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Algebraic Investigations on Anti-Fuzzy Soft Boolean Ring Theory

D. Ramesh[1], Gadde Sambasiva Rao[2], Aiyared Iampan[3,*], Shake Baji[4], P. Rajani[5], B. Satyanarayana[6]

[1] *Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Educational Foundation, Vaddeswaram, Andhra Pradesh-522302, India*
[2] *Department of Mathematics, Sree Dattha Group of Institutions, Sheriguda, Ibrahimpatnam, Ranga Reddy, Telangana-501510, India*
[3] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*
[4] *Department of Mathematics, Sir C.R. Reddy College of Engineering, Eluru, Andhra Pradesh-534007, India*
[5] *Department of Basic Sciences and Humanities, Seshadri Rao Gudlavalleru Engineering College, Seshadri Rao Knowledge Village, Gudlavalleru, Andhra Pradesh-521356, India*
[6] *Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, Andhra Pradesh-522510, India*

Abstract. In this paper, we introduce the concept of anti-fuzzy soft Boolean rings (AFSBRs), which serve as a complementary extension to fuzzy soft Boolean rings. While fuzzy soft structures have proven effective in modeling uncertainty through degrees of membership, they often overlook the critical role of non-membership or rejection—an essential aspect in contexts involving contradictions, conflict resolution, or decision-making under opposition. Motivated by this gap, the anti-fuzzy soft approach emphasizes the non-membership aspects of elements under uncertainty, offering a dual and more balanced perspective. We formally define the structure of AFSBRs, present basic operations, and explore their fundamental properties through illustrative examples and closure theorems. This study not only deepens the understanding of fuzzy algebraic systems but also provides a robust algebraic framework for modeling negative information in areas such as computational logic, artificial intelligence, and soft computing.

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Key Words and Phrases: Boolean ring, fuzzy soft set, anti-fuzzy soft Boolean ring, fuzzy soft sub Boolean ring, fuzzy ideal, fuzzy soft ideal

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*Corresponding author.
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Email addresses: `ram.fuzzy@gmail.com` (D. Ramesh), `gaddesambasivarao1@gmail.com` (G. S. Rao), `aiyared.ia@up.ac.th` (A. Iampan), `shakebaji6@gmail.com` (S. Baji), `rajanipapers@gmail.com` (P. Rajani), `drbsn63@yahoo.co.in` (B. Satyanarayana)

1. Introduction

Zadeh [1] created fuzzy set theory in 1965 as a mathematical approach for simulating uncertainty and ambiguity. Molodtsov [2] later introduced the concept of soft sets as a general framework for handling parameterised uncertainties that are difficult to handle with conventional techniques. These two ideas were combined to generate FSSs, which have been the focus of a lot of research due to their many applications in information systems, decision-making, and algebraic structures.

Soft set theory, introduced by Maji et al. [3, 4], established a flexible parameterized framework for modeling uncertainty, later expanded through fuzzy soft sets that integrate the vagueness of fuzzy sets with the structural adaptability of soft sets. Earlier algebraic generalizations, such as fuzzy rings, provided a foundation for embedding fuzziness into ring theory [5]. Subsequent works deepened these connections, with Ahmat and Kharal [6] formalizing fuzzy soft sets and Acar et al. [7] introducing soft rings as a bridge to algebraic applications. Despite these advances, most studies focus on membership-oriented information, leaving limited attention to the complementary notion of non-membership or opposition—an essential perspective for modeling contradictions. This gap motivates the study of anti-fuzzy soft Boolean rings as a natural extension of fuzzy soft algebraic systems.

Additional developments include the study of anti-fuzzy h-ideals in hemirings by Akram and Dar [8], the study of anti-fuzzy ideals of BCK-algebras by Hong and Jun [9], and other extensions of fuzzy ideals and anti-fuzzy ideals in ordered semigroups, Γ-semirings, and related algebraic structures. These works illustrate the growing importance of anti-fuzzy ideas, which provide an opposite perspective by representing non-membership or opposing information. Soft set theory has also been successfully applied to a variety of uncertainty models, such as intuitionistic fuzzy sets, neutrosophic sets, and bipolar fuzzy sets, leading to the development of intuitionistic fuzzy soft sets, neutrosophic soft sets, and bipolar fuzzy soft sets, respectively. These hybrid models have improved the theoretical and practical aspects of uncertainty modeling. However, the related notion of fuzziness in the soft set framework, anti-fuzzy soft sets, has not received much attention.

Rao et al. have made more contributions in this field by analysing soft Boolean near-rings [10], introducing fuzzy soft Boolean rings [11], and investigating the structure of soft intersection Boolean near-rings [12]. As an extension of fuzzy soft algebraic structures, they introduced $(\in, \in \vee q_k)$-fuzzy soft Boolean near-rings [13] and developed fuzzy soft Boolean near-rings [14] with their idealistic versions to improve the algebraic basis for soft computing. Additionally, Rao et al. [15] presented $(\in, \in \vee q_k)$-intuitionistic fuzzy soft Boolean near-rings, which combine generalised membership ideas with intuitionistic fuzzy logic. Further developments extended these ideas to intuitionistic fuzzy soft Boolean rings [16], which incorporate intuitionistic fuzzy sets into the soft Boolean framework, thereby enriching the treatment of dual membership and non-membership information. More recently, algebraic aspects of bipolar fuzzy soft Boolean rings [17] have been studied, capturing both positive and negative degrees of membership simultaneously and offering a richer perspective for uncertainty representation. Further developments on classical Boolean rings have also contributed to the field: Hamsa et al. [18] investigated central Boolean rings and introduced Boolean-type fuzzy ideals, enhancing the interplay between fuzzy algebra and classical Boolean logic. Chalapathi and Madhavi [19] proposed neutrosophic Boolean rings, integrating indeterminacy as a formal element into the Boolean structure. Similarly, Ameri et al. [20] formulated Boolean rings based on multirings, extending the classical ring framework to encompass multivalued logic and offering new interpretations for algebraic reasoning. Collectively, these contributions demonstrate a progressive effort to expand the algebraic foundations of fuzzy soft systems and to address increasingly complex forms of uncertainty in mathematical structures.

In this study, we introduce the concepts of AFSBRs and AFSIs within the framework of Boolean rings (BRs), and establish their fundamental algebraic properties. By extending the scope of fuzzy soft structures to incorporate non-membership information, this work contributes to strengthening the theoretical foundations of soft algebraic systems. Moreover, the results are intended to provide a clear and accessible framework that can support further exploration by students and young researchers in the field of algebraic approaches to uncertainty.

2. Preliminaries

To begin, we will present basic definitions.

Definition 1. *For a set $\aleph$, when two binary operations are available, namely addition $+$ and multiplication $\cdot$, if any of the following characteristics apply, it is considered to be a ring:*
(i) $\aleph$ *is a group under* $+$,
(ii) $\aleph$ *is a semigroup under* $\cdot$,
(iii) $(\mathfrak{g}+\mathfrak{s})\mathfrak{r} = \mathfrak{g}\mathfrak{r} + \mathfrak{s}\mathfrak{r}$ *and* $\mathfrak{g}(\mathfrak{s}+\mathfrak{r}) = \mathfrak{g}\mathfrak{s} + \mathfrak{g}\mathfrak{r}$, $\forall \mathfrak{g}, \mathfrak{s}, \mathfrak{r} \in \aleph$.

Definition 2. *If* $\mathfrak{x}^2 = \mathfrak{x}, \forall \mathfrak{x} \in \aleph$, *then a ring* $\aleph$ *is a Boolean ring (BR).*

Definition 3. *Let A symbolise a starting universe, E symbolise a set of parameters, and I symbolise the closed unit interval, or $I = [0,1]$. $P(A)$ denotes the power set of A. A function with a set value is $\gimel : E \to I^A$, where I^A indicates the total number of all the fuzzy sets on A.*

Definition 4. *A pair of FSSs, $(\gimel, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$, with $\mathfrak{U} \cap \mathfrak{H} \neq \emptyset$, are considered. If $\mathfrak{S} = \mathfrak{U} \cap \mathfrak{H}$ and $\Omega_{\mathfrak{x}} = \gimel_{\mathfrak{x}} \wedge \Xi_{\mathfrak{x}}$, $\forall \mathfrak{x} \in \mathfrak{U}$, the FSS $(\Omega, \mathfrak{S})$ is generated by the intersection of $(\gimel, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$. The formula $(\gimel, \mathfrak{U}) \cap (\Xi, \mathfrak{H}) = (\Omega, \mathfrak{S})$ can be represented.*

Definition 5. *A pair of FSSs, $(\gimel, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$. The union of $(\gimel, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$ forms the FSS $(\Omega, \mathfrak{S})$, where $\mathfrak{S} = \mathfrak{U} \cup \mathfrak{H}$ and* $\Omega_{\mathfrak{x}} = \begin{cases} \gimel_{\mathfrak{x}} & \text{if} \quad \mathfrak{x} \in \mathfrak{U} - \mathfrak{H} \\ \Xi_{\mathfrak{x}} & \text{if} \quad \mathfrak{x} \in \mathfrak{H} - \mathfrak{U} \\ \gimel_{\mathfrak{x}} \vee \Xi_{\mathfrak{x}} & \text{if} \quad \mathfrak{x} \in \mathfrak{U} \cap \mathfrak{H} \end{cases}$, $\forall \mathfrak{x} \in \mathfrak{S}$.
Next, we will write $(\gimel, \mathfrak{U}) \cup (\Xi, \mathfrak{H}) = (\Omega, \mathfrak{S})$.

Definition 6. *Consider* $(\gimel, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *to be two FSSs. Then,* $(\gimel, \mathfrak{U})$ *AND* $(\Xi, \mathfrak{H})$ *are symbolised by* $(\gimel, \mathfrak{U}) \wedge (\Xi, \mathfrak{H})$*, and it is suggested by* $(\Omega, \mathfrak{U} \times \mathfrak{H})$*, where* $\Omega_{(\mathfrak{x},\mathfrak{y})} = \gimel_{\mathfrak{x}} \wedge \Xi_{\mathfrak{x}}$ *for each* $(\mathfrak{x}, \mathfrak{y}) \in \mathfrak{U} \times \mathfrak{H}$*.*

Definition 7. *Consider* $(\gimel, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *to be two FSSs. Then,* $(\gimel, \mathfrak{U})$ *OR* $(\Xi, \mathfrak{H})$ *are symbolised by* $(\gimel, \mathfrak{U}) \vee (\Xi, \mathfrak{H})$*, and it is suggested by* $(\Omega, \mathfrak{U} \times \mathfrak{H})$*, where* $\Omega(\mathfrak{x}, \mathfrak{y}) = \gimel_{\mathfrak{x}} \vee \Xi_{\mathfrak{x}}$ *for each* $(\mathfrak{x}, \mathfrak{y}) \in \mathfrak{U} \times \mathfrak{H}$*.*

Definition 8. *Let* $(\gimel, \mathfrak{U})$ *be an FSS. A known support of the FSS* $(\gimel, \mathfrak{U})$ *is the set*

$$\mathrm{Supp}(\gimel, \mathfrak{U}) = \{\mathfrak{x} \in \mathfrak{U} : \gimel(\mathfrak{x}) = \gimel_{\mathfrak{x}} \neq \emptyset\}.$$

An FSS $(\gimel, \mathfrak{U})$ *is said to be non-null if its support is non-empty, i.e.,*

$$\mathrm{Supp}(\gimel, \mathfrak{U}) \neq \emptyset.$$

Definition 9. *Let* $(\gimel, \mathfrak{U})$ *be an FSS that is non-null. If* $\gimel(\mathfrak{a}) = \gimel_{\mathfrak{a}}$ *is an F-sub-BR of* $\aleph$ *for each* $\mathfrak{a} \in \mathfrak{U}$*, then* $(\gimel, \mathfrak{U})$ *is an FSBR of* $\aleph$*, i.e.,*
(i) $\gimel_{\mathfrak{a}}(\mathfrak{x} - \mathfrak{y}) \geq \gimel_{\mathfrak{a}}(\mathfrak{x}) \wedge \gimel_{\mathfrak{a}}(\mathfrak{y})$*,*
(ii) $\gimel_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \geq \gimel_{\mathfrak{a}}(\mathfrak{x}) \wedge \gimel_{\mathfrak{a}}(\mathfrak{y})$*,* $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$*.*

Definition 10. *An FSBR of* $\aleph$ *is assumed to be* $(\gimel, \mathfrak{U})$*. If the following criteria are met, an FSS* $(\Xi, \mathfrak{H})$ *will be referred to as a fuzzy soft ideal (FSI) of* $(\gimel, \mathfrak{U})$*, symbolised by* $(\Xi, \mathfrak{H}) \lhd (\gimel, \mathfrak{U})$*, i.e.,*
(i) $\mathfrak{H} \subseteq \mathfrak{U}$*,*
(ii) for each $\mathfrak{a} \in \mathrm{Supp}(\Xi, \mathfrak{H})$*, the fuzzy set* $\Xi_{\mathfrak{a}}$ *is a fuzzy ideal (FI) of the fuzzy Boolean ring* $\gimel_{\mathfrak{a}}$*, i.e.,*
(i) $\Xi_{\mathfrak{a}}(\mathfrak{x} - \mathfrak{y}) \geq \Xi_{\mathfrak{a}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{a}}(\mathfrak{y})$*,*
(ii) $\Xi_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \geq \Xi_{\mathfrak{a}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{a}}(\mathfrak{y})$*,*
(iii) $\Xi_{\mathfrak{a}}(\mathfrak{x}) \leq \gimel_{\mathfrak{a}}(\mathfrak{x})$*,* $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$*.*

3. Anti-Fuzzy Soft Boolean Rings

The concept of FSBRs was proposed by Rao et al. [11]. In this section, we define AFSBRs and discuss some of their fundamental properties. $\aleph$ denotes a BR from now on, and all FSSs are preferred over $\aleph$.

Definition 11. *An FSS* $(\gimel, \mathfrak{U})$ *over* $\aleph$ *is called an anti-fuzzy soft Boolean ring (AFSBR) of* $\aleph$ *if*
(i) $\gimel_{\mathfrak{a}}(\mathfrak{x} + \mathfrak{y}) \leq \gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})$*,*
(ii) $\gimel_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \leq \gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})$*,* $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$*.*

Example 1. *Let* $\aleph = \{0, \mathfrak{a}^*, \mathfrak{c}^*, \mathfrak{n}^*\}$ *be a non-empty set with two binary operations* $+$ *and* $\cdot$ *defined as follows:*

$+$	0	$\mathfrak{a}^*$	$\mathfrak{c}^*$	$\mathfrak{n}^*$
0	0	$\mathfrak{a}^*$	$\mathfrak{c}^*$	$\mathfrak{n}^*$
$\mathfrak{a}^*$	$\mathfrak{a}^*$	0	$\mathfrak{n}^*$	$\mathfrak{c}^*$
$\mathfrak{c}^*$	$\mathfrak{c}^*$	$\mathfrak{n}^*$	0	$\mathfrak{a}^*$
$\mathfrak{n}^*$	$\mathfrak{n}^*$	$\mathfrak{c}^*$	$\mathfrak{a}^*$	0

$\cdot$	0	$\mathfrak{a}^*$	$\mathfrak{c}^*$	$\mathfrak{n}^*$
0	0	0	0	0
$\mathfrak{a}^*$	0	$\mathfrak{a}^*$	$\mathfrak{n}^*$	$\mathfrak{c}^*$
$\mathfrak{c}^*$	0	$\mathfrak{n}^*$	$\mathfrak{c}^*$	$\mathfrak{a}^*$
$\mathfrak{n}^*$	0	$\mathfrak{c}^*$	$\mathfrak{a}^*$	$\mathfrak{n}^*$

Let $\mathfrak{U} = \{\iota_1^1, \iota_2^1, \iota_3^1\}$ *be the set of parameters and now define a FSS* $(\gimel, \mathfrak{U})$ *over* $\aleph$ *as follows:*

$$\begin{aligned}
\gimel(\iota_1^1) &= \{(0, 0.9), (\mathfrak{a}^*, 0.6), (\mathfrak{c}^*, 0.4), (\mathfrak{n}^*, 0.6)\}\\
\gimel(\iota_2^1) &= \{(0, 0.8), (\mathfrak{a}^*, 0.5), (\mathfrak{c}^*, 0.5), (\mathfrak{n}^*, 0.5)\}\\
\gimel(\iota_3^1) &= \{(0, 0.7), (\mathfrak{a}^*, 0.3), (\mathfrak{c}^*, 0.3), (\mathfrak{n}^*, 0.1)\}
\end{aligned}$$

Hence, $(\gimel, \mathfrak{U})$ *an AFSBR of* $\aleph$*.*

Theorem 1. *Let* $(\gimel, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *be two AFSBRs. If* $(\gimel, \mathfrak{U}) \wedge (\Xi, \mathfrak{H})$ *is non-null, then it's an AFSBR.*

Proof. Let us take $(\gimel, \mathfrak{U}) \wedge (\Xi, \mathfrak{H}) = (\Omega, \mathfrak{S})$ respectively, where $\mathfrak{S} = \mathfrak{U} \times \mathfrak{H}$ and $\Omega(\mathfrak{a}, \mathfrak{b}) = \gimel(\mathfrak{a}) \wedge \Xi(\mathfrak{b})$, $\forall(\mathfrak{a}, \mathfrak{b}) \in \mathfrak{S}$. Since $(\gimel, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$ are AFSBRs of $\aleph$, we have $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$,

$$\begin{aligned}
\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x} + \mathfrak{y}) &= \gimel_{\mathfrak{a}}(\mathfrak{x} + \mathfrak{y}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{x}) + \mathfrak{y})\\
&\leq (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})) \wedge (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y}))\\
&= (\gimel_{\mathfrak{a}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\gimel_{\mathfrak{a}}(\mathfrak{y}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{y}))\\
&= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}),
\end{aligned}$$

$$\begin{aligned}
\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}\mathfrak{y}) &= \gimel_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{x}\mathfrak{y})\\
&\leq (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})) \wedge (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y}))\\
&= (\gimel_{\mathfrak{a}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\gimel_{\mathfrak{a}}(\mathfrak{y}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{y}))\\
&= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}).
\end{aligned}$$

Hence, $(\gimel, \mathfrak{U}) \wedge (\Xi, \mathfrak{H})$ is an AFSBR of $\aleph$.

Theorem 2. *Let* $(\gimel, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *be two AFSBRs. If* $(\gimel, \mathfrak{U}) \vee (\Xi, \mathfrak{H})$ *is non-null, then it's an AFSBR.*

Proof. Let us take $(\gimel, \mathfrak{U}) \vee (\Xi, \mathfrak{H}) = (\Omega, \mathfrak{S})$ respectively, where $\mathfrak{S} = \mathfrak{U} \times \mathfrak{H}$ and $\Omega(\mathfrak{a}, \mathfrak{b}) = \gimel(\mathfrak{a}) \vee \Xi(\mathfrak{b})$, $\forall(\mathfrak{a}, \mathfrak{b}) \in \mathfrak{S}$. Since $(\gimel, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$ are AFSBRs of $\aleph$, we have $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$,

$$\begin{aligned}
\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x} + \mathfrak{y}) &= \gimel_{\mathfrak{a}}(\mathfrak{x} + \mathfrak{y}) \vee \Xi_{\mathfrak{b}}(\mathfrak{x} + \mathfrak{y})\\
&\leq (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})) \vee (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y}))\\
&= (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\gimel_{\mathfrak{a}}(\mathfrak{y}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y}))\\
&= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}),
\end{aligned}$$

$$\begin{aligned}\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}\mathfrak{y}) &= \gimel_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \vee \Xi_{\mathfrak{b}}(\mathfrak{x}\mathfrak{y})\\ &\leq (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})) \vee (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y}))\\ &= (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\gimel_{\mathfrak{a}}(\mathfrak{y}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y}))\\ &= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}).\end{aligned}$$

Hence, $(\gimel, \mathfrak{U}) \vee (\Xi, \mathfrak{H})$ is an AFSBR of $\aleph$.

Theorem 3. *Let* $(\gimel, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *be two AFSBRs. If* $(\gimel, \mathfrak{U}) \cap (\Xi, \mathfrak{H})$ *is non-null, then it's an AFSBR.*

Proof. Let us take $(\gimel, \mathfrak{U}) \cap (\Xi, \mathfrak{H}) = (\Omega, \mathfrak{S})$ respectively, where $\mathfrak{S} = \mathfrak{U} \cap \mathfrak{H}$ and $\Omega(\mathfrak{a}, \mathfrak{b}) = \gimel(\mathfrak{a}) \cap \Xi(\mathfrak{b})$, $\forall (\mathfrak{a}, \mathfrak{b}) \in \mathfrak{S}$. Since $(\gimel, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$ are AFSBRs of $\aleph$, we have $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$,

$$\begin{aligned}\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}+\mathfrak{y}) &= \gimel_{\mathfrak{a}}(\mathfrak{x}+\mathfrak{y}) \cap \Xi_{\mathfrak{b}}(\mathfrak{x}+\mathfrak{y})\\ &\leq (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})) \cap (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y}))\\ &= (\gimel_{\mathfrak{a}}(\mathfrak{x}) \cap \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\gimel_{\mathfrak{a}}(\mathfrak{y}) \cap \Xi_{\mathfrak{b}}(\mathfrak{y}))\\ &= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}),\end{aligned}$$

$$\begin{aligned}\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}\mathfrak{y}) &= \gimel_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \cap \Xi_{\mathfrak{b}}(\mathfrak{x}\mathfrak{y})\\ &\leq (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})) \cap (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y}))\\ &= (\gimel_{\mathfrak{a}}(\mathfrak{x}) \cap \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\gimel_{\mathfrak{a}}(\mathfrak{y}) \cap \Xi_{\mathfrak{b}}(\mathfrak{y}))\\ &= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}).\end{aligned}$$

Hence, $(\gimel, \mathfrak{U}) \cap (\Xi, \mathfrak{H})$ is an AFSBR of $\aleph$.

Theorem 4. *Let* $(\gimel, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *be two AFSBRs. If* $(\gimel, \mathfrak{U}) \cup (\Xi, \mathfrak{H})$ *is non-null, then it's an AFSBR.*

Proof. For any $e \in \mathfrak{U} \cup \mathfrak{H}$, and $\mathfrak{x}, \mathfrak{y} \in \aleph$, we consider the subsequent scenarios.
Case I: If $\mathfrak{e} \in \mathfrak{U} - \mathfrak{H}$, then

$$\begin{aligned}\Omega_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) &= \gimel_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y})\\ &\leq \gimel_{\mathfrak{e}}(\mathfrak{x}) \vee \gimel_{\mathfrak{e}}(\mathfrak{y})\\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}),\end{aligned}$$

$$\begin{aligned}\Omega_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) &= \gimel_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y})\\ &\leq \gimel_{\mathfrak{e}}(\mathfrak{x}) \vee \gimel_{\mathfrak{e}}(\mathfrak{y})\\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}).\end{aligned}$$

Case II: If $\mathfrak{e} \in \mathfrak{H} - \mathfrak{U}$, then

$$\Omega_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) = \Xi_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y})$$

$$\begin{aligned} &\leq \Xi_{\mathfrak{e}}(\mathfrak{x}) \vee \Xi_{\mathfrak{e}}(\mathfrak{y}) \\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}), \end{aligned}$$

$$\begin{aligned} \Omega_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) &= \Xi_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) \\ &\leq \Xi_{\mathfrak{e}}(\mathfrak{x}) \vee \Xi_{\mathfrak{e}}(\mathfrak{y}) \\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}). \end{aligned}$$

Case III: If $\mathfrak{e} \in \mathfrak{U} \cap \mathfrak{H}$, then

$$\begin{aligned} \Omega_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) &= \beth_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) \cup \Xi_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) \\ &\leq (\beth_{\mathfrak{e}}(\mathfrak{x}) \vee \beth_{\mathfrak{e}}(\mathfrak{y})) \cup (\Xi_{\mathfrak{e}}(\mathfrak{x}) \vee \Xi_{\mathfrak{e}}(\mathfrak{y})) \\ &= (\beth_{\mathfrak{e}}(\mathfrak{x}) \cup \Xi_{\mathfrak{e}}(\mathfrak{x})) \vee (\beth_{\mathfrak{e}}(\mathfrak{y}) \cup \Xi_{\mathfrak{e}}(\mathfrak{y})) \\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}), \end{aligned}$$

$$\begin{aligned} \Omega_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) &= \beth_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) \cup \Xi_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) \\ &\leq (\beth_{\mathfrak{e}}(\mathfrak{x}) \vee \beth_{\mathfrak{e}}(\mathfrak{y})) \cup (\Xi_{\mathfrak{e}}(\mathfrak{x}) \vee \Xi_{\mathfrak{e}}(\mathfrak{y})) \\ &= (\beth_{\mathfrak{e}}(\mathfrak{x}) \cup \Xi_{\mathfrak{e}}(\mathfrak{x})) \vee (\beth_{\mathfrak{e}}(\mathfrak{y}) \cup \Xi_{\mathfrak{e}}(\mathfrak{y})) \\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}). \end{aligned}$$

Hence, $(\beth, \mathfrak{U}) \cup (\Xi, \mathfrak{H})$ is an AFSBR of $\aleph$.

4. Anti-Fuzzy Soft Ideals over Boolean Rings

In this section, we define AFSIs and discuss some of their fundamental properties.

Definition 12. *An FSS $(\beth, \mathfrak{U})$ over $\aleph$ is called an anti-fuzzy soft ideal (AFSI) of $\aleph$ if*
(i) $\beth_{\mathfrak{a}}(\mathfrak{x}+\mathfrak{y}) \leq \beth_{\mathfrak{a}}(\mathfrak{x}) \vee \beth_{\mathfrak{a}}(\mathfrak{y})$,
(ii) $\beth_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \geq \beth_{\mathfrak{a}}(\mathfrak{x}) \wedge \beth_{\mathfrak{a}}(\mathfrak{y})$, $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$.

Example 2. *Let $\aleph = \{0, \mathfrak{a}^*, \mathfrak{c}^*, \mathfrak{n}^*\}$ be a non-empty set with two binary operations $+$ and $\cdot$ defined as follows:*

$+$	0	$\mathfrak{a}^*$	$\mathfrak{c}^*$	$\mathfrak{n}^*$
0	0	$\mathfrak{a}^*$	$\mathfrak{c}^*$	$\mathfrak{n}^*$
$\mathfrak{a}^*$	$\mathfrak{a}^*$	0	$\mathfrak{c}^*$	$\mathfrak{n}^*$
$\mathfrak{c}^*$	$\mathfrak{c}^*$	$\mathfrak{c}^*$	0	$\mathfrak{a}^*$
$\mathfrak{n}^*$	$\mathfrak{n}^*$	$\mathfrak{n}^*$	$\mathfrak{a}^*$	0

$\cdot$	0	$\mathfrak{a}^*$	$\mathfrak{c}^*$	$\mathfrak{n}^*$
0	0	0	0	0
$\mathfrak{a}^*$	0	$\mathfrak{a}^*$	0	$\mathfrak{a}^*$
$\mathfrak{c}^*$	0	0	$\mathfrak{c}^*$	$\mathfrak{c}^*$
$\mathfrak{n}^*$	0	$\mathfrak{a}^*$	$\mathfrak{c}^*$	$\mathfrak{n}^*$

Let $\mathfrak{U} = \{\iota_1^1, \iota_2^1, \iota_3^1\}$ be the set of parameters and now define a FSS $(\beth, \mathfrak{U})$ over $\aleph$ as follows:

$$\beth(\iota_1^1) = \{(0, 0.9), (\mathfrak{a}^*, 0.7), (\mathfrak{c}^*, 0.8), (\mathfrak{n}^*, 0.7)\}$$
$$\beth(\iota_2^1) = \{(0, 0.8), (\mathfrak{a}^*, 0.5), (\mathfrak{c}^*, 0.5), (\mathfrak{n}^*, 0.8)\}$$
$$\beth(\iota_3^1) = \{(0, 0.4), (\mathfrak{a}^*, 0.4), (\mathfrak{c}^*, 0.6), (\mathfrak{n}^*, 0.8)\}$$

Hence, $(\beth, \mathfrak{U})$ *is an AFSI of* $\aleph$.

Theorem 5. *Let* $(\beth, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *be two AFSIs. If* $(\beth, \mathfrak{U}) \wedge (\Xi, \mathfrak{H})$ *is non-null, then it's an AFSI.*

Proof. Let us take $(\beth, \mathfrak{U}) \wedge (\Xi, \mathfrak{H}) = (\Omega, \mathfrak{S})$ respectively, where $\mathfrak{S} = \mathfrak{U} \times \mathfrak{H}$ and $\Omega(\mathfrak{a}, \mathfrak{b}) = \beth(\mathfrak{a}) \wedge \Xi(\mathfrak{b})$, $\forall (\mathfrak{a}, \mathfrak{b}) \in \mathfrak{S}$. Since $(\beth, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$ are AFSIs of $\aleph$, we have $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$,

$$\begin{aligned}
\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x} + \mathfrak{y}) &= \beth_{\mathfrak{a}}(\mathfrak{x} + \mathfrak{y}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{x} + \mathfrak{y}) \\
&\leq (\beth_{\mathfrak{a}}(\mathfrak{x}) \vee \beth_{\mathfrak{a}}(\mathfrak{y})) \wedge (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y})) \\
&= (\beth_{\mathfrak{a}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\beth_{\mathfrak{a}}(\mathfrak{y}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{y})) \\
&= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}),
\end{aligned}$$

$$\begin{aligned}
\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}\mathfrak{y}) &= \beth_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{x}\mathfrak{y}) \\
&\geq (\beth_{\mathfrak{a}}(\mathfrak{x}) \wedge \beth_{\mathfrak{a}}(\mathfrak{y})) \wedge (\Xi_{\mathfrak{b}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{y})) \\
&= (\beth_{\mathfrak{a}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{x})) \wedge (\beth_{\mathfrak{a}}(\mathfrak{y}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{y})) \\
&= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}).
\end{aligned}$$

Hence, $(\beth, \mathfrak{U}) \wedge (\Xi, \mathfrak{H})$ is an AFSI of $\aleph$.

Theorem 6. *Let* $(\beth, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *be two AFSIs. If* $(\beth, \mathfrak{U}) \vee (\Xi, \mathfrak{H})$ *is non-null, then it's an AFSI.*

Proof. Let us take $(\beth, \mathfrak{U}) \vee (\Xi, \mathfrak{H}) = \Omega, \mathfrak{S})$ respectively, where $\mathfrak{S} = \mathfrak{U} \times \mathfrak{H}$ and $\Omega(\mathfrak{a}, \mathfrak{b}) = \beth(\mathfrak{a}) \vee \Xi(\mathfrak{b})$, $\forall (\mathfrak{a}, \mathfrak{b}) \in \mathfrak{S}$. Since $(\beth, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$ are AFSIs of $\aleph$, we have $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$,

$$\begin{aligned}
\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x} + \mathfrak{y}) &= \beth_{\mathfrak{a}}(\mathfrak{x} + \mathfrak{y}) \vee \Xi_{\mathfrak{b}}(\mathfrak{x} + \mathfrak{y}) \\
&\leq (\beth_{\mathfrak{a}}(\mathfrak{x}) \vee \beth_{\mathfrak{a}}(\mathfrak{y})) \vee (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y})) \\
&= (\beth_{\mathfrak{a}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\beth_{\mathfrak{a}}(\mathfrak{y}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y})) \\
&= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}),
\end{aligned}$$

$$\begin{aligned}
\Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}\mathfrak{y}) &= \beth_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \vee \Xi_{\mathfrak{b}}(\mathfrak{x}\mathfrak{y}) \\
&\geq (\beth_{\mathfrak{a}}(\mathfrak{x}) \wedge \beth_{\mathfrak{a}}(\mathfrak{y})) \vee (\Xi_{\mathfrak{b}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{y})) \\
&= (\beth_{\mathfrak{a}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{x})) \wedge (\beth_{\mathfrak{a}}(\mathfrak{y}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y})) \\
&= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \wedge \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}).
\end{aligned}$$

Hence, $(\beth, \mathfrak{U}) \vee (\Xi, \mathfrak{H})$ is an AFSI of $\aleph$.

Theorem 7. *Let* $(\gimel, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *be two AFSIs. If* $(\gimel, \mathfrak{U}) \cap (\Xi, \mathfrak{H})$ *is non-null, then it's an AFSI.*

Proof. Let us take $(\gimel, \mathfrak{U}) \cap (\Xi, \mathfrak{H}) = (\Omega, \mathfrak{S})$ respectively, where $\mathfrak{S} = \mathfrak{U} \cap \mathfrak{H}$ and $\Omega(\mathfrak{a}, \mathfrak{b}) = \gimel(\mathfrak{a}) \cap \Xi(\mathfrak{b})$, $\forall (\mathfrak{a}, \mathfrak{b}) \in \mathfrak{S}$. Since $(\gimel, \mathfrak{U})$ and $(\Xi, \mathfrak{H})$ are AFSIs of $\aleph$, we have $\forall \mathfrak{x}, \mathfrak{y} \in \aleph$,

$$\begin{aligned} \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}+\mathfrak{y}) &= \gimel_{\mathfrak{a}}(\mathfrak{x}+\mathfrak{y}) \cap \Xi_{\mathfrak{b}}(\mathfrak{x}+\mathfrak{y}) \\ &\leq (\gimel_{\mathfrak{a}}(\mathfrak{x}) \vee \gimel_{\mathfrak{a}}(\mathfrak{y})) \cap (\Xi_{\mathfrak{b}}(\mathfrak{x}) \vee \Xi_{\mathfrak{b}}(\mathfrak{y})) \\ &= (\gimel_{\mathfrak{a}}(\mathfrak{x}) \cap \Xi_{\mathfrak{b}}(\mathfrak{x})) \vee (\gimel_{\mathfrak{a}}(\mathfrak{y}) \cap \Xi_{\mathfrak{b}}(\mathfrak{y})) \\ &= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \vee \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}), \end{aligned}$$

$$\begin{aligned} \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}\mathfrak{y}) &= \gimel_{\mathfrak{a}}(\mathfrak{x}\mathfrak{y}) \cap \Xi_{\mathfrak{b}}(\mathfrak{x}\mathfrak{y}) \\ &\geq (\gimel_{\mathfrak{a}}(\mathfrak{x}) \wedge \gimel_{\mathfrak{a}}(\mathfrak{y})) \cap (\Xi_{\mathfrak{b}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{b}}(\mathfrak{y})) \\ &= (\gimel_{\mathfrak{a}}(\mathfrak{x}) \cap \Xi_{\mathfrak{b}}(\mathfrak{x})) \wedge (\gimel_{\mathfrak{a}}(\mathfrak{y}) \cap \Xi_{\mathfrak{b}}(\mathfrak{y})) \\ &= \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{x}) \wedge \Omega_{(\mathfrak{a},\mathfrak{b})}(\mathfrak{y}). \end{aligned}$$

Hence, $(\gimel, \mathfrak{U}) \cap (\Xi, \mathfrak{H})$ is an AFSI of $\aleph$.

Theorem 8. *Let* $(\gimel, \mathfrak{U})$ *and* $(\Xi, \mathfrak{H})$ *be two AFSIs. If* $(\gimel, \mathfrak{U}) \cup (\Xi, \mathfrak{H})$ *is non-null, then it's an AFSI.*

Proof. For any $\mathfrak{e} \in \mathfrak{U} \cup \mathfrak{H}$, and $\mathfrak{x}, \mathfrak{y} \in \aleph$, we consider the subsequent scenarios.
Case I: If $\mathfrak{e} \in \mathfrak{U} - \mathfrak{H}$, then

$$\begin{aligned} \Omega_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) &= \gimel_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) \\ &\leq \gimel_{\mathfrak{e}}(\mathfrak{x}) \vee \gimel_{\mathfrak{e}}(\mathfrak{y}) \\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}), \end{aligned}$$

$$\begin{aligned} \Omega_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) &= \gimel_{\mathfrak{e}}(xy) \\ &\geq \gimel_{\mathfrak{e}}(\mathfrak{x}) \wedge \gimel_{\mathfrak{e}}(\mathfrak{y}) \\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \wedge \Omega_{\mathfrak{e}}(\mathfrak{y}). \end{aligned}$$

Case II: If $\mathfrak{e} \in \mathfrak{H} - \mathfrak{U}$, then

$$\begin{aligned} \Omega_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) &= \Xi_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) \\ &\leq \Xi_{\mathfrak{e}}(\mathfrak{x}) \vee \Xi_{\mathfrak{e}}(\mathfrak{y}) \\ &= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}), \end{aligned}$$

$$\begin{aligned} \Omega_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) &= \Xi_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) \\ &\geq \Xi_{\mathfrak{e}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{e}}(\mathfrak{y}) \end{aligned}$$

$$= \Omega_{\mathfrak{e}}(\mathfrak{x}) \wedge \Omega_{\mathfrak{e}}(\mathfrak{y}).$$

Case III: If $\mathfrak{e} \in \mathfrak{U} \cap \mathfrak{H}$, then

$$\begin{aligned}
\Omega_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) &= \gimel_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) \cup \Xi_{\mathfrak{e}}(\mathfrak{x}+\mathfrak{y}) \\
&\leq (\gimel_{\mathfrak{e}}(\mathfrak{x}) \vee \gimel_{\mathfrak{e}}(\mathfrak{y})) \cup (\Xi_{\mathfrak{e}}(\mathfrak{x}) \vee \Xi_{\mathfrak{e}}(\mathfrak{y})) \\
&= (\gimel_{\mathfrak{e}}(\mathfrak{x}) \cup \Xi_{\mathfrak{e}}(\mathfrak{x})) \vee (\gimel_{\mathfrak{e}}(\mathfrak{y}) \cup \Xi_{\mathfrak{e}}(\mathfrak{y})) \\
&= \Omega_{\mathfrak{e}}(\mathfrak{x}) \vee \Omega_{\mathfrak{e}}(\mathfrak{y}),
\end{aligned}$$

$$\begin{aligned}
\Omega_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) &= \gimel_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) \cup \Xi_{\mathfrak{e}}(\mathfrak{x}\mathfrak{y}) \\
&\geq (\gimel_{\mathfrak{e}}(\mathfrak{x}) \wedge \gimel_{\mathfrak{e}}(\mathfrak{y})) \cup (\Xi_{\mathfrak{e}}(\mathfrak{x}) \wedge \Xi_{\mathfrak{e}}(\mathfrak{y})) \\
&= (\gimel_{\mathfrak{e}}(\mathfrak{x}) \cup \Xi_{\mathfrak{e}}(\mathfrak{x})) \wedge (\gimel_{\mathfrak{e}}(\mathfrak{y}) \cup \Xi_{\mathfrak{e}}(\mathfrak{y})) \\
&= \Omega_{\mathfrak{e}}(\mathfrak{x}) \wedge \Omega_{\mathfrak{e}}(\mathfrak{y}).
\end{aligned}$$

Hence, $(\gimel, \mathfrak{U}) \cup (\Xi, \mathfrak{H})$ is an AFSI of $\aleph$.

5. Conclusion

In this work, we introduced and rigorously investigated the theory of anti-fuzzy soft Boolean rings (AFSBRs) and anti-fuzzy soft ideals (AFSIs), defining their structure, algebraic operations, and core properties. Motivated by the need to formally capture non-membership and opposing information—often overlooked in classical fuzzy soft frameworks—this study provides a dual extension that enriches the algebraic modeling of uncertainty. Through precise definitions, illustrative examples, and closure theorems, we established the internal consistency and structural robustness of AFSBRs under standard set-theoretic operations. These findings affirm the potential of anti-fuzzy soft algebraic systems as foundational tools for representing rejection, contradiction, and negative knowledge in decision-making scenarios. Future work may expand on this foundation by exploring morphisms, deeper characterizations, and algorithmic implementations, as well as practical applications in non-classical logic, artificial intelligence, and soft computing environments.

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References

[1] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[2] D. Molodtsov. Soft set theory - first results. *Computers and Mathematics with Applications*, 37(4-5):19–31, 1999.

[3] P. K. Maji, R. Biswas, and A. R. Roy. Fuzzy soft sets. *Journal of Fuzzy Mathematics*, 9(3):589–602, 2001.

[4] P. K. Maji, R. Biswas, and A. R. Roy. Soft set theory. *Computers and Mathematics with Applications*, 45(4-5):555–562, 2003.

[5] V. N. Dixit, R. Kumar, and N. Ajmal. On fuzzy rings. *Fuzzy Sets and Systems*, 49(2):205–213, 1992.

[6] B. Ahmat and A. Kharal. On fuzzy soft sets. *Advances in Fuzzy Systems*, 2009:Article ID 586507, 6 pages, 2009.

[7] U. Acar, F. Koyuncu, and B. Tanay. Soft sets and soft rings. *Computers and Mathematics with Applications*, 59:3458–3463, 2010.

[8] M. Akram and K. H. Dar. On anti fuzzy left h-ideals in hemirings. *International Mathematical Forum*, 2(46):2295–2304, 2007.

[9] S. M. Hong and Y. B. Jun. Anti fuzzy ideals in BCK-algebras. *Kyungpook Mathematical Journal*, 38(1):145–150, 1998.

[10] G. S. Rao, P. Kolluru, and B. P. Munagala. A note on soft Boolean near-rings. *AIP Conference Proceedings*, 2707:020013, 2023.

[11] G. S. Rao, D. Ramesh, A. Iampan, and B. Satyanarayana. Fuzzy soft Boolean rings. *International Journal of Analysis and Applications*, 21:60, 2023.

[12] G. S. Rao, P. Kolluru, and N. Thandu. Soft intersection Boolean near-rings with its applications. *AIP Conference Proceedings*, 2707:020012, 2023.

[13] G. S. Rao, D. Ramesh, and B. Satyanarayana. $(\in, \in \vee q_k)$-Fuzzy soft Boolean near rings. *Asia Pacific Journal of Mathematics*, 10:50, 2023.

[14] G. S. Rao, D. Ramesh, A. Iampan, and B. Satyanarayana. Fuzzy soft Boolean near-rings and idealistic fuzzy soft Boolean near-rings. *ICIC Express Letters*, 18(7):677–684, 2024.

[15] G. S. Rao, D. Ramesh, A. Iampan, G. Vijaya Lakshmi, and B. Satyanarayana. $(\in, \in \vee q_k)$-Intuitionistic fuzzy soft Boolean near-rings. *International Journal of Analysis and Applications*, 23:91, 2025.

[16] G. S. Rao, D. Ramesh, A. Iampan, B. Satyanarayana, and P. Rajani. Intuitionistic fuzzy soft Boolean rings. *International Journal of Analysis and Applications*, 23:43, 2025.

[17] G. S. Rao, V. P. Kolanchinathan, K. Jhansi Rani, A. Iampan, K. Hemabala, D. Ramesh, and B. Satyanarayana. Algebraic aspects of bipolar fuzzy soft Boolean rings. *European Journal of Pure and Applied Mathematics*, 18(3):6459, 2025.

[18] N. Hamsa, K. B. Srinivas, and K. S. Prasad. On central Boolean rings and Boolean type fuzzy ideals. *Kuwait Journal of Science*, 46(4):23–32, 2019.

[19] T. Chalapathi and L. Madhavi. Neutrosophic Boolean rings. *Neutrosophic Sets and Systems*, 33:59–66, 2020.

[20] R. Ameri, M. Hamidi, and A. A. Tavakoli. Boolean rings based on multirings. *Journal of Sciences, Islamic Republic of Iran*, 32(2):159–168, 2021.