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An Extended TOPSIS Technique in Cubic Vague Soft Set Relations with Its Application in Renewable Energy Sources

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Abstract. In this work, we address multi-criteria decision-making (MCDM) problems under high uncertainty by developing an extended Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) within the framework of **cubic vague soft set relations** (CVSSR). The suggested model incorporates interval-based truth and falsity membership functions to more accurately represent dual uncertainty than current methods, which handle cubic or vague soft sets separately. In order to provide a rigorous mathematical basis for the extended TOPSIS method, the paper presents formal definitions and properties of cubic vague soft relations, equivalence relations, and functions. The model improves the accuracy of alternative ranking by redefining the calculation of positive and negative ideal solutions (PIS and NIS) under the cubic vague environment. The framework's resilience and interpretability in actual decision-making situations are demonstrated by a real-world application in the selection of renewable energy sources. Thus, the suggested method offers a strong decision-support tool for complex systems with imprecision and vagueness as well as a theoretical breakthrough in soft set mathematics.

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1. Introduction

In various fields of mathematics, artificial intelligence, and decision sciences, modeling uncertainty has proven to be a continuous challenge. Numerous extensions of fuzzy set theory have been implemented to deal with ambiguous, uncertain, and incomplete data since Zadeh's groundbreaking presentation of this idea [1]. Among these, vague sets [2] resolve philosophical ambiguity more elegantly than classical fuzzy sets by separating truth-membership and falsity-membership functions. By adding ranges and degrees of hesitation, interval-valued fuzzy sets [3] and Atanassov's intuitionistic fuzzy sets [4] enhanced this framework even more. Building on these frameworks, Jun et al. [5] presented cubic sets, which capture greater uncertainty data by combining interval-valued and single-valued fuzzy membership. while vague sets and cubic sets are combined, the result is cubic vague sets, which have more ability to express when there is hesitation or dual uncertainty since truth and falsity are represented as intervals.

Zadeh was the first to introduce the idea of fuzzy subsets of a set in [1]. By defining a fuzzy subset of a set X as a mapping from X into the unit interval [0,1], he expanded on the idea of a characteristic function. The study of vague soft relations and their composition was later started in [6]. A mapping from the Cartesian product $(F, A) \times (G, B)$ to a vague soft set, where $\Re$ is the relation's value set, was called a vague soft relation from (F, A) to (G, B). Since then, the literature has put forth a number of definitions for cubic vague soft functions.

Recent investigations have significantly enriched neutrosophic theory through novel functional extensions. Palanikumar et al. [7] introduced a reciprocal floor function based algebraic structure to extend the complex logarithmic neutrosophic set using averaging and geometric operators, enhancing uncertainty modeling accuracy . Also Palanikumar et al. [8] proposed a decision support system employing a Diophantine spherical fuzzy normal interval-valued set for selecting suitable artificial intelligence tools, thereby improving robustness in handling high dimensional and uncertain decision data. These studies collectively demonstrate the growing potential of hybrid fuzzyneutrosophic models for solving complex multi criteria decisionmaking problems across diverse application domains.

Along with these developments, Molodtsov [9] presented soft set theory as an adaptable parameterized tool for uncertainty. Maji et al. [10] subsequently expanded this theory to fuzzy soft sets. Muhiuddin et al. [11] presented cubic soft sets, which extend cubic theory to soft settings and allow parameter-based decision analysis. The richer mathematical frameworks have been made possible by Abdullah et al.'s [12] thorough study of operations on cubic soft sets, including P-union, R-union, P-intersection, and R-intersection. The adaptability of cubic-based hybrid structures is demonstrated by the further development of these concepts into cubic vague sets [13], cubic intuitionistic fuzzy soft sets

[14], and neutrosophic cubic sets [15]. Recently performed studies, including applications of Pythagorean cubic fuzzy sets [16, 17], furthermore, highlight that cubic structures are becoming more and more significant in multiattribute decision making.

In multi-criteria decision making (MCDM), where decision makers have to rank alternatives across several conflicting criteria, these advancements are especially significant. The Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), first put forth by Hwang and Yoon [18], has become well-known among the several MCDM techniques because of its ease of use and resilience. The key principle of TOPSIS is that the most effective choice should be the one that is most far from the negative ideal solution (NIS) and closest to the positive ideal solution (PIS). In order to improve the modeling of uncertainty, TOPSIS variants have been presented under fuzzy sets [19], vague sets [20], intuitionistic fuzzy sets [21], and cubic intuitionistic fuzzy environment [22]. Devi et al. [23], for instance, expanded TOPSIS to include uncertain settings, whereas ahin & Yiider [24] and Karaaslan [25] investigated neutrosophic soft set-based TOPSIS. likewise, to show the possibilities of cubic-based frameworks, Saqlain et al. [14] created a cubic intuitionistic fuzzy soft TOPSIS, and Khan et al. [17] suggested a Pythagorean cubic fuzzy TOPSIS model.

One of the most promising developments in renewable energy is photovoltaic (PV) technology [26]. Because of its great availability, solar energy is a very alluring source of electricity. According to the United Nations Development Program's 2000 World Energy Assessment, solar energy has a year-round potential of 1,57549,837 exajoules (EJ) [27]. Solar energy radiation is at least three times greater than the world's total energy consumption in 2012, which was 559.8 EJ. The cost of PV panels is decreasing annually, which when combined with the untapped potential of solar energy will make PV electricity more affordable and widely available than electricity from non-renewable sources [28]. Therefore, PV systems in the future can produce a huge amount of electrical energy from solar power and can be an interesting solution in places when wind farm cannot be located [29, 30].

Selecting the exact PV model that is required is a challenging task because of the large number of options and the diversity of selection criteria. In this scenario, the most suited approaches for resolving are fuzzy logic or multi-criteria solution analysis (MCDA), which based on alternatives data would rank them and as a result, present the optimal alternative [31]. It will be challenging to choose the best PV model manually or based just on a few variables because there are many of them, such as price, pick power, area, and efficiency [32]. The MCDA techniques, which have demonstrated efficacy in assessing the sustainability of transportation, are employed to address sustainability-related issues. Since it takes into consideration every criterion of every possibility, the MCDA technique is therefore required. For instance, the PROMETHEE method with stability assessment (PROSA) was used to evaluate offshore farm wind sites [33, 34], the Analytic Network Process (ANP) and Analytic Hierarchy Process (AHP) were used to design wind farms [35].

In recent years, researchers have increasingly focused on advancing fuzzy set generalizations and their integration with multi-attribute decision-making (MADM) techniques to handle complex uncertainty in practical domains. Numerous studies have enhanced classical fuzzy and intuitionistic frameworks by incorporating higher-dimensional and dual evidential structures. Hussain et al. [36] introduced the complex cubic q-Rung Orthopair Fuzzy Model for web security assessment, demonstrating how cubic and q-rung orthopair properties can jointly represent multidimensional uncertainty with superior accuracy. In the field of energy management, Petchimuthu et al. [37] proposed a Complex q-Rung picture fuzzy generalized power prioritized Yager operator to improve decision reliability in power and energy transformation problems, while Shahin et al. [38] developed an interval-valued circular intuitionistic fuzzy MARCOS method for renewable-energy source selection, confirming the growing importance of interval-valued and circular fuzzy models in sustainable decision analysis. Complementing these methodological advances, Sahoo et al. [39, 40] presented comprehensive reviews of multi-criteria decision-making (MCDM) applications in sustainable renewable-energy development, highlighting that recent progress primarily centers on fuzzy, intuitionistic, and q-rung frameworks combined with MCDM methods such as TOPSIS, VIKOR, and MARCOS.

Despite these advancements, a clear research gap persists. Current fuzzy-based MCDM models often capture either interval-valued uncertainty or dual membership evidence, but they rarely integrate these features with parameterized flexibility within a single unified framework. Most existing approaches lack the ability to simultaneously model interval uncertainty (cubic structure), truthfalsity duality (vague component), and parameter dependence (soft set environment). This limitation restricts their applicability in real-world problems such as renewable-energy evaluation where expert judgments are typically imprecise, hesitant, and context dependent. The present study addresses this gap by introducing a cubic vague soft set (CVSS) integrated with the TOPSIS technique, offering a unified decision-making framework capable of representing multi-level uncertainty and improving the robustness and interpretability of rankings in renewable-energy selection problems.

1.1. Problem statement

Real-world multi-criteria decision-making (MCDM) problems are typically defined by human judgments, available information, and inherent uncertainties, imprecision, and ambiguity. Although existing MCDM techniques like TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) offer a strong framework for ranking alternatives, and multiple extensions of fuzzy set theory, such as vague sets and cubic sets, have been developed to better capture these complexities, a significant challenge remains in efficiently integrating these sophisticated uncertainty-handling structures with exact decision-making methodologies.

In particular, existing methods frequently fail in the following way:

(i) Comprehensive modeling of complex uncertainty: More advanced modeling techniques are required because current cubic vague soft set relations might not adequately capture the multi-layered uncertainties in real-world facts and arbitrary human preferences.

(ii) Durable integration with TOPSIS: the direct application of TOPSIS to cubic vague soft sets poses methodological difficulties, since conventional procedures must be modified in order to process data efficiently. Data may not be properly utilized by current integration techniques, which could result in less-than-ideal or erroneous decision-making.

(iii) Computation efficiency and adaptability: developing effective methods for large-scale decision support systems is necessary since integrating cubic vague soft sets with TOPSIS in MCDM situations can be computationally taxing.

With an emphasis on providing a solid foundation for precise and useful decision making, this study seeks to reinforce multi-criteria decision-making under cubic vague soft set relations using an improved TOPSIS technique.

1.2. Motivation

Expert knowledge that is interval-valued, contradictory, and incomplete is a hallmark of decision-making in complex systems. The range and contradiction that frequently coexist in expert assessments cannot be represented by traditional fuzzy and intuitionistic frameworks because they approach uncertainty through single-valued membership degrees. The cubic vague soft set (CVSS) framework, which integrates three complementary modelling capabilities and used in this study to overcome this constraint.

The main strengths of this research are summarized as follows:

(i) Cubic representation : enables each criterion assessment to be expressed as an interval, accommodating lower and upper bounds of expert confidence.

(ii) Vague dual structure : distinguishes the truth and falsity degrees and preserves hesitation explicitly, thereby modelling contradictory evidence more faithfully.

(iii) Soft set parameterisation : allows flexible description of evaluations with respect to independent decision parameters without altering the universal set.

When these components are combined, CVSS is especially well-suited for multi-attribute decision analysis, which involves information that is both uncertain and reliant on parameters. For instance, experts may report maintenance reliability with varying degrees in [0.1, 0.25] and efficiency in the interval [0.75, 0.9] in the context of renewable energy; CVSS may simultaneously support both.

By establishing positive and negative ideal solutions on dual intervals of truth and falsity, CVSS expands the traditional distance-based ranking procedure when used in conjunction with the TOPSIS technique. In addition to producing rankings that are more reliable and consistent under data volatility, this dual-interval distance computation preserves more uncertainty information. Because CVSS offers a practically significant and mathematically rigorous framework that captures interval-valued vagueness and facilitates more reliable decision-making.

1.3. Contributions of the Study.

The key contributions of this research can be summarized as follows:

(i) We develop a new theoretical framework for cubic vague soft set relations (CVSSR), introducing rigorous definitions, properties, and proofs that generalize classical vague and cubic soft set structures.

(ii) We extend TOPSIS technique into the CVSSR environment, redefining the computation of positive and negative ideal solutions (PIS/NIS) and the similarity measures under dual-interval uncertainty.

(iii) We propose an enhanced multicriteria decision-making (MCDM) method capable of handling higher-order vagueness and hesitation degrees with improved accuracy.

(iv) We demonstrate the applicability of the proposed framework through a renewable energy source problem, validating the methods practical effectiveness.

This integration provides not a mere combination, but a comprehensive theoretical and applied advance in decision making under uncertainty.

1.4. Structure of the paper

Basic concepts of vague soft sets are reviewed in Section 2. In Section 3, Cartesian products and relations on cubic vague soft sets are studied. In this section, induced relations from the universal set and attribute set are introduced along with examples. Key findings are presented in Section 4, which examines partitions and equivalence relations on cubic vague soft sets. The composition of CVSSR is introduced in Section 5 and is backed up by theoretical proofs and examples. CVSSF are defined in Section 6, along with function composition and pertinent results. Application of Multicriteria Cubic Vague Soft Set in Decision Making Problem presented in Section 7. The conclusion wraps up the contributions and offers ideas for further study.

2. Preliminaries

In order to lay the groundwork for the following discussions, this section provides key definitions, characteristics, and proven results about vague soft set relations, functions, and cubic vague soft sets.

Alhazaymeh and Hassan [6] first proposed the idea of vague soft set relations and functions as an expansion of conventional soft set relations and functions. This development improves the modeling and analysis of issues with high levels of imprecision and uncertainty, especially in complex datasets. The following is an outline of the basic definitions and initial concepts.

Definition 1. *(See [6]) Let (F, A) and (G, B) denote two vague soft sets defined over a universal set U. Their Cartesian product, written as $(H, A \times B)$, is a vague soft set where* *the mapping* $H : A \times B \to V(U \times U)$ *assigns to each parameter pair* $(a, b) \in A \times B$, *i.e* $H(a, b) = F(a) \times G(b)$. *This set comprises all ordered pairs* (h_i, h_j) *such that* $h_i \in F(a)$ *and* $h_j \in G(b)$.

Definition 2. *(See [6]) Let* (F, A) *and* (G, B) *be vague soft sets over* U. *A vague soft relation between them is a vague soft subset of* $(F, A) \times (G, B)$, *denoted as* (H, S), *where* $S \subseteq A \times B$ *and* $\Re(a, b)$ *are defined for all* $(a, b) \in S$. *For a single vague soft set* (F, A), *such a relation is a subset of* $(F, A) \times (F, A)$. *Parametrically, it* $F(a) \, \Re \, F(b)$ *holds if and only if* $F(a) \times F(b) \subseteq \Re$, *where* $\Re$ *represents the relation structure.*
If $(F, A) = \{F(a), F(b), ...\}$, *then* $F(a)\Re F(b)$ *if and only if* $F(a) \times F(b) \in \Re$.

This formulation provides a structured representation of the relations within the framework of vague soft sets.

Definition 3. *(See [6]) Let* $\mathcal{R}$ *be a relation on a vague soft set* (F, A). *Then:*

- *(i)* $\Re$ *is reflexive if, for every parameter* $a \in A$, *the ordered pair* $\Re(a, a)$ *is contained in* $\Re$.
- *(ii)* $\Re$ *is symmetric if, for all* $(a, b) \in A \times A$, *whenever* $\Re(a, b) \in \Re$, *it follows that* $\Re(b, a) \in \Re$.
- *(iii)* $\Re$ *is transitive if, for any parameters* $a, b, c \in A$, *the inclusion of* $\Re(a, b)$ *and* $\Re(b, c)$ *in* $\Re$ *necessitates that* $\Re(a, c)$ *is also a member of* $\Re$.

Definition 4. *(See [6]) Let* (F, A), (G, B) *and* (H, C) *be three vague soft sets. Let* $\Re$ *be a vague soft relation from* (F, A) *to* (G, B) *and* S *be a vague soft set relation from* (G, B) *to* (H, C). *Then a new vague soft set relation, the composition of* $\Re$ *and* S *expressed as* S $o \, \Re$ *from* (F, A) *to* (H, C) *is defined as follows: If* $F(a)$ *is in* (F, A) *and* $H(c)$ *is in* (H, C) *then*
$F(a) S o \, \Re H(c)$ *iff there is some* $G(b)$ *in* (G, B) *such that* $F(a)\Re G(b)$ *and* $G(b) S H(c)$.
i.e., for $(a, b) \in A \times B$, $(b, c) \in B \times C$, $S \ o \, \Re(a, c) = \max[\Re \bullet S]$ *where* $\bullet$ *is the dot product.*

Definition 5. *(See [6]) Let* (F, A) *and* (G, B) *be two nonempty vague soft sets. Then a vague soft set relation* f *from* (F, A) *to* (G, B) *is called a vague soft set function if every element in the domain has a unique element in the range. If* $F(a) \, f \, G(b)$ *then we write* $f(F(a)) = G(b)$.

Definition 6. *(See [41]) Let* X *be a universal set. A cubic vague set* $\mathbb{A}^V$ *defined over the universal set* X *is an ordered pair which is defined as follows*

$$\mathbb{A}^V = \{\langle x, A_V(x), \lambda_V(x)\rangle : x \in X\}$$

where $A_V = \left\langle A_V^t, A_V^{1-f} \right\rangle = \{\langle x, [t_{A_V}^-(x), t_{A_V}^+(x)], [1 - f_{A_V}^-(x), 1 - f_{A_V}^+(x)]\rangle : x \in X\}$ *represents IVVS defined on* X *while* $\lambda_V = \{(x, t_{\lambda_V}(x), 1 - f_{\lambda_V}(x)) : x \in X\}$ *represents VS such that* $t_{A_V}^+(x) + f_{A_V}^+(x) \leq 1$ *and* $t_{\lambda_V}(x) + f_{\lambda_V}(x) \leq 1$. *For clarity, we denote the pairs as* $\mathbb{A}^V = \langle A_V, \lambda_V \rangle$, *where* $A_V = \langle [t_{A_V}^-, t_{A_V}^+], [1 - f_{A_V}^-, 1 - f_{A_V}^+]\rangle$ *and* $\lambda_V = (t_{\lambda_V}, 1 - f_{\lambda_V})$. C_V^X *denotes the sets of all cubic vague sets in* X.

Definition 7. *(See [41]) Let X be a universal set and V be a non-empty vague set. A cubic vague set $\mathbb{A}^V =< A_V, \lambda_V >$ is called an internal cubic vague set (brief. ICVS) if $A_V^-(x) \leq \lambda_V(x) \leq A_V^+(x)$ for all $x \in X$.*

Definition 8. *(See [41]) Let X be a universal set and V be a non-empty vague set. A cubic vague set $\mathbb{A}^V =< A_V, \lambda_V >$ is called an external cubic vague set (brief. ECVS) if $\lambda_V(x) \notin \left(A_V^-(x), A_V^+(x)\right)$ for all $x \in X$.*

3. Cubic Vague Soft Set Relations

In this section we define the concept of the relation of CVSS and study some of its properties.

Definition 9. *If a pair $(\tilde{F}, \mathbb{A}^V)$ and $(\tilde{G}, \mathbb{B}^V)$ are two cubic vague soft sets over U, then the Cartesian product of $\mathbb{A}^V$ and $\mathbb{B}^V$ is defined as, $\mathbb{A}^V \times \mathbb{B}^V = (H, \mathbb{A}^V \times \mathbb{B}^V)$, where $\tilde{H} : \mathbb{A}^V \times \mathbb{B}^V \to CV(U \times U)$ and $\tilde{H}(a, b) = \tilde{F}(a) \times \tilde{G}(b)$, where $(a, b) \in \mathbb{A}^V \times \mathbb{B}^V$, i.e., $H(a, b) = \{(h_i, h_j) : where\, h_i \in \tilde{F}(a) and\, h_j \in \tilde{G}(b)\}$.*

The Cartesian product of three or more nonempty vague soft sets can be defined by generalizing the definition of the Cartesian product of two vague soft sets. The Cartesian product $(\tilde{F}_1, \mathbb{A}^V) \times (\tilde{F}_2, \mathbb{A}^V) \times ... \times (\tilde{F}_n, \mathbb{A}^V)$ of the nonempty vague soft sets $(\tilde{F}_1, \mathbb{A}^V), (\tilde{F}_2, \mathbb{A}^V), ..., (\tilde{F}_n, \mathbb{A}^V)$ is the vague soft set of all ordered n-tuple $(h_1, h_2, ..., h_n)$ where $h_i \in \tilde{F}_i(a)$.

Example 1. *Let $U = \{s_1, s_2\}$ be a set universe and let $E = \{e_1, e_2, e_3\}$ be a set of parameters. Let $(\tilde{F}, \mathbb{A}^V)$ and $(\tilde{G}, \mathbb{B}^V)$ be two CVSSs over the common universe U. Let $(\tilde{F}, \mathbb{A}^V)$ and $(\tilde{G}, \mathbb{B}^V)$ describe the "learning" and " learning outcomes" respectively.*

Suppose that $\mathbb{A}^V = \{a_1 =$ distance learning, $a_2 =$ blended, $a_3 =$ class room$\}$ and $\mathbb{B}^V = \{b_1 =$"result", $b_2 =$"conduct", $b_3 =$"games and sports performances"$\}$.

Suppose $(\tilde{F}, \mathbb{A}^V)$ and $(\tilde{G}, \mathbb{A}^V)$ are defined as the following:

$$\tilde{F}(a_1) = \left\{ \frac{s_1}{\langle[0.10, 0.30], [0.30, 0.70]\rangle, (0.50, 0.70)}, \frac{s_2}{\langle[0.30, 0.40], [0.50, 0.60]\rangle, (0.10, 0.30)} \right\},$$
$$\tilde{F}(a_2) = \left\{ \frac{s_1}{\langle[0.20, 0.30], [0.30, 0.45]\rangle, (0.15, 0.35)}, \frac{s_2}{\langle[0.35, 0.45], [0.45, 0.50]\rangle, (0.20, 0.40)} \right\},$$
$$\tilde{F}(a_3) = \left\{ \frac{s_1}{\langle[0.15, 0.25], [0.20, 0.35]\rangle, (0.20, 0.25)}, \frac{s_2}{\langle[0.30, 0.40], [0.40, 0.50]\rangle, (0.30, 0.35)} \right\},$$
$$\tilde{G}(b_1) = \left\{ \frac{s_1}{\langle[0.10, 0.30], [0.20, 0.40]\rangle, (0.15, 0.25)}, \frac{s_2}{\langle[0.30, 0.40], [0.45, 0.55]\rangle, (0.35, 0.40)} \right\},$$
$$\tilde{G}(b_2) = \left\{ \frac{s_1}{\langle[0.25, 0.45], [0.30, 0.45]\rangle, (0.10, 0.40)}, \frac{s_2}{\langle[0.20, 0.35], [0.40, 0.50]\rangle, (0.15, 0.40)} \right\},$$
$$\tilde{G}(b_3) = \left\{ \frac{s_1}{\langle[0.20, 0.35], [0.40, 0.50]\rangle, (0.15, 0.40)}, \frac{s_2}{\langle[0.15, 0.25], [0.20, 0.35]\rangle, (0.20, 0.25)} \right\}.$$

Now, $(\tilde{F}, \mathbb{A}^V) \times (\tilde{G}, \mathbb{B}^V) = (\tilde{H}, \mathbb{A}^V \times \mathbb{B}^V)$ where a typical element will look like

$$\tilde{H}(a_1, b_1) = \left\{ \frac{s_1}{\langle[0.10, 0.30], [0.30, 0.70]\rangle, (0.50, 0.70)}, \frac{s_2}{\langle[0.30, 0.40], [0.50, 0.60]\rangle, (0.10, 0.30)} \right\} \times$$

$$\left\{\frac{s_1}{\langle[0.10,0.30],[0.20,0.40]\rangle,(0.15,0.25)},\frac{s_2}{\langle[0.30,0.40],[0.45,0.55]\rangle,(0.35,0.40)}\right\}$$
$$=\{(s_1,s_1),(s_1,s_2),(s_2,s_1),(s_2,s_2)\}$$
$$=\left\{\frac{(s_1,s_1)}{\langle[0.10,0.30],[0.30,0.70]\rangle,(0.50,0.70)},\frac{(s_1,s_2)}{\langle[0.10,0.30],[0.45,0.70]\rangle,(0.35,0.70)},\right.$$
$$\left.\frac{(s_2,s_1)}{\langle[0.10,0.30],[0.50,0.60]\rangle,(0.10,0.30)},\frac{(s_2,s_2)}{\langle[0.20,0.35],[0.50,0.60]\rangle,(0.10,0.40)}\right\}.$$

is a relation in vague soft set defined in terms of ordered pairs.

$$\Re=\left\{\tilde H(a_1,b_1),\tilde H(a_1,b_2),\tilde H(a_1,b_3),\tilde H(a_2,b_1),\tilde H(a_2,b_2),\tilde H(a_2,b_3),\tilde H(a_3,b_1),\tilde H(a_3,b_2),\tilde H(a_3,b_3)\right\}.$$

As in our example the pair (s_i,s_j) for $i,j=1,2$ is the relation between students and their activities with the vague soft set such as: $\frac{(s_1,s_1)}{\langle[0.10,0.30],[0.30,0.70]\rangle,(0.50,0.70)}$ meaning s_1 is the first item from the students and s_1 is the first item from the activity with cubic vague soft set $\langle[0.10,0.30],[0.30,0.70]\rangle,(0.50,0.70)$.

Definition 10. *Let $(\tilde F,\mathbb{A}^V)$ and $(\tilde G,\mathbb{B}^V)$ be two CVSS defined over the universe U. A relation from $(\tilde F,\mathbb{A}^V)$ to $(\tilde G,\mathbb{B}^V)$ is described as a cubic vague soft subset of $(\tilde F,\mathbb{A}^V)\times(\tilde G,\mathbb{B}^V)$. This relation takes the form $(\tilde H_1,\mathbb{S}^V)$, where $\mathbb{S}^V\subseteq\mathbb{A}^V\times\mathbb{B}^V$ and $\tilde H_1(a,b)$ is defined for all $(a,b)\in\mathbb{S}^V$. Any subset of $(\tilde F,\mathbb{A}^V)\times(\tilde F,\mathbb{A}^V)$ defines a relation on $(\tilde F,\mathbb{A}^V)$, which is written in parameterized form as:*

$$(\tilde F,\mathbb{A}^V)=\{\tilde F(a),\tilde F(b),\ldots\},\ \text{then } \tilde F(a)\,\Re\,\tilde F(b)\iff\tilde F(a)\times\tilde F(b)\in\Re.$$

Definition 11. *Let $\Re$ be a CVSS relation from $(\tilde F,\mathbb{A}^V)$ to $(\tilde G,\mathbb{A}^V)$. The domain of $\Re$ is denoted as dom $\Re$ and is defined as the CVSS $(\tilde D,\mathbb{A}_1^V)$ where $\mathbb{A}_1^V=\{a\in\mathbb{A}^V:\tilde H(a,b)\in\Re$ for some $b\in\mathbb{B}^V\}$ and $\tilde D(a_1)=\tilde F(a_1),\forall a_1\in\mathbb{A}^V$. The range of $\Re$ denoted by ran $\Re$, is defined as a CVSS $(\tilde T,\mathbb{B}_1^V)$, where $\mathbb{B}_1^V\subset\mathbb{B}^V$ and $\mathbb{B}_1^V=\{b\in\mathbb{B}^V:\tilde H(a,b)\in\Re$ for some $a\in\mathbb{A}^V\}$ and $\tilde T(b_1)=\tilde G(b_1)\forall b_1\in\mathbb{B}^V$, where $ran\Re=\tilde T$.*

Definition 12. *Let $(\tilde F,\mathbb{A}^V)$ be a CVSS defined on the universal set and $\Re$ be a relation defined on the cubic vague set of U (i.e., $\Re\subset CV(U\times U)$). The induced CVSS relation $\Re_U$ on $(\tilde F,\mathbb{A}^V)$ is defined as follows: $\tilde F(a)\Re_U\tilde F(b)\Leftrightarrow u\Re v$ for every $u\in\tilde F(a)$ and $v\in\tilde F(b)$.*

Definition 13. *Let $(\tilde F,\mathbb{A}^V)$ be a CVSS defined on the universal set and $\Re$ be a relation defined on $\mathbb{A}^V$. (i.e., $\Re\subset\mathbb{A}^V\times\mathbb{A}^V$)). The induced CVSS relation $\Re_{\mathbb{A}^V}$ on $(\tilde F,\mathbb{A}^V)$ is defined as follows: $\tilde F(a)\Re_{\mathbb{A}^V}\tilde F(b)\Leftrightarrow a\Re b$.*

Example 2. *Suppose that $U=\{s_1,s_2,s_3\}$ is the set of students who have online courses and $\mathbb{A}^V$ denotes the average performance of these students in their exams are given as $\mathbb{A}^V=\{$ excellent, very good, good, poor$\}$ (i.e., $\mathbb{A}^V=\{e,v,g,p\}$). Then the CVSS $(\tilde F,\mathbb{A}^V)$ is to point out the results of these students. Let $\Re$ be a relation defined on the CVSS U as $s_i\Re s_j$ if and only if s_i and s_j come under the same conditions.*

$\tilde{F}(e) = \left\{ \frac{s_1}{\langle [0.15, 0.25], [0.20, 0.35] \rangle, (0.20, 0.25)}, \frac{s_2}{\langle [0.30, 0.40], [0.40, 0.50] \rangle, (0.30, 0.35)}, \right.$
$\left. \frac{s_3}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)} \right\},$

$\tilde{F}(v) = \left\{ \frac{s_1}{\langle [0.10, 0.30], [0.30, 0.70] \rangle, (0.50, 0.70)}, \frac{s_2}{\langle [0.30, 0.40], [0.50, 0.60] \rangle, (0.10, 0.30)}, \right.$
$\left. \frac{s_3}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)} \right\},$

$\tilde{F}(g) = \left\{ \frac{s_1}{\langle [0.20, 0.30], [0.30, 0.45] \rangle, (0.15, 0.35)}, \frac{s_2}{\langle [0.35, 0.45], [0.45, 0.50] \rangle, (0.20, 0.40)}, \right.$
$\left. \frac{s_3}{\langle [0.30, 0.40], [0.45, 0.55] \rangle, (0.35, 0.40)} \right\},$

$\tilde{F}(p) = \left\{ \frac{s_1}{\langle [0.10, 0.30], [0.20, 0.40] \rangle, (0.15, 0.25)}, \frac{s_2}{\langle [0.30, 0.40], [0.45, 0.55] \rangle, (0.35, 0.40)}, \right.$
$\left. \frac{s_3}{\langle [0.30, 0.40], [0.45, 0.55] \rangle, (0.35, 0.40)} \right\}.$

Then the induced relation $\Re_U$ *on* $(\tilde{F}, \mathbb{A}^V)$ *is given by*

$\Big\{ \tilde{F}(e) \times \tilde{F}(e), \tilde{F}(e) \times \tilde{F}(v), \tilde{F}(v) \times \tilde{F}(e), \tilde{F}(v) \times \tilde{F}(v), \tilde{F}(g) \times \tilde{F}(g), \tilde{F}(g) \times \tilde{F}(p), \tilde{F}(p) \times \tilde{F}(g), \tilde{F}(p) \times \tilde{F}(p) \Big\}.$

$F(e) \times F(e) = \left\{ \frac{(s_1, s_1)}{\langle [0.15, 0.25], [0.20, 0.35] \rangle, (0.20, 0.50)}, \frac{(s_1, s_2)}{\langle [0.150.25], [0.40, 0.50] \rangle, (0.20, 0.35)}, \right.$
$\frac{(s_1, s_3)}{\langle [0.15, 0.25], [0.40, 0.50] \rangle, (0.15, 0.40)}, \frac{(s_2, s_1)}{\langle [0.150.25], [0.40, 0.50] \rangle, (0.20, 0.35)},$
$\frac{(s_2, s_2)}{\langle [0.30, 0.40], [0.40, 0.50] \rangle, (0.30, 0.35)}, \frac{(s_2, s_3)}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)},$
$\frac{(s_3, s_1)}{\langle [0.15, 0.25], [0.40, 0.50] \rangle, (0.15, 0.40)}, \frac{(s_3, s_2)}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)},$
$\left. \frac{(s_3, s_3)}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)} \right\}.$

We have $\Re_U = \left\{ \frac{(s_1, s_1)}{\langle [0.15, 0.25], [0.20, 0.35] \rangle, (0.20, 0.50)}, \frac{(s_1, s_2)}{\langle [0.150.25], [0.40, 0.50] \rangle, (0.20, 0.35)}, \right.$
$\frac{(s_1, s_3)}{\langle [0.15, 0.25], [0.40, 0.50] \rangle, (0.15, 0.40)}, \frac{(s_2, s_1)}{\langle [0.150.25], [0.40, 0.50] \rangle, (0.20, 0.35)},$
$\frac{(s_2, s_2)}{\langle [0.30, 0.40], [0.40, 0.50] \rangle, (0.30, 0.35)}, \frac{(s_2, s_3)}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)},$
$\frac{(s_3, s_1)}{\langle [0.15, 0.25], [0.40, 0.50] \rangle, (0.15, 0.40)}, \frac{(s_3, s_2)}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)},$
$\left. \frac{(s_3, s_3)}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)}, \cdots \right\}$

The candidate (s_i, s_j) *for* $i, j = 1, 2, 3$ *represents the Cartesian product of the students, where* s_i *refers to the student of the first universe and* s_j *refers to the student of the second universe. However, the CVSS* $\langle t, 1 - f \rangle$ *represents the Cartesian product between*

CVSS s_i *and CVSS* s_j *such as:* $\frac{(s_1,s_1)}{\langle[0.15,0.25],[0.20,0.35]\rangle,(0.20,0.50)}$ *where* s_1 *is the first student of the first universe and* s_2 *is the second student of the second universe with the CVSS* $\langle[0.15,0.25],[0.20,0.35]\rangle,(0.20,0.50)$.

4. Equivalence Relations and Partitions on Cubic Vague Soft Sets

Definition 14. *Let* $\Re$ *be a relation on CVSS* $(\tilde{F},\mathbb{A}^V)$. *Then*

(i) $\Re$ *is reflexive if* $\tilde{H}_1(a,a)\in\Re,\forall a\in\mathbb{A}^V$.

(ii) $\Re$ *is symmetric if* $\tilde{H}_1(a,b)\in\Re\Rightarrow\tilde{H}_1(b,a)\in\Re,\forall(a,b)\in\mathbb{A}^V\times\mathbb{A}^V$.

(iii) $\Re$ *is transitive if* $\tilde{H}_1(a,b)\in\Re$, $\tilde{H}_1(b,c)\in\Re\Rightarrow\tilde{H}_1(a,c)\in\Re,\forall a,b,c\in\mathbb{A}^V$.

Definition 15. *A CVSS relation* $\Re$ *on a CVSS* $(\tilde{F},\mathbb{A}^V)$ *is called an equivalence relation if it is reflexive, symmetric and transitive.*

Example 3. *Consider a cubic vague soft set* $(\tilde{F},\mathbb{A}^V)$ *over* U *where* $U=\{u_1,u_2,u_3\}$, $\mathbb{A}^V=\{a_1,a_2,a_3\}$ *and*

$$\tilde{F}(a_1)=\left\{\frac{u_1}{\langle[0.15,0.25],[0.20,0.35]\rangle,(0.20,0.25)},\frac{u_2}{\langle[0.30,0.40],[0.40,0.50]\rangle,(0.30,0.35)},\frac{u_3}{\langle[0.20,0.35],[0.40,0.50]\rangle,(0.15,0.40)}\right\},$$

$$\tilde{F}(a_2)=\left\{\frac{u_1}{\langle[0.10,0.30],[0.30,0.70]\rangle,(0.50,0.70)},\frac{u_2}{\langle[0.30,0.40],[0.50,0.60]\rangle,(0.10,0.30)},\frac{u_3}{\langle[0.20,0.35],[0.40,0.50]\rangle,(0.15,0.40)}\right\},$$

$$\tilde{F}(a_3)=\left\{\frac{u_1}{\langle[0.20,0.30],[0.30,0.45]\rangle,(0.15,0.35)},\frac{u_2}{\langle[0.35,0.45],[0.45,0.50]\rangle,(0.20,0.40)},\frac{u_3}{\langle[0.30,0.40],[0.45,0.55]\rangle,(0.35,0.40)}\right\}.$$

Consider a relation $\Re$ *defined on* $(\tilde{F},\mathbb{A}^V)$ *as* $\{\tilde{F}(a_1)\times\tilde{F}(a_2),\tilde{F}(a_2)\times\tilde{F}(a_3),\tilde{F}(a_3)\times\tilde{F}(a_1),\tilde{F}(a_3)\times\tilde{F}(a_2),\tilde{F}(a_1)\times\tilde{F}(a_3),\tilde{F}(a_2)\times\tilde{F}(a_1),\tilde{F}(a_1)\times\tilde{F}(a_1),\tilde{F}(a_2)\times\tilde{F}(a_2),\tilde{F}(a_3)\times\tilde{F}(a_3)\}$. *This relation is a CVSS equivalence relation.*

Definition 16. *Let* $(\tilde{F},\mathbb{A}^V)$ *be a CVSS. Then the equivalence class of* $\tilde{F}(a)$ *denoted by* $[\tilde{F}(a)]$ *is defined as* $[\tilde{F}(a)]=\{\tilde{F}(b):\tilde{F}(b)\Re F(a)\}$.

Example 4. *Consider Example 3 We have*

$$[\tilde{F}(a_1)]=\{\tilde{F}(a_1),\tilde{F}(a_2)\}=[\tilde{F}(a_2)].$$

Lemma 1. *Let* $\Re$ *be an equivalence relation on a CVSS* $(\tilde{F},\mathbb{A}^V)$. *For any* $\tilde{F}(a),\tilde{F}(b)\in(\tilde{F},\mathbb{A}^V)$, $\tilde{F}(a)\Re\tilde{F}(b)$ *iff* $[\tilde{F}(a)]=[\tilde{F}(b)]$.

Proof. Suppose $[\tilde{F}(a)] = [\tilde{F}(b)]$. Since $\Re$ is reflexive $\tilde{F}(b)\Re\tilde{F}(b)$, hence $\tilde{F}(b) \in [\tilde{F}(b)] = [\tilde{F}(a)]$ which gives $\tilde{F}(a)\Re\tilde{F}(b)$.
Conversely suppose $\tilde{F}(a)\Re\tilde{F}(b)$. Let $\tilde{F}(a_1) \in [\tilde{F}(a)]$. Then $\tilde{F}(a_1)\Re\tilde{F}(a)$. Using the transitive property of $\Re$ this gives $\tilde{F}(a_1) \in [\tilde{F}(b)]$. Hence $[\tilde{F}(a)] \subseteq [\tilde{F}(b)]$. Using a similar argument $[\tilde{F}(b)] \subseteq [\tilde{F}(a)]$. Hence $[\tilde{F}(a) = [\tilde{F}(b)]$.

Definition 17. *A collection of nonempty cubic vague soft subsets $\mathbb{P}^V = \{(\tilde{F}_i, \mathbb{A}_i^V), i \in I\}$ of a cubic vague soft set $(\tilde{F}, \mathbb{A}^V)$ is called a partition of $(\tilde{F}, \mathbb{A}^V)$ such that*

(i) $(\tilde{F}, \mathbb{A}^V) = \tilde{\bigcup\limits_i}(\tilde{F}_i, \mathbb{A}_i^V)$ *and*

(ii) $\mathbb{A}_i^V \cap \mathbb{A}_j^V = \phi$, *whenever* $i \neq j$.

Theorem 1. *Let $\{(\tilde{F}_i, \mathbb{A}_i^V), i \in I\}$ be a partition of CVSS $(\tilde{F}, \mathbb{A}^V)$. The CVSS relation defined on $(\tilde{F}, \mathbb{A}^V)$ as $\tilde{F}(a)\Re\tilde{F}(b)$ iff $\tilde{F}(a)$ and $\tilde{F}(b)$ are the members belonging to the same equivalence relation.*

Proof. Reflexive: Let $\tilde{F}(a)$ be any element of $(\tilde{F}, \mathbb{A}^V)$. It is clear that $\tilde{F}(a)$ is in the same relation in itself. Hence $\tilde{F}(a)\Re\tilde{F}(a)$.

Symmetric: If $\tilde{F}(a)\Re\tilde{F}(b)$, then $\tilde{F}(a)$ and $\tilde{F}(b)$ are in the same relation. Therefore $\tilde{F}(b)\Re\tilde{F}(a)$.

Transitive: If $\tilde{F}(a)\Re\tilde{F}(b)$, $\tilde{F}(b)\Re\tilde{F}(c)$ then $\tilde{F}(a)$, $\tilde{F}(b)$ and $\tilde{F}(c)$ must lie in the same relation. Then $\tilde{F}(a)\Re\tilde{F}(c)$. Therefore $\tilde{F}(a)\Re\tilde{F}(b)$ is an equivalence relation.

Theorem 2. *Corresponding to every equivalence relation defined on a CVSS $(\tilde{F}, \mathbb{A}^V)$, there exists a partition on $(\tilde{F}, \mathbb{A}^V)$ and this partition precisely consists of the equivalence classes of $\Re$.*

Proof. Let $[\tilde{F}(a)]$ be the equivalence class with respect to a relation $\Re$ on $(\tilde{F}, \mathbb{A}^V)$. Let $\mathbb{A}_a^V$ denote all those elements in $\mathbb{A}^V$ corresponding to $[\tilde{F}(a)]$ (*i.e.*, $\mathbb{A}_a^V = \{b \in \mathbb{A}^V : \tilde{F}(b)\Re\tilde{F}(a)\}$). Thus we can denote $[\tilde{F}(a)]$ as $(\tilde{F}, \mathbb{A}_a^V)$. We have to show that the collection $\{(\tilde{F}, \mathbb{A}_a^V) : a \in \mathbb{A}^V\}$ of such distinct sets forms a partition $\mathbb{P}^V$ of $(\tilde{F}, \mathbb{A}^V)$. In order to prove this we should prove

(i) $(\tilde{F}, \mathbb{A}^V) = \tilde{\bigcup\limits_{a \in \mathbb{A}^V}} (\tilde{F}, \mathbb{A}_a^V)$.

(ii) If $\mathbb{A}_a^V$, $\mathbb{A}_b^V$, are not identical then $\mathbb{A}_a^V \cap \mathbb{A}_b^V = \phi$.

$\tilde{F}(a)\Re\tilde{F}(a) \forall a \in \mathbb{A}^V$. Since $\Re$ is reflexive then $(\tilde{F}, \mathbb{A}^V) = \tilde{\bigcup\limits_{a \in \mathbb{A}^V}} (\tilde{F}, \mathbb{A}_a^V)$.
Now for the second part
Let $x \in \mathbb{A}_a^V \cap \mathbb{A}_b^V$. Then $\tilde{F}(x) \in (\tilde{F}, \mathbb{A}_a^V)$ and $\tilde{F}(x) \in (\tilde{F}, \mathbb{A}_b^V)$
$\Rightarrow \tilde{F}(x)\Re\tilde{F}(a)$ and $\tilde{F}(x)\Re\tilde{F}(b)$.

Using the transitive property of $\Re$ we have $\tilde{F}(a)\Re\tilde{F}(b)$. Using Lemma 1 we have $[\tilde{F}(a)] = [\tilde{F}(b)]$. This gives $\mathbb{A}_a^V = \mathbb{A}_b^V$ (contradiction). Since $\mathbb{A}_a^V$ and $\mathbb{A}_b^V$ are not identical then $\mathbb{A}_a^V \cap \mathbb{A}_b^V = \phi$.

Motivation of Composition Cubic Vague Soft Set

The need to depict multiple phases and interdependent decision-making under uncertainty is what drives the effort to design a composition for CVSSs. Real-world issues frequently call for the blending of information, such as merging symptoms with test results to arrive at a diagnosis, even though this paradigm is effective in capturing complicated and ambiguous data in a single step. This vital step of combining and connecting data from many sources is still unattainable without a formal composition operation. In order to fully utilize CVSSs for intricate, real-world applications, it is necessary to define composition to go from static evaluation to dynamic, relational reasoning.

5. Composition of Cubic Vague Soft Set

In this section we define a composition of CVSS and its properties.

Definition 18. *Consider $(\tilde{F}, \mathbb{A}^V)$, $(\tilde{G}, \mathbb{B}^V)$ and $(\tilde{H}, \mathbb{C}^V)$ be three CVSS. Let $\Re$ be a cubic vague soft relation from $(\tilde{F}, \mathbb{A}^V)$ to $(\tilde{G}, \mathbb{B}^V)$ and S be a CVSS relation from $(\tilde{G}, \mathbb{B}^V)$ to $(\tilde{H}, \mathbb{C}^V)$. Then a new CVSS relation, the composition of $\Re$ and S expressed as $S\ o\ \Re$ from $(\tilde{F}, \mathbb{A}^V)$ to $(\tilde{H}, \mathbb{C}^V)$ is defined as follows: If $\tilde{F}(a)$ is in $(\tilde{F}, \mathbb{A}^V)$ and $\tilde{H}(c)$ is in $(\tilde{H}, \mathbb{C}^V)$ then $\tilde{F}(a) S o\ \Re \tilde{H}(c)$ iff there is some $\tilde{G}(b)$ in $(\tilde{G}, \mathbb{B}^V)$ such that $\tilde{F}(a)\Re\tilde{G}(b)$ and $\tilde{G}(b) S \tilde{H}(c)$. (i.e., for $(a,b) \in \mathbb{A}^V \times \mathbb{B}^V$, $(b,c) \in \mathbb{B}^V \times \mathbb{C}^V$, $S\ o\ \Re(a,c) = b \in \mathbb{B}^V \bigvee T\left(\Re(a,b), S(b,c)\right)$ where T is a chosen t-norm and $\bigvee$ denotes max.)*

Example 5. *Let $(\tilde{F}, \mathbb{A}^V)$, $(\tilde{G}, \mathbb{B}^V)$ and $(\tilde{H}, \mathbb{C}^V)$ be three cubic vague soft sets and consider relations $\Re$ from $(\tilde{F}, \mathbb{A}^V)$ to $(\tilde{G}, \mathbb{B}^V)$ and S from $(\tilde{G}, \mathbb{B}^V)$ to $(\tilde{H}, \mathbb{C}^V)$. Let $\mathbb{A}^V = \{a_1, a_2, a_3\}$, $\mathbb{B}^V = \{b_1, b_2, b_3\}$, $\mathbb{C}^V = \{c_1, c_2\}$ and $U = \{u_1, u_2, u_3\}$. Suppose that*

$$\tilde{F}(a_1) = \left\{ \frac{u_1}{\langle [0.10, 0.30], [0.30, 0.70] \rangle, (0.50, 0.70)}, \frac{u_2}{\langle [0.30, 0.40], [0.50, 0.60] \rangle, (0.10, 0.30)} \right\},$$
$$\left. \frac{u_3}{\langle [0.20, 0.35], [0.40, 0.50] \rangle, (0.15, 0.40)} \right\},$$
$$\tilde{F}(a_2) = \left\{ \frac{u_1}{\langle [0.20, 0.30], [0.30, 0.45] \rangle, (0.15, 0.35)}, \frac{u_2}{\langle [0.35, 0.45], [0.45, 0.50] \rangle, (0.20, 0.40)} \right\},$$
$$\left. \frac{u_3}{\langle [0.30, 0.40], [0.40, 0.50] \rangle, (0.30, 0.35)} \right\},$$
$$\tilde{F}(a_3) = \left\{ \frac{u_1}{\langle [0.15, 0.25], [0.20, 0.35] \rangle, (0.20, 0.25)}, \frac{u_2}{\langle [0.30, 0.40], [0.40, 0.50] \rangle, (0.30, 0.35)} \right\},$$

$$\left.\frac{u_3}{\langle [0.30,0.40],[0.45,0.55]\rangle,(0.35,0.40)}\right\}$$

$$\tilde{G}(b_1)=\left\{\frac{u_1}{\langle [0.10,0.30],[0.20,0.40]\rangle,(0.15,0.25)},\frac{u_2}{\langle [0.30,0.40],[0.45,0.55]\rangle,(0.35,0.40)}\right\},$$

$$\left.\frac{u_3}{\langle [0.30,0.40],[0.40,0.50]\rangle,(0.30,0.35)}\right\}$$

$$\tilde{G}(b_2)=\left\{\frac{u_1}{\langle [0.25,0.45],[0.30,0.45]\rangle,(0.10,0.40)},\frac{u_2}{\langle [0.20,0.35],[0.40,0.50]\rangle,(0.15,0.40)}\right\},$$

$$\left.\frac{u_3}{\langle [0.20,0.35],[0.40,0.50]\rangle,(0.15,0.40)}\right\},$$

$$\tilde{G}(b_3)=\left\{\frac{u_1}{\langle [0.20,0.35],[0.40,0.50]\rangle,(0.15,0.40)},\frac{u_2}{\langle [0.15,0.25],[0.20,0.35]\rangle,(0.20,0.25)}\right\}.$$

$$\left.\frac{u_3}{\langle [0.30,0.40],[0.45,0.55]\rangle,(0.35,0.40)}\right\},$$

and

$$\tilde{H}(c_1)=\left\{\frac{u_1}{\langle [0.10,0.30],[0.30,0.70]\rangle,(0.50,0.70)},\frac{u_2}{\langle [0.30,0.40],[0.50,0.60]\rangle,(0.10,0.30)},\right.$$

$$\left.\frac{u_3}{\langle [0.20,0.35],[0.40,0.50]\rangle,(0.15,0.40)}\right\},$$

$$\tilde{H}(c_2)=\left\{\frac{u_1}{\langle [0.20,0.30],[0.30,0.45]\rangle,(0.15,0.35)},\frac{u_2}{\langle [0.35,0.45],[0.45,0.50]\rangle,(0.20,0.40)},\right.$$

$$\left.\frac{u_3}{\langle [0.30,0.40],[0.45,0.55]\rangle,(0.35,0.40)}\right\}.$$

The relation $\Re$ from $(\tilde{F},\mathbb{A}^V)$ to $(\tilde{G},\mathbb{B}^V)$ is defined as $(\tilde{\Re},\mathbb{D}^V)\subseteq(\tilde{F},\mathbb{A}^V)\times(\tilde{G},\mathbb{B}^V)$ and the relation S from $(\tilde{G},\mathbb{B}^V)$ to $(\tilde{H},\mathbb{C}^V)$ is defined as $(\tilde{S},\mathbb{Z}^V)=(\tilde{G},\mathbb{B}^V)\times(\tilde{H},\mathbb{C}^V)$.
In matrix notation, we write

$$(\tilde{F},\mathbb{A}^V)\times(\tilde{G},\mathbb{B}^V)=\begin{bmatrix}(a_1,b_1) & (a_1,b_2) & (a_1,b_3)\\ (a_2,b_1) & (a_2,b_2) & (a_2,b_3)\\ (a_3,b_1) & (a_3,b_2) & (a_3,b_3)\end{bmatrix}$$

where

$$(a_i,b_j)=\left\{\frac{u_x}{\left\langle \min\left(t_{a_i}(u_x),t_{b_j}(u_x)\right),\max\left(1-f_{a_i}(u_x),1-f_{b_j}(u_x)\right)\right\rangle,\left(\min t(u_x),\max 1-f(u_x)\right)}\right\},$$

for $x=1,2,3,\ i=1,2,3$ and $j=1,2,3$.
In the matrix notation, we write

$$(\tilde{G},\mathbb{B}^V)\times(\tilde{H},\mathbb{C}^V)=\begin{bmatrix}(b_1,c_1) & (b_1,c_2)\\ (b_2,c_1) & (b_2,c_2)\\ (b_3,c_1) & (b_3,c_2)\end{bmatrix}$$

where

$$(b_j,c_k)=\left\{\frac{u_x}{\left\langle \min\left(t_{b_j}(u_x),t_{c_k}(u_x)\right),\max\left(1-f_{b_j}(u_x),1-f_{c_k}(u_x)\right)\right\rangle,\left(\min t(u_x),\max 1-f(u_x)\right)}\right\},$$

for $x = 1, 2, 3,\ j = 1, 2, 3$ *and* $k = 1, 2$.
Then $(\tilde{S}, \mathbb{Z}^V)\ o\ (\tilde{\Re}, \mathbb{D}^V)$ *can be written as*

$$(\tilde{S}, \mathbb{Z}^V)\, o\, (\tilde{\Re}, \mathbb{D}^V) = \begin{bmatrix} (a_1, c_1) & (a_1, c_2) \\ (a_2, c_1) & (a_2, c_2) \\ (a_3, c_1) & (a_3, c_2) \end{bmatrix}$$

Where

$$(a_1, c_1) = \left\{ \frac{u_1}{\langle [0.10,0.30],[0.40,0.70] \rangle,(0.15,0.70)}, \frac{u_2}{\langle [0.15,0.25],[0.50,0.60] \rangle,(0.10,0.40)}, \frac{u_3}{\langle [0.20,0.35],[0.45,0.55] \rangle,(0.15,0.40)} \right\},$$
$$(a_2, c_1) = \left\{ \frac{u_1}{\langle [0.10,0.30],[0.40,0.70] \rangle,(0.10,0.70)}, \frac{u_2}{\langle [0.15,0.25],[0.50,0.60] \rangle,(0.10,0.40)}, \frac{u_3}{\langle [0.20,0.35],[0.40,0.50] \rangle,(0.15,0.40)} \right\},$$
$$(a_3, c_1) = \left\{ \frac{u_1}{\langle [0.10,0.25],[0.40,0.70] \rangle,(0.10,0.70)}, \frac{u_2}{\langle [0.15,0.25],[0.50,0.60] \rangle,(0.10,0.40)}, \frac{u_3}{\langle [0.20,0.35],[0.45,0.55] \rangle,(0.15,0.40)} \right\},$$
$$(a_1, c_2) = \left\{ \frac{u_1}{\langle [0.10,0.30],[0.30,0.70] \rangle,(0.10,0.70)}, \frac{u_2}{\langle [0.15,0.25],[0.50,0.60] \rangle,(0.10,0.40)}, \frac{u_3}{\langle [0.20,0.35],[0.45,0.55] \rangle,(0.15,0.40)} \right\},$$
$$(a_2, c_2) = \left\{ \frac{u_1}{\langle [0.10,0.30],[0.40,0.50] \rangle,(0.10,0.40)}, \frac{u_2}{\langle [0.15,0.25],[0.45,0.55] \rangle,(0.15,0.40)}, \frac{u_3}{\langle [0.20,0.35],[0.45,0.55] \rangle,(0.15,0.40)} \right\},$$
$$(a_3, c_2) = \left\{ \frac{u_1}{\langle [0.10,0.25],[0.40,0.50] \rangle,(0.10,0.40)}, \frac{u_2}{\langle [0.15,0.25],[0.45,0.55] \rangle,(0.15,0.40)}, \frac{u_3}{\langle [0.20,0.35],[0.40,0.55] \rangle,(0.15,0.40)} \right\}.$$

In general $S\, o\, \Re \neq \Re\, o\, S$.

Definition 19. *The inverse of a CVSS relation* $\Re$ *denoted as* $\Re^{-1}$ *is defined by* $\Re^{-1} = \{(\tilde{F}(b) \times \tilde{F}(a)) : \tilde{F}(a)\Re\tilde{F}(b)\}$.

Theorem 3. *Let* $\Re$ *be a CVSS relation from* $(\tilde{F}, \mathbb{A}^V)$ *to* $(\tilde{G}, \mathbb{B}^V)$ *and* S *be a CVSS relation from* $(\tilde{G}, \mathbb{B}^V)$ *to* $(\tilde{H}, \mathbb{C}^V)$. *Then* $(S\, o\, \Re)^{-1} = \Re^{-1}\, o\, S^{-1}$.

Proof. Clearly $(S\, o\, \Re)^{-1}$ is a CVSS relation from $(\tilde{H}, \mathbb{C}^V)$ to $(\tilde{F}, \mathbb{A}^V)$. Now let $\tilde{H}(c)$ be any element in $(\tilde{H}, \mathbb{C}^V)$ and $\tilde{F}(a)$ be any element in $(\tilde{F}, \mathbb{A}^V)$. Then $\tilde{H}(c)(S\, o\, \Re)^{-1}\tilde{F}(a)$ if $\tilde{F}(a)S\, o\, \Re\tilde{H}(C)$. This by definition exists if there is some $\tilde{G}(b)$ in $(\tilde{G}, \mathbb{B}^V)$ such that $\tilde{F}(a)\Re\tilde{G}(b)$ and $\tilde{G}(b)S\tilde{H}(c)$. This is equivalent to $\tilde{G}\Re^{-1}\tilde{F}(a)$ and $\tilde{H}(c)S^{-1}\tilde{G}(b)$. Then $\tilde{H}(c)\Re^{-1}\, o\, S^{-1}\tilde{F}(a)$. Hence $(S\, o\, \Re)^{-1} = \Re^{-1}\, o\, S^{-1}$.

Definition 20. *Let* $(\tilde{F}, \mathbb{A}^V)$ *be a CVSS. The identity relation on* $(\tilde{F}, \mathbb{A}^V)$, *denoted by* $I_{\tilde{F}\mathbb{A}^V}$, *is defined as* $I_{\tilde{F}\mathbb{A}^V} = \{ (\tilde{F}(a), \tilde{F}(b)) \mid a = b,\ a, b \in \mathbb{A}^V \}$. *Equivalently, for all* $a, b \in \mathbb{A}^V$, $\tilde{F}(a)\ I_{\tilde{F}\mathbb{A}^V}\ \tilde{F}(b) \iff a = b$.

6. Cubic Vague Soft Set Functions

In this section we introduce the concept of CVSS function and its properties.

Definition 21. *Let $(\tilde{F}, \mathbb{A}^V)$ and $(\tilde{G}, \mathbb{A}^V)$ be two nonempty cubic vague soft sets. Then a cubic vague soft set relation f from $(\tilde{F}, \mathbb{A}^V)$ to $(\tilde{G}, \mathbb{A}^V)$ is called a cubic vague soft set function if every element in the domain has a unique element in the range. If $\tilde{F}(a)\, f\, \tilde{G}(b)$ then we write $f(\tilde{F}(a)) = \tilde{G}(b)$.*

Example 6. *Let $U = \{u_1, u_2, u_3\}$, $\mathbb{A}^V = \{a_1, a_2, a_3\}$ and $\mathbb{B}^V = \{b_1, b_2\}$. Consider a cubic vague soft set $(\tilde{F}, \mathbb{A}^V)$ and $(\tilde{G}, \mathbb{B}^V)$ defined by*
$\tilde{F}(a_1) = \{\frac{u_1}{\langle 0.1,0.2\rangle}, \frac{u_2}{\langle 0.2,0.6\rangle}, \frac{u_3}{\langle 0.4,0.6\rangle}\}$,
$\tilde{F}(a_2) = \{\frac{u_1}{\langle 0.7,0.3\rangle}, \frac{u_2}{\langle 0.1,0.3\rangle}, \frac{u_3}{\langle 0.3,0.6\rangle}\}$,
$\tilde{F}(a_3) = \{\frac{u_1}{\langle 0.1,0.1\rangle}, \frac{u_2}{\langle 0.8,0.9\rangle}, \frac{u_3}{\langle 0.3,0.3\rangle}\}$,
$\tilde{G}(b_1) = \{\frac{u_1}{\langle 0.6,0.5\rangle}, \frac{u_2}{\langle 0.6,0.8\rangle}, \frac{u_3}{\langle 0,0\rangle}\}$, $\tilde{G}(b_2) = \{\frac{u_1}{\langle 0.4,0.5\rangle}, \frac{u_2}{\langle 0.9,0.9\rangle}, \frac{u_3}{\langle 1,1\rangle}\}$.

Then a CVSS function from $(\tilde{F}, \mathbb{A}^V)$ to $(\tilde{G}, \mathbb{B}^V)$ is given by $f = \{\tilde{F}(a_1)\times\tilde{G}(b_1), \tilde{F}(a_2)\times \tilde{G}(b_1), \tilde{F}(a_3) \times \tilde{G}(b_2)\}$.

Definition 22. *A function f from $(\tilde{F}, \mathbb{A}^V)$ to $(\tilde{G}, \mathbb{A}^V)$ is called injective (one to one) if $\tilde{F}(a_1) = \tilde{F}(a_2) \implies f(\tilde{F}(a_1)) = f(\tilde{F}(a_2)), \forall a_1, a_2 \in A$. f is called injective if each element of the range of f appears exactly once in the function.*

Definition 23. *A function f from $(\tilde{F}, \mathbb{A}^V)$ to $(\tilde{G}, \mathbb{B}^V)$ is called surjective (onto) if range of $f = (\tilde{G}, \mathbb{B}^V)$.*

Definition 24. *A function which is both injective and surjective is called a bijective function.*

Definition 25. *A constant CVSS function is a function in which every element in domf has the same image.*

Definition 26. *Identity CVSS function I on a CVSS $(\tilde{F}, \mathbb{A}^V)$ is defined by the function $I : (\tilde{F}, \mathbb{A}^V) \to (\tilde{F}, \mathbb{A}^V)$ as $I(\tilde{F}(a)) = \tilde{F}(a)$ for every $\tilde{F}(a)$ in $(\tilde{F}, \mathbb{A}^V)$.*

Theorem 4. *Let $f : (\tilde{F}, \mathbb{A}^V) \to (\tilde{G}, \mathbb{B}^V)$ be a CVSS function and $(\tilde{F}, \mathbb{A}^V_1)$ and $(\tilde{F}, \mathbb{A}^V_2)$ be a cubic vague soft subsets of $(\tilde{F}, \mathbb{A}^V)$. Then*

(i) $(\tilde{F}, \mathbb{A}^V_1) \subseteq (\tilde{F}, \mathbb{A}^V_2) \Rightarrow f(\tilde{F}, \mathbb{A}^V_1) \subseteq f(\tilde{F}, \mathbb{A}^V_2)$,

(ii) $f[(\tilde{F}, \mathbb{A}^V_1) \cup (\tilde{F}, \mathbb{A}^V_2)] = f(\tilde{F}, \mathbb{A}^V_1) \cup f(\tilde{F}, \mathbb{A}^V_2)$,

(iii) $f[(\tilde{F}, \mathbb{A}^V_1) \cap (\tilde{F}, \mathbb{A}^V_2)] \subseteq f(\tilde{F}, \mathbb{A}^V_1) \cap f(\tilde{F}, \mathbb{A}^V_2)$. *Equality holds if f is one to one.*

Proof. see Appendix 1

Now we will define and propose a few theorems on the composition of cubic vague soft set functions.

Definition 27. *Let* $f : (\tilde{F}, \mathbb{A}^V) \to (\tilde{G}, \mathbb{B}^V)$ *and* $g : (\tilde{G}, \mathbb{B}^V) \to (\tilde{H}, \mathbb{C}^V)$ *be two CVSS functions. Then* $g\, o\, f : (\tilde{F}, \mathbb{A}^V) \to (\tilde{H}, \mathbb{C}^V)$ *is also a CVSS function defined by* $(g\, o\, f)(\tilde{F}(a)) = g(f(\tilde{F}(a)))$.

Definition 28. *Let* f *be a bijective function from* $(\tilde{F}, \mathbb{A}^V)$ *to* $(\tilde{G}, \mathbb{B}^V)$ *. Then the inverse relation* f^{-1} *is called the inverse function.*

Theorem 5. *If* $f : (\tilde{F}, \mathbb{A}^V) \to (\tilde{G}, \mathbb{B}^V)$ *is bijective then* $f^{-1} : (\tilde{G}, \mathbb{B}^V) \to (\tilde{F}, \mathbb{A}^V)$ *is also a bijective function.*

Proof. see Appendix 2

Theorem 6. *Let* $f : (\tilde{F}, \mathbb{A}^V) \to (\tilde{G}, \mathbb{B}^V)$ *and* $g : (\tilde{G}, \mathbb{B}^V) \to (\tilde{H}, \mathbb{C}^V)$ *be two bijective cubic vague soft set functions. Then* $g\, o\, f : (\tilde{F}, \mathbb{A}^V) \to (\tilde{H}, \mathbb{C}^V)$ *is also bijective and* $(g\, o\, f)^{-1} = f^{-1}\, o\, g^{-1}$.

Proof. see Appendix 3

7. Application of Multicriteria Cubic Vague Soft Set in Decision Making Problem

Hwang and Yoon [18] introduced a method for solving multi-criteria decision-making (MCDM) problems called the TOPSIS technique. This approach is based on the concept of the shortest Euclidean distance and involves identifying a Positive Ideal Solution (PIS) and a Negative Ideal Solution (NIS), where each criterion is either maximized or minimized. They argued that TOPSIS effectively ranks alternatives based on how close they are to the ideal solution, helping to identify the best possible option among the available choices. The alternative closest to the PIS receives the highest rank (1), while the one nearest to the NIS is ranked lowest (0). All other alternatives fall somewhere in between, depending on their relative closeness to the ideal. When the same set of criteria is used for evaluation, proper weighting helps identify which condition or issue is most critical and requires attention. The TOPSIS method is structured around a series of steps and treats an MCDM problem with m alternatives as a geometric model in n-dimensional space [42]. The fundamental idea is that the best alternative should be the one closest to the PIS and farthest from the NIS [43]. When applying TOPSIS [44], it is typically assumed that the criteria are either monotonically increasing or decreasing, which simplifies the identification of PIS and NIS.

To overcome this difficulty, we represent uncertainty, imprecision, and hesitation in expert evaluations using CVSS theory in combination with multicriteria decision-making (MCDM) techniques (Figure 1). The TOPSIS technique is utilized to figure out suitable smartphone is optimal.

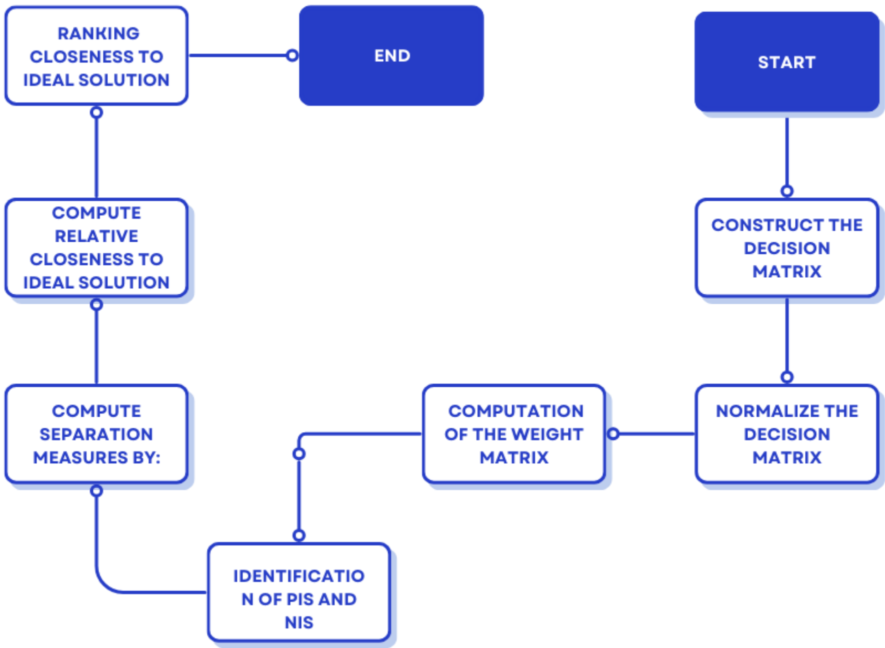

Figure 1: Graphical model of CVSS based TOPSIS method.

7.1. Parameter Selection and Sensitivity Considerations

In practical decision-making, the performance of the proposed extended TOPSIS method under cubic vague soft set relations (CVSSR) depends on the appropriate selection of parameters such as the membership interval ranges, weighting coefficients, and normalization schemes. These parameters should be determined according to the characteristics of the decision problem and the nature of uncertainty in the data. For instance, broader membership intervals may be used when expert opinions are highly inconsistent or when uncertainty is significant, while narrower intervals are suitable for more precise evaluations. Weighting coefficients can be obtained using methods such as entropy weighting, pairwise comparison, or expert judgment, ensuring that each criterion reflects its real-world importance. Furthermore, normalization parameters should be selected to maintain comparability among criteria measured in different units. Sensitivity analysis results presented in Section 7 demonstrate that moderate variations in these parameters do not significantly affect the final ranking of alternatives, confirming the stability and robustness of the proposed approach.

7.2. Mathematical Models of TOPSIS under Cubic Vague Soft Set (CVSS)

To process the complex data gathered from our method, we developed an algorithm for the MADM technique. The steps of this process, which are also shown in Algorithim 1, are described below, offering readers and specialists alike a simple and understandable manual.

Algorithm 1 CVSS–TOPSIS Evaluation Procedure

1: **Establishment of the Decision Matrix (DM).**

$$DM = \begin{bmatrix} \langle A_{V11}, \lambda_{V11} \rangle & \langle A_{V12}, \lambda_{V12} \rangle & \cdots & \langle A_{V1q}, \lambda_{V1q} \rangle \\ \langle A_{V21}, \lambda_{V21} \rangle & \langle A_{V22}, \lambda_{V22} \rangle & \cdots & \langle A_{V2q}, \lambda_{V2q} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle A_{Vp1}, \lambda_{Vp1} \rangle & \langle A_{Vp2}, \lambda_{Vp2} \rangle & \cdots & \langle A_{Vpq}, \lambda_{Vpq} \rangle \end{bmatrix}$$

$$A_V = \left\langle A_V^t, A_V^{1-f} \right\rangle = \left\{ \langle x, [t^-_{A_V}(x), t^+_{A_V}(x)], [1 - f^-_{A_V}(x), 1 - f^+_{A_V}(x)] \rangle : x \in X \right\},$$

$$\lambda_V = \{(x, t_{\lambda_V}(x), 1 - f_{\lambda_V}(x)) : x \in X\}.$$

2: **Normalization of the Decision Matrix.**

Normalize each Cubic Vague value as:

$$t^u_{ij} = \frac{t^-_{ij}}{\max(t^-_{ij})}, \quad (1 - f^-_{ij})^u = \frac{1 - f^-_{ij}}{\max(1 - f^-_{ij})}, \quad t^u_{\lambda_V} = \frac{t_{\lambda_V}}{\max(t_{\lambda_V})},$$

$$t^l_{ij} = \frac{t^+_{ij}}{\max(t^+_{ij})}, \quad (1 - f^+_{ij})^l = \frac{1 - f^+_{ij}}{\max(1 - f^+_{ij})}, \quad (1 - f_{\lambda_V})^l = \frac{1 - f_{\lambda_V}}{\max(1 - f_{\lambda_V})}.$$

where:

(i) $\max(t^-_j)$ is the highest vague truth value in row j;

(ii) $\max(1 - f^-_j)$ is the highest vague falsity value in row j;

(iii) $\max(t_{\lambda_V})$ is the highest cubic vague value in row j.

3: **Construction of the Weighted Normalized Decision Matrix.**

$$\mathbf{V}_i = \left(t^{\oplus}_{i1}, 1 - f^{\oplus}_{i1}, t^{\oplus}_{i2}, 1 - f^{\oplus}_{i2}, \ldots, t^{\oplus}_{ij}, 1 - f^{\oplus}_{ij}, t^{\oplus}_{\lambda_V,i}, 1 - f^{\oplus}_{\lambda_V,i}\right).$$

4: **Identification of Positive and Negative Ideal Solutions.**

Positive Ideal Solution (PIS):

$$P^+_i = \begin{cases} \max_i V_{ij} & \text{for components corresponding to truth } (t^{\oplus}), \\ \min_i V_{ij} & \text{for components corresponding to falsity } (1 - f^{\oplus}), \end{cases}$$

$$P^-_i = \begin{cases} \min_i V_{ij} & \text{for truth components,} \\ \max_i V_{ij} & \text{for falsity components.} \end{cases}$$

5: **Computation of Separation Measures.**

$$D^+_i = \sqrt{\sum_{j=1}^{n} \left(V_{ij} - P^+_i\right)^2}, \quad D^-_i = \sqrt{\sum_{j=1}^{n} \left(V_{ij} - P^-_i\right)^2}.$$

6: **Calculation of the Relative Closeness Coefficient.**

$$C_i = \frac{D^-_i}{D^+_i + D^-_i}, \quad 0 \le C_i \le 1.$$

7: **Ranking of Alternatives.**

Rank all alternatives in descending order of C_i. Alternatives with higher C_i values are preferred.

7.3. Application Of TOPSIS method based on Cubic Vague Soft Set

In order to increase efficiency in energy use and accomplish sustainability objectives, a renewable energy company seeks to choose the most beneficial solar system. Four options are assessed during the decision-making process: Longi is x_1, Canadian is x_2, Jinko is x_3, and JA is x_4. These options are evaluated according to a number of significant factors, such as c_1: energy efficiency, c_2: cost of installation, c_3: durability, and c_4: maintenance needs. $W = [0.3, 0.1, 0.2, 0.4]$ deontes the weight of the criteria derived from expert judgments using a direct rating approach , where $\sum_{i=1}^{n} w_j = 1$. Every one of these factors is essential in assessing the solar system's overall viability and performance. For example, installation costs impact the initial financial burden, whereas energy efficiency has a direct impact on electricity output and return on investment. In a similar vein, longevity guarantees long-term dependability, and maintenance needs establish how simple and expensive it will be to maintain the system going forward.

Solution by CVSS-TOPSIS

Step 1 : Construct the decision matrix

Table 1: Decision Matrix D = $[x_{ij}]_{m\times n}$

Solar Panel	Energy Efficiency	Cost of Installation	Durability	Maintenance
Longi	([0.3,0.4],[0.6,0.5]), [0.7,0.9]	([0.7,0.4],[0.1,0.5]), [0.8,0.9]	([0.2,0.6],[0.4,0.4]), [0.8,0.8]	([0.5,0.2],[0.3,0.5]), [0.4,0.6]
Canadian	([0.2,0.6],[0.4,0.4]), [0.3,0.6]	([0.5,0.2],[0.3,0.5]), [0.5,0.8]	([0.6,0.5],[0.3,0.4]), [0.1,0.6]	([0.2,0.2],[0.1,0.6]), [0.4,0.8]
Jinko	([0.5,0.8],[0.3,0.3]), [0.7,0.9]	([0.6,0.8],[0.2,0.4]), [0.7,0.8]	([0.3,0.4],[0.6,0.5]), [0.2,0.2]	([0.3,0.1],[0.4,0.5]), [0.8,0.9]
JA	([0.2,0.6],[0.4,0.4]), [0.8,0.8]	([0.5,0.2],[0.3,0.5]), [0.7,0.7]	([0.2,0.6],[0.4,0.4]), [0.5,0.5]	([0.2,0.6],[0.4,0.4]), [0.4,0.5]

Step 2 : Normalize the Decision Matrix

Table 2: Calculating $[t^u_{ij}], [t^l_{ij}], [t^u_{\lambda_V}], [1 - f^-_{ij}]^u, [1 - f^+_{ij}]^l, [1 - f_{\lambda_V}]$

Solar Panel	Energy Efficiency	Cost of Installation	Durability	Maintenance
Longi	[0.428,0.666],[1.0,1.0] [0.875,1.0]	[1.0,0.666],[0.166,1.0] [1.0,1.0]	[0.285,0.666],[0.666,0.8] [1.0,0.888]	[0.714,0.333],[0.5,1.0] [0.5,0.666]
Canadian	[0.5,1.0],[1.0,0.8] [0.6,0.75]	[0.833,0.333],[0.75,0.833] [1.0,1.0]	[1.0,0.833],[0.75,0.666] [0.2,0.75]	[0.333,0.333],[0.25,1.0] [0.8,1.0]
Jinko	[0.833,1.0],[0.5,0.6] [1.0,1.0]	[1.0,1.0],[0.5,0.8] [0.875,0.8888]	[0.5,0.5],[1.0,1.0] [0.25,0.222]	[0.5,0.125],[0.666,1.0] [1.0,1.0]
JA	[0.4,1.0],[1.0,0.8] [1.0,1.0]	[1.0,0.333],[0.75,1.0] [0.875,0.875]	[0.4,1.0],[1.0,0.8] [0.625,0.625]	[0.4,1.0],[1.0,0.8] [0.5,0.625]

To normalize the decision matrix, divide each entry for example:

$t^u_{ij} = \frac{t^-_{ij}}{\max(t^-_{ij})}$, $(1-f^-_{ij})^u = \frac{1-f^-_{ij}}{\max(1-f^-_{ij})}$, $t^u_{\lambda_V} = \frac{t_{\lambda_V}}{\max(t_{\lambda_V})}$. $t^u_{11} = \frac{0.3}{\max(0.3,0.7,0.2,0.5)} = \frac{0.3}{0.7} = 0.428$,
$(1-f^-_{11})^u = \frac{0.6}{\max(0.6,0.1,0.4,0.3)} = \frac{0.6}{0.6} = 1.0$, $t^u_{\lambda_V} = \frac{0.7}{\max(0.7,0.8,0.8,0.4)} = \frac{0.7}{0.8} = 0.875$.
$t^l_{ij} = \frac{t^+_{ij}}{\max(t^+_{ij})}$,
$(1-f^+_{ij})^l = \frac{1-f^+_{ij}}{\max(1-f^+_{ij})}$, $(1-f_{\lambda_V})^l = \frac{1-f_{\lambda_V}}{\max(1-f_{\lambda_V})}$. $t^l_{11} = \frac{0.4}{\max(0.4,0.4,0.6,0.2)} = \frac{0.4}{0.6} = 0.666$,
$(1-f^+_{ij})^l = \frac{0.5}{\max(0.5,0.5,0.4,0.5)} = \frac{0.5}{0.5} = 1.0$, $(1-f_{\lambda_V})^l = \frac{0.9}{\max(0.9,0.9,0.8,0.6)} = \frac{0.9}{0.9} = 1.0$.

Step 3: Computation of the Weight Matrix

The weights assigned by the experts (decision makers) to the criteria are given by the matrix:

$$W = \left[w_1 \text{ (EF)} = 0.3, \quad w_2 \text{ (Cost of Installation)} = 0.1, \quad w_3 \text{ (Durability)} = 0.2, \quad w_4 \text{ (Maintenance)} = 0.4\right]$$

Table 3: Weighted Normalized Decision Matrix

Weights w_j	0.3	0.1	0.2	0.4
	EF	**Cost of Installation**	**Durability**	**Maintenance**
Longi	[0.128,0.199],[0.3,0.3] [0.262,0.3]	[0.1,0.066],[0.0166,0.1] [0.025,0.025]	[0.057,0.133],[0.133,0.16] [0.2,0.177]	[0.2856,0.1332],[0.2,0.4] [0.2,0.2664]
Canadian	[0.15,0.3],[0.3,0.24] [0.18,0.225]	[0.0833,0.033],[0.075,0.0833] [0.025,0.025]	[0.2,0.166],[0.15,0.133] [0.05,0.044]	[0.1332,0.1332],[0.1,0.4] [0.32,0.1]
Jinko	[0.249,0.3],[0.15,0.18] [0.3,0.3]	[0.1,0.1],[0.05,0.08] [0.0875,0.088]	[0.1,0.1],[0.2,0.2] [0.114,0.044]	[0.2,0.05],[0.2664,0.4] [0.4,0.4]
JA	[0.123,0.09],[0.3,0.24] [0.3,0.3]	[0.1,0.033],[0.107587,0.1] [0.0875,0.0875]	[0.08,0.2],[0.2,0.16] [0.15,0.075]	[0.16, 0.4],[0.4,0.32] [0.2,0.25]

$$\mathbf{V}_i = \left(t^{\oplus}_{i1}, 1-f^{\oplus}_{i1}, t^{\oplus}_{i2}, 1-f^{\oplus}_{i2}, \ldots, t^{\oplus}_{ij}, 1-f^{\oplus}_{ij}, t^{\oplus}_{\lambda_V,i}, 1-f^{\oplus}_{\lambda_V,i}\right).$$

Step 4 : Identification of PIS and NIS

Compute PIS and NIS :

$$P^+_i = \begin{cases} \max_i V_{ij} & \text{for components corresponding to truth } (t^{\oplus}), \\ \min_i V_{ij} & \text{for components corresponding to falsity } (1-f^{\oplus}), \end{cases}$$

$$P^-_i = \begin{cases} \min_i V_{ij} & \text{for truth components,} \\ \max_i V_{ij} & \text{for falsity components.} \end{cases}$$

To find the PIS and NIS P^+, P^-

$$P^{+} = \begin{bmatrix} ([0.2856, 0.066], [0.3, 0.1]), [0.262, 0.025], \\ ([0.2, 0.033], [0.3, 0.0833]), [0.32, 0.025], \\ ([0.249, 0.05], [0.2664, 0.08]), [0.04, 0.044], \\ ([0.16, 0.033], [0.04, 0.1]), [0.3, 0.025] \end{bmatrix}$$

$$P^{-} = \begin{bmatrix} ([0.057, 0.199], [0.0166, 0.4]), [0.025, 0.3], \\ ([0.0833, 0.3], [0.075, 0.4]), [0.025, 0.225], \\ ([0.1, 0.3], [0.05, 0.4]), [0.0875, 0.4], \\ ([0.08, 0.4], [0.107587, 0.32]), [0.0875, 0.3] \end{bmatrix}$$

Step 5 : Compute Separation Measures by:

$$D_i^{+} = \sqrt{\sum_{j=1}^{n} \left(V_{ij} - P_i^{+}\right)^2}, \quad D_i^{-} = \sqrt{\sum_{j=1}^{n} \left(V_{ij} - P_i^{-}\right)^2}.$$

Table 4: Calculation of D_i^+, D_i^-

	D_i^+	D_i^-
Longi	(0.9482)	(0.8072)
Canadian	(0.9659)	(0.8056)
Jinko	(0.9624)	(0.7840)
JA	(0.9466)	(0.7869)

Step 6: Relative closeness to ideal solution

RCC to the ideal solution C_i is computed as follows:

$$C_{Longi} = \frac{D_1^-}{D_1^+ + D_1^-} = \frac{0.8072}{0.8072 + 0.9482} = 0.4598$$

Similarly, we can get

$$C_{Canadian} = 0.4547,$$
$$C_{Jinko} = 0.4489,$$
$$C_{JA} = 0.4539.$$

Step 7: Ranking closeness to ideal solution

The final ranking shows that the "Longi" emerged as the top choice, achieving the highest closeness coefficient among all the alternatives evaluated. This suggests that the "Longi" delivered the most favorable overall performance in all criteria, including energy efficiency, cost, durability and maintenance. "Canadian" and "JA" followed closely behind, while "Jinko" ranked slightly lower. The preference for the "Longi" indicates that decision-makers valued its strong all-around performance, where even if it involved trade-offs in specific factors, its overall reliability and balanced attributes made it the most desirable option.

7.4. Comparative Analysis

To demonstrate the practical relevance of the proposed extended TOPSIS method based on Cubic Vague Soft Set Relations (CVSSR), we perform a comparative numerical analysis. In this case study, the ranking outcomes of the proposed model are compared with those obtained using classical fuzzy TOPSIS and intuitionistic fuzzy TOPSIS approaches. The decision problem considers multiple renewable energy alternatives (e.g., solar, wind, hydro, biomass) evaluated across several criteria such as cost, efficiency, sustainability, and environmental impact.

The results show that the CVSSRTOPSIS model produces more stable and consistent rankings when parameter uncertainty and hesitation degrees are present. Specifically, the dual-interval structure of the CVSSR framework allows finer differentiation between close alternatives, leading to more reliable decision outcomes under imprecise information. This comparative analysis confirms that the proposed method enhances both decision accuracy and interpretability, demonstrating its superiority over traditional fuzzy-based approaches.

We investigated an analysis between our suggested method (CVSS-based TOPSIS), WASPAS [45], and TODIM [46]. The results are displayed in Table (5). Although there are some differences in the specific ranking orders (see Figure 2), it is clear that the three approaches consistently yield the best option. In particular, C_{Longi} is chosen as the best option by all technique. The MCDM problem in fuzzy environments is successfully addressed by the suggested CVSS-based TOPSIS technique. Furthermore, by constantly placing C_{Longi} at the top, our method exhibits robustness compared to the conventional WASPAS and TODIM methods, offering more reliable decision assistance in the alternative selection process.

Table 5: Comparison of ranking results based on different methods.

Method	**Ranking Results (C_i values)**	**Ranking Order**
TODIM [45]	$C_{\text{Longi}} = 0.6581,\ C_{\text{Cand}} = 0.5632,\ C_{\text{Jinko}} = 0.4421,\ C_{\text{JA}} = 0.2969$	$C_{\text{Longi}} \succ C_{\text{Cand}} \succ C_{\text{Jinko}} \succ C_{\text{JA}}$
WASPAS [46]	$C_{\text{Longi}} = 0.7403,\ C_{\text{Cand}} = 0.7155,\ C_{\text{Jinko}} = 0.6978,\ C_{\text{JA}} = 0.6741$	$C_{\text{Longi}} \succ C_{\text{Cand}} \succ C_{\text{Jinko}} \succ C_{\text{JA}}$
Proposed Method	$C_{\text{Longi}} = 0.4598,\ C_{\text{Cand}} = 0.4547,\ C_{\text{Jinko}} = 0.4489,\ C_{\text{JA}} = 0.4539$	$C_{\text{Longi}} \succ C_{\text{Cand}} \succ C_{\text{JA}} \succ C_{\text{Jinko}}$

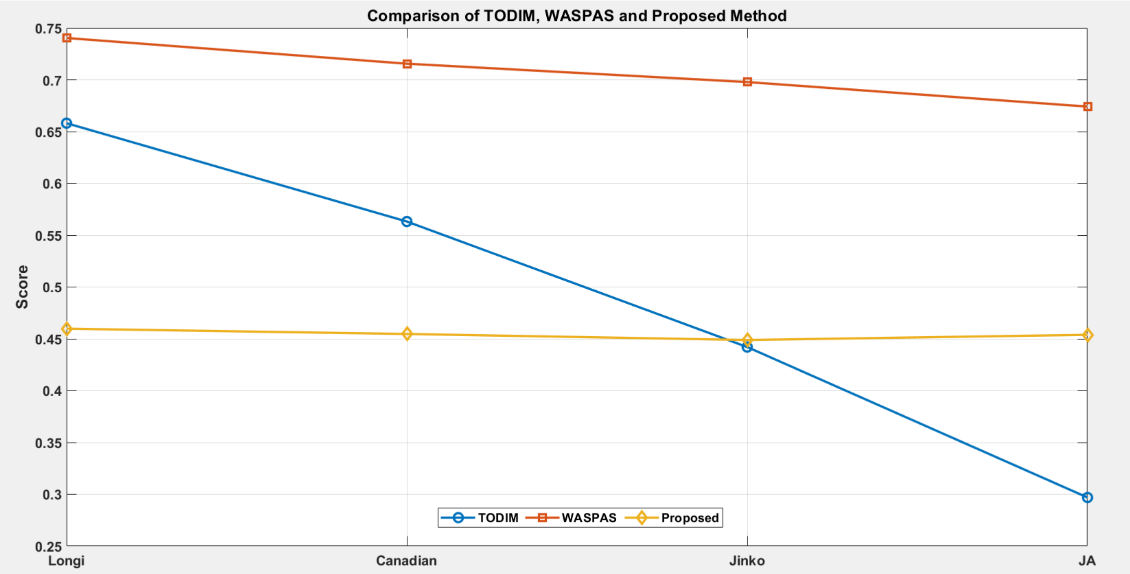

Figure 2: Comparison of ranking results based on different methods.

7.5. Advantages and Limitations of TOPSIS in Comparison with Other MADM Methods.

7.5.1. Advantages of TOPSIS in the Cubic Vague Soft Set Framework

The Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) offers several notable advantages when integrated with the Cubic Vague Soft Set (CVSS) model. Its greatest strength lies in its geometric simplicity and interpretability it measures the relative closeness of each alternative to a positive ideal solution (PIS) and a negative ideal solution (NIS), both of which can be naturally defined within the dual interval truth and falsity structure of CVSS. Unlike MABAC and MAIRCA, which rely on complex boundary or reference matrices that require additional transformations for interval-valued data, TOPSIS computes distances directly in the cubicvague domain without any modification of the membership structure. Similarly, compared with VIKOR, which depends on a compromise coefficient and requires balancing group utility and individual regret, TOPSIS remains parameter-free and therefore avoids sensitivity to arbitrary tuning values. Furthermore, the RAFSI approach depends on ratio-based normalization and linear aggregation, which can distort dual-interval data, while TOPSIS preserves both the truth and falsity intervals during the normalization and distance computation stages. Its computational efficiency, ease of integration with other fuzzy generalizations, and wide recognition in the MCDM literature make it an ideal candidate for extending to the CVSS environment. Consequently, the CVSSTOPSIS combination provides a mathematically consistent and practically interpretable decision-support tool for handling dual-interval and hesitant information in renewable energy evaluations.

7.5.2. Limitations of TOPSIS Compared with Other MADM Methods

Despite its strengths, the TOPSIS method also has several limitations when compared with MABAC, MAIRCA, VIKOR, and RAFSI. A key assumption of TOPSIS is that criteria are independent and equally stable, which may not hold in real decision systems where trade-offs or correlations exist among criteria such as cost, efficiency, and maintenance. Methods such as VIKOR and MAIRCA sometimes better capture compromise behavior and allow for flexible weighting or preference modeling. In addition, TOPSIS applies deterministic normalization and Euclidean distance metrics, which may oversimplify non-linear relationships between interval-valued and vague components. In contrast, MABAC and RAFSI can incorporate different aggregation rules or benefit cost separation, offering richer modeling flexibility in some contexts. Nevertheless, these alternative methods become mathematically cumbersome when extended to the CVSS domain, as they require redefining boundary areas, ratio normalizations, or reference ideals under dual-interval uncertainty. Therefore, while TOPSIS may not fully capture interactive or compensatory effects among criteria, its mathematical compatibility with the CVSS structure, conceptual transparency, and stable ranking performance justify its use in this study as the most suitable and computationally efficient approach for multi-criteria decision-making under complex vague interval uncertainty.

8. Conclusion

The soft set theory provides a general framework for addressing problems that involve uncertainty. In this paper, we focus on the theoretical foundations of cubic vague soft sets. Specifically, we expand existing concepts of relations and functions within the context of vague soft sets. We also explore how certain theories about relations and functions can be reinterpreted through this lens. These foundational ideas serve as essential tools for further research and development in vague soft set theory.

Inspired by the concepts discussed in this work, one could explore the ideas of multi-soft sets and soft multi-sets, where multi-sets allow repeated elements. Delving deeper into the theoretical aspects of these extended models could prove valuable and deserves greater attention. This line of research may help build stronger theoretical foundations for applications in soft computing. Furthermore, future work could investigate the topological structures generated by vague soft-set relations, opening the door to studying the topological characteristics of soft sets.

Moreover, the concept of cubic vague soft set relations can be extended to other frameworks such as intuitionistic soft sets, enabling more reliable solutions in real-world decision-making problems similar to generalized vague soft expert sets [47, 48] and other cubic set models [49–51]. Our future goal is to further develop these ideas by applying cubic soft set relations to areas such as Q-neutrosophic models [52], stock portfolio analysis [53], and numerical convergence studies [54–58].

List of Symbol

For clarity, the principal symbols and notations employed throughout this study are summarized in Table 6.

Table 6: List of Symbols and Their Descriptions

Symbol	Description
U	Universal set of elements (e.g., alternatives or objects).
A, B, C	Sets of parameters associated with vague or cubic vague soft sets.
(F, A)	Vague soft set defined over the universe U.
$(\tilde{F}, A_V)$	Cubic vague soft set (CVSS) with truth- and falsity-membership intervals.
$t^-_{A_V}(x)$, $t^+_{A_V}(x)$	Lower and upper bounds of the truth-membership degree for element x.
$f^-_{A_V}(x)$, $f^+_{A_V}(x)$	Lower and upper bounds of the falsity-membership degree for element x.
$\tilde{R}$	Cubic vague soft relation between two CVSSs.
$S \circ R$	Composition of two cubic vague soft relations.
PIS, NIS	Positive and negative ideal solutions in the extended TOPSIS framework.
D^+_i, D^-_i	Euclidean distances of the i-th alternative from the PIS and NIS, respectively.
RC_i	Relative closeness coefficient used for final ranking of alternatives.

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Appendix

Appendix 1. *Proof.*

(i) Let $\tilde{G}(b) \in f(\tilde{F}, \mathbb{A}_1^V)$.
Let $\tilde{G}(b) = f(\tilde{F}(a))$ such that $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}_1^V)$.
Since $(\tilde{F}, \mathbb{A}_1^V) \subseteq (\tilde{F}, \mathbb{A}_2^V)$, then $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}_2^V)$ and $\tilde{G}(b) = f(\tilde{F}(a)) \in f(\tilde{F}, \mathbb{A}_2^V)$.
Therefore $f(\tilde{F}, \mathbb{A}_1^V) \subseteq f(\tilde{F}, \mathbb{A}_2^V)$.

(ii) Let $\tilde{G}(b) \in f[(\tilde{F}, \mathbb{A}_1^V) \cup (\tilde{F}, \mathbb{A}_2^V)]$.
Then $\tilde{G}(b) = f(\tilde{F}(a))$ such that $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}_1^V) \cup (\tilde{F}, \mathbb{A}_2^V)$.
By using union definition, we have $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}_1^V)$ or $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}_2^V)$.
$\Rightarrow \tilde{G}(b) \in f(\tilde{F}, \mathbb{A}_1^V)$ or $\tilde{G}(b) \in f(\tilde{F}, \mathbb{A}_2^V)$.
$\Rightarrow \tilde{G}(b) \in f(\tilde{F}, \mathbb{A}_1^V) \cup f(\tilde{F}, \mathbb{A}_2^V)$.
Therefore $f[(\tilde{F}, \mathbb{A}_1^V) \cup (\tilde{F}, \mathbb{A}_2^V)] = f(\tilde{F}, \mathbb{A}_1^V) \cup f(\tilde{F}, \mathbb{A}_2^V)$.

(iii) Let $\tilde{G}(b) \in f[(\tilde{F}, \mathbb{A}_1^V) \cap (\tilde{F}, \mathbb{A}_2^V)]$.
Let $\tilde{G}(b) = f(\tilde{F}(a))$ such that $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}_1^V) \cap (\tilde{F}, \mathbb{A}_2^V)$.
By using intersection definition, we have $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}_1^V)$ and $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}_2^V)$.
$\Rightarrow \tilde{G}(b) \in f(\tilde{F}, \mathbb{A}_1^V)$ and $\tilde{G}(b) \in f(\tilde{F}, \mathbb{A}_2^V)$.
$\Rightarrow \tilde{G}(b) \in f(\tilde{F}, \mathbb{A}_1^V) \cap f(\tilde{F}, \mathbb{A}_2^V)$.
Therefore $f[(\tilde{F}, \mathbb{A}_1^V) \cap (\tilde{F}, \mathbb{A}_2^V)] \subseteq f(\tilde{F}, \mathbb{A}_1^V) \cap f(\tilde{F}, \mathbb{A}_2^V)$.

Conversely suppose $\tilde{G}(b) \in f(\tilde{F}, \mathbb{A}_1^V) \cap f(\tilde{F}, \mathbb{A}_2^V)$.
By using intersection definition, we have
$\tilde{G}(b) \in (\tilde{F}, \mathbb{A}_1^V)$ and $\tilde{G}(b) \in (\tilde{F}, \mathbb{A}_2^V)$.
Let $\tilde{G}(b) = f(\tilde{F}(a_1))$ such that $\tilde{F}(a_1) \in (\tilde{F}, \mathbb{A}_1^V)$ and $\tilde{G}(b) = f(\tilde{F}(a_2))$ such that $\tilde{F}(a_2) \in (\tilde{F}, \mathbb{A}_2^V)$.
Case 1:$f(\tilde{F}(a_1)) \neq f(\tilde{F}(a_2)) \Rightarrow \tilde{F}(a_1) \neq \tilde{F}(a_2)$
$\Rightarrow \tilde{F}(a_1) \in (\tilde{F}, \mathbb{A}_2^V)$ and $\tilde{F}(a_2) \notin (\tilde{F}, \mathbb{A}_1^V)$,
Then $\exists\, \tilde{G}(b) \notin f((\tilde{F}, \mathbb{A}_1^V) \cap (\tilde{F}, \mathbb{A}_2^V))$.
Therefore $f[(\tilde{F}, \mathbb{A}_1^V) \cap (\tilde{F}, \mathbb{A}_2^V)] \subseteq f(\tilde{F}, \mathbb{A}_1^V) \cap f(\tilde{F}, \mathbb{A}_2^V)$.
Case 2:$f(tildeF(a_1)) = f(\tilde{F}(a_2)) \Rightarrow \tilde{F}(a_1) = \tilde{F}(a_2)$.
$\Rightarrow \tilde{F}(a_1) \in (\tilde{F}, \mathbb{A}_1^V)$ and $\tilde{F}(a_1) \in (tildeF, \mathbb{A}_2^V)$,
Then $\tilde{G}(b) \in f((\tilde{F}, \mathbb{A}_1^V) \cap (\tilde{F}, \mathbb{A}_2^V))$.
Therefore $f[(\tilde{F}, \mathbb{A}_1^V) \cap (\tilde{F}, \mathbb{A}_1^V)] = f(\tilde{F}, \mathbb{A}_1^V) \cap f(\tilde{F}, \mathbb{A}_2^V)$, when f is one to one.

Appendix 2. *Proof.* Let $\tilde{G}(b_1) \neq \tilde{G}(b_2)$ for $\tilde{G}(b_1)$ and $\tilde{G}(b_2)$ in $(\tilde{G}, \mathbb{B}^V)$.
Let $f^{-1}(\tilde{G}(b_1)) = \tilde{F}(a_1)$ and $f^{-1}(\tilde{G}(b_2)) = \tilde{F}(a_2)$.
Then $f(\tilde{F}(a_1)) = \tilde{G}(b_1)$ and $f(\tilde{F}(a_2)) = \tilde{G}(b_2)$.
Thus $f(F(a_1)) \neq f(F(a_2))$
$\Rightarrow \tilde{F}(a_1) \neq \tilde{F}(a_2)$ since f is one to one
$\Rightarrow f^{-1}(\tilde{G}(b_1)) \neq f^{-1}(\tilde{G}(b_2))$. Hence f is one to one.

Now $\tilde{F}(a)$ is an element of $(\tilde{F}, \mathbb{A}^V)$. Since f is surjective there exists a unique element $\tilde{G}(b)$ in $(\tilde{G}, \mathbb{B}^V)$ such that $f(\tilde{F}(a)) = \tilde{G}(b)$
$\Rightarrow \tilde{F}(a) = f^{-1}(\tilde{G}(b))$ for $\tilde{F}(a)$ in $(\tilde{F}, \mathbb{A}^V)$.
Thus f^{-1} is onto. Hence f^{-1} is bijective.

Appendix 3. *Proof.* Let $\tilde{H}(c) \in (\tilde{H}, \mathbb{C}^V)$, $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}^V)$ and $\tilde{G}(b) \in (\tilde{G}, \mathbb{B}^V)$.
To prove $(g\,o\,f)$ is bijective, we need to show $(g\,o\,f)$ is surjective and injective.
To show $(g\,o\,f)$ is surjective, we need to prove $\forall\, \tilde{H}(c) \in (\tilde{H}, \mathbb{C}^V)$, $\exists$ at least one element $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}^V)$ such that $(g\,o\,f)(\tilde{F}(a)) = \tilde{H}(c)$.
Since f is onto, then $\forall \tilde{G}(^V) \in (\tilde{G}, \mathbb{B}^V)$, $\exists$ at least one element $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}^V)$ such that $f(\tilde{F}(a)) = \tilde{G}(b) \in (\tilde{G}, ^V)$.
Since g is onto, then $\forall \tilde{H}(c) \in (\tilde{H}, \mathbb{C}^V)$, $\exists$ at least one element $\tilde{G}(b) \in (\tilde{G}, \mathbb{B}^V)$ such that $g(\tilde{G}(b)) = \tilde{H}(c) \in (\tilde{H}, \mathbb{C}^V)$.
And $(g\,o\,f)(\tilde{F}(a)) = g(f(\tilde{F}(a))) = g(\tilde{G}(b)) = \tilde{H}(c)$.
Therefore $\forall \tilde{H}(c) \in (\tilde{H}, \mathbb{C}^V)$, $\exists$ at least one element $\tilde{F}(a) \in (\tilde{F}, \mathbb{A}^V)$ such that $(g\,o\,f)(\tilde{F}(a)) = \tilde{H}(c)$.
Hence $(g\,o\,f)$ is surjective.
To show $(g\,o\,f)$ is injective, we need to prove
$(g\,o\,f)(\tilde{F}(a_1)) = (g\,o\,f)(\tilde{F}(a_2))$, where both elements in $(\tilde{H}, \mathbb{C}^V)$, if $\tilde{F}(a_1) = \tilde{F}(a_2)$.
$(g\,o\,f)(\tilde{F}(a_1)) = g(f(\tilde{F}(a_1))) = g(f(\tilde{F}(a_2)))$, since f is injective.
But then $g(f(\tilde{F}(a_2))) = (g\,o\,f)(\tilde{F}(a_2))$, since g is injective.
Thus it is shown that $(g\,o\,f)$ is injective. We can then conclude that $(g\,o\,f)$ is bijective since it is proven to be both surjective and injective. Since f, g and $g\,o\,f$ are bijective, they are invertible and for any relation $\Re$ and S we have $(S\,o\,\Re)^{-1} = \Re^{-1}\,o\,S^{-1}$. Thus we have in this case $(g\,o\,f)^{-1} = f^{-1}\,o\,g^{-1}$.