathcal{H}(\alpha_1)$ then $\mathcal{F} \in \mathcal{H}(\alpha_2)$.

Corollary 2. $\mathcal{H}(\alpha) \subset \mathcal{H}$

Proof. By Theorem 2, the proof follows directly from the fact that $\alpha \leq 1$.

Remark 1. *Based on Corollary 2 and the starlikeness of the class $\mathcal{H}$ (see Singh and Singh [2]), we conclude that all functions in the class $\mathcal{H}(\alpha)$ are starlike in E.*

4. Applications of the Pascal and the Poisson distributions

Theorem 3 below provides a necessary and sufficient condition for Pascal distribution series $\Delta_p^n(\zeta)$ to be in the class $\mathcal{H}(\alpha)$.

Theorem 3. *The series* $\Delta_p^n(\zeta)$ *given by (7) is in the class* $\mathcal{H}(\alpha)$ *if and only if*

$$pn(pn+1) + (1-p)^2[(1-p)^n - 1] < (2\alpha-1)\left\{(1-p)^{n+2} - pn[p(n-1)+2]\right\}. \tag{15}$$

Proof. By Theorem 1, we need to show that

$$\sum_{m=2}^{\infty} m\left[2(m\alpha-1)-(2\alpha-1)\right]\binom{m+n-2}{n-1}p^{m-1} < 2\alpha-1.$$

Now, we can write

$$\begin{aligned}
&\sum_{m=2}^{\infty} m\left[2(m\alpha-1)-(2\alpha-1)\right]\binom{m+n-2}{n-1}p^{m-1}\\
=\ & 2\alpha\sum_{m=3}^{\infty}(m-1)(m-2)\binom{m+n-2}{n-1}p^{m-1}\\
&+(4\alpha-1)\sum_{m=2}^{\infty}(m-1)\binom{m+n-2}{n-1}p^{m-1} - \sum_{m=2}^{\infty}\binom{m+n-2}{n-1}p^{m-1}\\
=\ & \frac{2\alpha p^2 n(n+1)}{(1-p)^{n+2}} + \frac{(4\alpha-1)pn}{(1-p)^{n+1}} - \left(\frac{1}{(1-p)^n}-1\right)\\
=\ & \frac{2\alpha p^2 n(n+1) + (1-p)\left((4\alpha-1)pn + (1-p)[(1-p)^n-1]\right)}{(1-p)^{n+2}}.
\end{aligned}$$

The last expression is less than $2\alpha-1$ if and only if condition (15) is fulfilled, which concludes the proof of the theorem.

Putting $\alpha = 1$ in Theorem 3, we get Corollary 3 below.

Corollary 3. *The series* $\Delta_p^n(\zeta)$ *given by (7) is in the class* $\mathcal{H}$ *if and only if*

$$pn\left[p(2n-1)+3\right] < (1-p)^2.$$

Theorem 4 below provides a sufficient and necessary condition for $N_k(\zeta)$ to be in the class $\mathcal{H}(\alpha)$.

Theorem 4. *Let* $k>0$, *then* $N_k(\zeta)$ *given by (11) is in the class* $\mathcal{H}(\alpha)$ *if and only if*

$$k(k+1) - (1-e^{-k}) < (2\alpha-1)\left[1-k(k+2)\right]. \tag{16}$$

Proof. According to Theorem 1, we need to show

$$\sum_{m=2}^{\infty} m\left[2(m\alpha-1)-(2\alpha-1)\right]\frac{k^{m-1}}{(m-1)!}e^{-k} < 2\alpha-1.$$

Now, we can write

$$\begin{aligned}
&\sum_{m=2}^{\infty} m\left[2(m\alpha-1)-(2\alpha-1)\right]\frac{k^{m-1}}{(m-1)!}e^{-k}\\
=\;& e^{-k}\sum_{m=2}^{\infty}[2\alpha(m-1)(m-2)+(m-1)(4\alpha-1)-1]\frac{k^{m-1}}{(m-1)!}\\
=\;& e^{-k}\left(2\alpha k^2\sum_{m=3}^{\infty}\frac{k^{m-3}}{(m-3)!}+(4\alpha-1)k\sum_{m=2}^{\infty}\frac{k^{m-2}}{(m-2)!}-\sum_{m=2}^{\infty}\frac{k^{m-1}}{(m-1)!}\right)\\
=\;& e^{-k}[2\alpha k^2e^k+(4\alpha-1)ke^k-(e^k-1)]=2\alpha k^2+(4\alpha-1)k-(1-e^{-k})\\
=\;& (2\alpha-1)k(k+2)+k(k+1)-(1-e^{-k})
\end{aligned}$$

The last expression is less than $2\alpha-1$ if and only if condition (16) holds, which completes the proof.

Putting $\alpha=1$ in Theorem 4, we get Corollary 4 below.

Corollary 4. *Let $k>0$, then $N_k(\zeta)$ given by (11) is in the class $\mathcal{H}$ if and only if*

$$k(2k+3)+e^{-k}<2.$$

5. Integral operators

This section establishes the necessary and sufficient conditions for the integral operators defined by

$$\mathcal{G}_p^n(\zeta)=\int_0^\zeta\frac{\Delta_p^n(t)}{t}dt,\quad\text{and}\quad\mathcal{M}_k(\zeta)=\int_0^\zeta\frac{N_k(t)}{t}dt \tag{17}$$

to belong to the class $\mathcal{H}(\alpha)$. In Theorem 5 we provide a the necessary and sufficient condition for the integral operators $\mathcal{G}_p^n(\zeta)$ to be in the class $\mathcal{H}(\alpha)$.

Theorem 5. *Let the integral operator $\mathcal{G}_p^n(\zeta)$ given by (17). Then it belongs to the class $\mathcal{H}(\alpha)$ if and only if*

$$pn+(1-p)[(1-p)^n-1]<(2\alpha-1)\left[(1-p)^{n+1}-pn\right]. \tag{18}$$

Proof. From (17), have

$$\mathcal{G}_p^n(\zeta)=\zeta+\sum_{m=2}^{\infty}\binom{m+n-2}{n-1}p^{m-1}\frac{\zeta^m}{m}\qquad(\zeta\in E).$$

By Theorem 1, it suffices to show that

$$\sum_{m=2}^{\infty}\frac{1}{m}\left\{m\left[2(m\alpha-1)-(2\alpha-1)\right]\binom{m+n-2}{n-1}p^{m-1}\right\}<2\alpha-1.$$

Now, we can rewrite the sum as

$$\begin{aligned}
&\sum_{m=2}^{\infty}[2(m\alpha-1)-(2\alpha-1)]\binom{m+n-2}{n-1}p^{m-1}\\
=&\sum_{m=2}^{\infty}[2\alpha(m-1)-1]\binom{m+n-2}{n-1}p^{m-1}\\
=&\ 2\alpha\sum_{m=2}^{\infty}(m-1)\binom{m+n-2}{n-1}p^{m-1}-\sum_{m=2}^{\infty}\binom{m+n-2}{n-1}p^{m-1}\\
=&\ 2\alpha\frac{pn}{(1-p)^{n+1}}-\left(\frac{1}{(1-p)^{n}}-1\right)\\
=&\ \frac{(2\alpha-1)pn+(1-p)[(1-p)^{n}-1]+pn}{(1-p)^{n+1}}.
\end{aligned}$$

Finally, the last expression is less than $2\alpha-1$ if and only if condition (18) holds. This completes the proof.

Putting $\alpha=1$ in Theorem 5, we get Corollary 5 below.

Corollary 5. *Let the integral operator $\mathcal{G}_p^n(\zeta)$ given by (17). Then it belongs to the class $\mathcal{H}$ if and only if*

$$2pn<1-p.$$

Theorem 6 below provides the necessary and sufficient condition for the integral operator $\mathcal{M}_k(\zeta)$ to be in the class $\mathcal{H}(\alpha)$.

Theorem 6. *Let $k>0$,then $\mathcal{M}_k(\zeta)$ given by (17) belongs to the class $\mathcal{H}(\alpha)$ if and only if*

$$e^{-k}\leq 2\alpha(1-k). \tag{19}$$

Proof. From (17), we can write

$$\mathcal{M}_k(\zeta)=\zeta+\sum_{m=2}^{\infty}\frac{k^{m-1}}{m!}e^{-k}\zeta^{m}\qquad(\zeta\in E).$$

According to Theorem 1, we need to show

$$\sum_{m=2}^{\infty}m[2(m\alpha-1)-(2\alpha-1)]\frac{k^{m-1}}{m!}e^{-k}<2\alpha-1.$$

Now, we can write

$$\sum_{m=2}^{\infty}m[2(m\alpha-1)-(2\alpha-1)]\frac{k^{m-1}}{m!}e^{-k}$$

$$
\begin{aligned}
&= e^{-k}\sum_{m=2}^{\infty}[2\alpha(m-1)-1]\frac{k^{m-1}}{(m-1)!} \\
&= e^{-k}\left(\sum_{m=2}^{\infty}2\alpha k\frac{k^{m-2}}{(m-2)!}-\sum_{m=2}^{\infty}\frac{k^{m-1}}{(m-1)!}\right) \\
&= e^{-k}[2\alpha k e^{k}-(e^{k}-1)] = 2\alpha k-1+e^{-k}
\end{aligned}
$$

The last expression is less than $2\alpha - 1$ if and only if condition (19) holds. Thus, the proof is concluded.

Putting $\alpha = 1$ in Theorem 6, we get Corollary 6 below.

Corollary 6. *Let $k > 0$, then $\mathcal{M}_k(\zeta)$ given by (17) is in the class $\mathcal{H}$ if and only if*

$$e^{-k} \leq 2(1-k).$$

Conclusion 1. *In this paper, we investigate a subclass of analytic and close-to-convex functions introduced by Singh and Singh [2]. For this subclass, coefficient inequalities and inclusion relations are derived, and it is proved that all functions belonging to this class are starlike in the open unit disc. Furthermore, inspired by earlier studies connecting subclasses of analytic and univalent functions with hypergeometric and Bessel functions, as well as with the Poisson and Pascal distributions, we determine the necessary and sufficient conditions for the Pascal and Poisson distributions to belong to this subclass. In addition, the necessary and sufficient conditions for certain integral operators associated with these distributions to belong to the same class are established.*

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