EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 7023
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Generalization of an Integral Related to Stieltjes Moment Problem

Irshad Ayoob

Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

---

Abstract. In connection with the non-uniqueness of the Stieltjes moment problem on $(0, \infty)$, Stieltjes constructed the nontrivial function $f(x) = e^{-x^{1/4}} \sin(x^{1/4})$ satisfying $\int_0^\infty x^n f(x)\, dx = 0$ for all integers $n \geq 0$. We extend this by considering

$$I(k) = \int_0^\infty e^{-x^{1/4}} \sin(x^{1/4})\, x^k\, dx$$

for real $k \geq 0$, evaluating $I(k)$ explicitly and proving $I(k) = 0$ if and only if $k \in \mathbb{Z}_{\geq 0}$. More generally, for parameters $m > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, $q, k \in \mathbb{R}$ we analyze

$$I_{m,q}(k, \alpha, \beta) = \int_0^\infty e^{-\alpha x^{1/m}} \sin(\beta x^{1/m})\, x^k (x^{1/m})^q\, dx,$$

derive a closed form, and give necessary and sufficient conditions for its vanishing. We also establish cosine analogues, both for the Stieltjes example and for the generalized integral mentioned above. As a consequence, we obtain integral representations of $\Gamma(A)$ for suitable $A > 0$, as well as integral formulas for several classical constants arising from gamma function. To understand the importance of integrals that vanish for every value of a continuous parameter, we will also discuss Salem's equivalence of the Riemann hypothesis, which is formulated in terms of such a parameter-dependent integral.

2020 Mathematics Subject Classifications: 44A60, 33B15, 26B15

Key Words and Phrases: Moment problem, gamma function, definite integral

---

1. Introduction

Definite integrals are widely used in both pure and applied mathematics. While many can be solved with basic methods like substitution or integration by parts, numerous integrals are non-elementary and cannot be expressed using standard functions. Evaluating these requires advanced techniques, such as transformations (Laplace, Mellin), complex

DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.7023

Email address: `iayoub@psu.edu.sa` (I. Ayoob)

analysis (contour integration, residues), and special functions (Gamma, Beta, hypergeometric). Exact solutions are valuable for precise predictions, deeper theoretical insight, and verifying numerical methods. Consequently, research on advanced methods for definite integrals remains active, with comprehensive compilations and recent studies highlighting ongoing developments and continued interest in the field. For the extensive list of definite integrals and some recent recsults, one may look for the references [1–10]. This paper deals with the evaluation of an integral related to the Steiltjes moment problem. We recall few definitions from the moment problem.

The classical moment problem asks [11]: given a sequence of real numbers

$$\{m_n\}_{n=0}^{\infty},$$

does there exist a positive Borel measure μ on some subset of $\mathbb{R}$ such that

$$m_n = \int x^n \, d\mu(x), \qquad n = 0, 1, 2, \ldots?$$

Depending on the support allowed for μ, one obtains three classical versions as stated below.

Definition 1 (Hamburger moment problem)**.** *This problem seeks the measure μ on the entire real line $(-\infty, \infty)$. That is, given $\{m_n\}$, does there exist a positive measure μ with*

$$m_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \qquad n = 0, 1, 2, \ldots?$$

This is the most general form of the problem.

Definition 2 (Stieltjes moment problem)**.** *This problem seeks the measure μ on the half–line $[0, \infty)$. That is, one asks whether there exists μ with*

$$m_n = \int_{0}^{\infty} x^n \, d\mu(x), \qquad n = 0, 1, 2, \ldots?$$

Such sequences $\{m_n\}$ are called Stieltjes moment sequences. This version is closely connected to continued fractions and orthogonal polynomials on $[0, \infty)$.

Definition 3 (Hausdorff moment problem)**.** *Here the measure μ is confined to the compact interval $[0, 1]$. That is, does there exist μ with*

$$m_n = \int_{0}^{1} x^n \, d\mu(x), \qquad n = 0, 1, 2, \ldots?$$

In every version of the moment problem, the issue of uniqueness is crucial. For the Stieltjes and Hamburger cases, whenever a solution exists it need not be unique, there may be infinitely many measures producing the same moment sequence (the indeterminate case). By contrast, the Hausdorff moment problem, when solvable, always yields a single measure, making it determinate. Thus, in the indeterminate situations, one encounters infinitely many distinct measures that share the same prescribed moments. The non-uniqueness of moments is classically studied through Carleman's condition and Krein's condition. Related to Hausdorff moment problem, we have the classical result proved through Weierstrass approximation theorem,

Proposition 1. *[12] Let $f \in C([0,1])$. If*

$$\int_0^1 x^n f(x)\,\mathrm{d}x = 0 \qquad \textit{for all } n = 0,1,2,\dots,$$

then $f \equiv 0$ on $[0,1]$.

About the infinite case, we may pose a similar question:

Question 1. *Suppose $f \in C([0,\infty))$ and*

$$\int_0^\infty x^n f(x)\,\mathrm{d}x = 0 \qquad \textit{for all } n = 0,1,2,\dots.$$

Does it follow that $f \equiv 0$ on $[0,\infty)$?

In relation to the non-uniqueness of the moment problem, Stieltjes showed [13](see p.506) that function f in Question 1 may be non-zero by giving the following example:

Example 1. *The integral*

$$I(n) = \int_0^\infty x^n f(x)\,\mathrm{d}x = 0 \qquad (n = 0,1,2,\dots), \tag{1}$$

for the non-trivial function,

$$f(x) = e^{-x^{1/4}} \sin\left(x^{1/4}\right), \qquad x \geq 0. \tag{2}$$

This paper studies the following questions related to Stieltjes Example 1:

Question 2. *The integral $I(n) = \int_0^\infty x^n f(x)\,\mathrm{d}x = 0$ for $f(x) = e^{-x^{1/4}} \sin\left(x^{1/4}\right), x \geq 0$ for all $n = 0,1,2,\dots$. What if non-negative integer n is replaced by non-negative real number k, that is, for what real numbers $k \geq 0$, we have that $I(k) = \int_0^\infty x^k e^{-x^{1/4}} \sin\left(x^{1/4}\right)\mathrm{d}x = 0$.*

The second question deals with the following generalization of the Steiltjes integral.

Question 3. *Evaluate the following generalized integral explicitly (for appropriate parameters) and determine the necessary and sufficient conditions on the parameters so that the integral vanishes,*

$$I_{m,q}(k) = \int_0^\infty e^{-\alpha x^{1/m}} \sin\bigl(\beta x^{1/m}\bigr)\, x^k\, (x^{1/m})^q\, dx, \tag{3}$$

where $k \geq 0$ is a real number.

We will solve the integrals $I(k)$ and $I_{m,q}(k)$ given in Questions 2 and 3 explicitly, and establish the necessary and sufficient conditions for the vanishing of these of integrals. This question is interesting as there exist integrals which vanish for all values of the continuous parameter (not only on discrete subsets of reals as in the case of Example 1 with parameter n.)
For instance,

Example 2.

$$I(a) = \int_0^\infty \frac{\sin(ax) - \sin((a+1)x)}{x}\, dx = 0, \qquad a > 0.$$

This follows from the Dirichlet integral $\int_0^\infty \frac{\sin(\alpha x)}{x}\, dx = \frac{\pi}{2}\operatorname{sgn}(\alpha)$*, and since* $a > 0$ *and* $a + 1 > 0$*,*

$$I(a) = \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

Thus

$$I(a) = 0 \quad \text{for } a > 0.$$

Another motivating example for studying Question 1 and Question 2, and more broadly, integrals that vanish for every value of a continuous parameter, is Salem's equivalence of the Riemann Hypothesis. This equivalence is expressed through an integral involving a continuous parameter, as stated below:

Theorem 1. *[14] The Riemann hypothesis is true if and only if the integral equation*

$$\int_0^\infty \frac{x^{r-1} f(x)}{e^{xt} + 1}\, dx = 0, \qquad \text{for all } t > 0,$$

admits only the trivial bounded measurable solution $f(x) \equiv 0$*, for a fixed parameter* r *satisfying* $\frac{1}{2} < r < 1$*.*

Equivalently, we we may write Theorem 1 as following:

Theorem 2. *The Riemann hypothesis is false if and only if there exists a nontrivial bounded measurable function* $f(x) \not\equiv 0$ *satisfying the integral equation*

$$\int_0^\infty \frac{x^{r-1} f(x)}{e^{xt} + 1}\, dx = 0, \qquad \text{for all } t > 0,$$

for fixed parameter r *such that* $\frac{1}{2} < r < 1$*.*

Salem's criterion has been the subject of recent investigations by several authors ([15–18]) using a variety of integral transform techniques. In particular, the following result was obtained in [17] (see Corollary 3.3):

Theorem 3. *Let* $f(x)$ *be a bounded measurable function on* $\mathbb{R}_+$ *such that* $f(x) = O(x^{1/2})$ *as* $x \to 0^+$*. If* $f(x)$ *satisfies the integral equation given in Theorem 1 for given* k *with* $\frac{1}{2} < k < 1$ *and for all* $t > 0$*, then* $f(x)$ *vanishes almost everywhere on* $\mathbb{R}_+$*.*

It is important to note that if there exists a nontrivial bounded measurable function $f(x)$ satisfying the integral equation presented in Theorem 2 for all continuous parameters $t > 0$, then such an $f(x)$ would constitute a counterexample to Salem's equivalence of the Riemann Hypothesis. Emphasis should be placed on the fact that this nontrivial bounded function must cause the integral on the left-hand side of Theorem 2 to vanish for all $t > 0$. Consequently, the study of such parameter-dependent improper integrals that vanish for every value of a continuous parameter becomes particularly interesting depending upon the context. This paper examines a parameter-dependent improper integral, inspired by the Stieltjes moment problem, which vanishes for all continuous parameter values.

2. Main Results

We begin with the explicit evaluation of Steiltjes integral.

Theorem 4. *For a real number $k > -1$ define*

$$I(k) = \int_0^\infty e^{-x^{1/4}} \sin\bigl(x^{1/4}\bigr)\, x^k \, dx. \tag{4}$$

Then $I(k)$ is absolutely convergent for $k > -1$ and,

$$I(k) = 2^{-2k}\, \Gamma(4k+4)\, \sin\bigl((k+1)\pi\bigr), \tag{5}$$

and, in particular, $I(k) = 0$ if and only if $k \in \{0, 1, 2, \dots\}$.

Proof. Absolute convergence of (4) holds for $k > -1$ because on $[0, 1]$,

$$\bigl|e^{-x^{1/4}} \sin(x^{1/4})\, x^k\bigr| \le x^{1/4} x^k = x^{k+1/4} \le x^k, \qquad \int_0^1 x^k \, dx = \frac{1}{k+1} < \infty,$$

and on $[1, \infty)$ we have $|\sin(x^{1/4})| \le 1$ and the factor $e^{-x^{1/4}}$ decays super–polynomially, so $\int_1^\infty e^{-x^{1/4}} x^k \, dx < \infty$. We make the substitution,

$$u = x^{1/4}, \qquad x = u^4, \qquad dx = 4u^3 \, du, \tag{6}$$

to obtain

$$I(k) = \int_0^\infty e^{-u} \sin(u)\, (u^4)^k\, (4u^3)\, du = 4 \int_0^\infty e^{-u} \sin(u)\, u^{4k+3} \, du. \tag{7}$$

We denote real and imaginary parts by $\Re$ and $\Im$, and write $\sin u = \Im(e^{iu})$, we have

$$e^{-u} \sin u = \Im\bigl(e^{-u} e^{iu}\bigr) = \Im\bigl(e^{-(1-i)u}\bigr), \quad \Rightarrow \quad \int_0^\infty u^{4k+3} e^{-u} \sin u \, du = \Im \int_0^\infty u^{4k+3} e^{-(1-i)u} \, du. \tag{8}$$

Combining (7) and (8) yields the identity,

$$I(k) = 4\, \Im \int_0^\infty u^{4k+3}\, e^{-(1-i)u} \, du. \tag{9}$$

Now we set,

$$a = 4k + 4 > 0, \qquad \lambda = 1 - i \quad (\Re\lambda = 1 > 0). \tag{10}$$

By the Gamma function (with the substitution $t = \lambda u$), we obtain

$$\int_0^\infty u^{a-1} e^{-\lambda u}\, du = \int_0^\infty \left(\frac{t}{\lambda}\right)^{a-1} e^{-t}\, \frac{dt}{\lambda} = \lambda^{-a} \int_0^\infty t^{a-1} e^{-t}\, dt = \frac{\Gamma(a)}{\lambda^a}, \tag{11}$$

we have, with $a = 4k + 4$,

$$\int_0^\infty u^{4k+3} e^{-(1-i)u}\, du = \frac{\Gamma(4k+4)}{(1-i)^{\,4k+4}}. \tag{12}$$

Next we compute the complex power explicitly using the polar form $1 - i = \sqrt{2}\, e^{-i\pi/4}$:

$$(1-i)^{-(4k+4)} = \left(\sqrt{2}\, e^{-i\pi/4}\right)^{-(4k+4)} = 2^{-(2k+2)}\, e^{\,i(4k+4)\pi/4} = 2^{-(2k+2)}\, e^{\,i(k+1)\pi}. \tag{13}$$

Substituting (12) and (13) into (9) and taking imaginary parts gives,

$$\begin{aligned} I(k) &= 4\, \Im\Big(\Gamma(4k+4)\, (1-i)^{-(4k+4)}\Big) \\ &= 4\, \Gamma(4k+4)\, 2^{-(2k+2)}\, \Im\big(e^{\,i(k+1)\pi}\big) \\ &= 2^{-2k}\, \Gamma(4k+4)\, \sin\big((k+1)\pi\big), \end{aligned} \tag{14}$$

which proves (5). Since $2^{-2k} > 0$ and $\Gamma(4k+4) > 0$ for $k > -1$, the equality $I(k) = 0$ holds if and only if $\sin((k+1)\pi) = 0$, i.e.,

$$(k+1)\pi = m\pi \text{ for some } m \in \mathbb{Z} \quad \Longleftrightarrow \quad k = m - 1.$$

Under the constraint $k > -1$, this forces $m \geq 1$, so precisely $k \in \{0, 1, 2, \ldots\}$ yield $I(k) = 0$, and conversely any such k makes $\sin((k+1)\pi) = 0$, establishing the stated equivalence.

Now we have the following generalization of Theorem 4.

Remark 1. *The above theorem provides an explicit closed-form evaluation of the integral $I(k)$ in terms of the Gamma function and the sine function. It demonstrates that the vanishing of $I(k)$ occurs only for nonnegative integer values of k.*

Theorem 5. *Let $m > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, $q \in \mathbb{R}$, and $k \in \mathbb{R}$. Define*

$$I_{m,q}(k, \alpha, \beta) = \int_0^\infty e^{-\alpha x^{1/m}}\, \sin\big(\beta x^{1/m}\big)\, x^k\, (x^{1/m})^q\, dx, \tag{15}$$

and set

$$A = mk + q + m. \tag{16}$$

Then the integral in (15) *converges for every* $A > -1$*. Moreover, if* $A > 0$ *(which we assume for the evaluation below), one has the closed form*

$$I_{m,q}(k,\alpha,\beta) = m\,\Gamma(A)\,(\alpha^2+\beta^2)^{-A/2}\,\sin\Big(A\,\arctan\frac{\beta}{\alpha}\Big). \tag{17}$$

Consequently, if $\beta \neq 0$ *then* $I_{m,q}(k,\alpha,\beta) = 0$ *if and only if*

$$A\,\arctan\frac{\beta}{\alpha} \in \pi\mathbb{Z}; \tag{18}$$

if $\beta = 0$ *then* $I_{m,q}(k,\alpha,0) = 0$ *for all admissible parameters.*

Proof. Write the integrand as $e^{-\alpha x^{1/m}}\sin(\beta x^{1/m})\,x^{k+q/m}$. For convergence near $x = 0$, use $|\sin t| \le \min\{1, |t|\}$. As $x \downarrow 0$, we have $t = x^{1/m} \downarrow 0$, thus $|\sin(\beta x^{1/m})| \le |\beta|\,x^{1/m}$, which gives

$$\big|e^{-\alpha x^{1/m}}\sin(\beta x^{1/m})\,x^{k+q/m}\big| \ll x^{k+q/m+1/m} \qquad (x \downarrow 0). \tag{19}$$

Hence $\int_0^1 |e^{-\alpha x^{1/m}}\sin(\beta x^{1/m})\,x^{k+q/m}|\,dx < \infty$ provided $k + q/m + 1/m > -1$, i.e.

$$mk + q + m > -1 \iff A > -1. \tag{20}$$

As $x \to \infty$ we have $|\sin(\beta x^{1/m})| \le 1$ and the factor $e^{-\alpha x^{1/m}}$ decays faster than any power, so

$$\int_1^\infty \big|e^{-\alpha x^{1/m}}\sin(\beta x^{1/m})\,x^{k+q/m}\big|\,dx < \infty \quad \text{for every } k, q \text{ when } \alpha > 0. \tag{21}$$

Therefore (15) converges for all $A > -1$.

For evaluation assume $A > 0$. We make the substitution

$$u = x^{1/m}, \qquad x = u^m, \qquad dx = m\,u^{m-1}\,du, \tag{22}$$

and note $x^k(x^{1/m})^q = u^{mk+q}$. Then

$$\begin{aligned} I_{m,q}(k,\alpha,\beta) &= \int_0^\infty e^{-\alpha u}\,\sin(\beta u)\,u^{mk+q}\,(m\,u^{m-1})\,du \\ &= m\int_0^\infty e^{-\alpha u}\,\sin(\beta u)\,u^{A-1}\,du. \end{aligned} \tag{23}$$

Using $\sin(\beta u) = \Im(e^{i\beta u})$ and the absolute integrability guaranteed by $A > -1$ and $\alpha > 0$, we may pass the imaginary part inside the integral:

$$e^{-\alpha u}\sin(\beta u) = \Im\big(e^{-(\alpha-i\beta)u}\big) \implies I_{m,q}(k,\alpha,\beta) = m\,\Im\int_0^\infty u^{A-1}\,e^{-(\alpha-i\beta)u}\,du. \tag{24}$$

Put

$$\lambda = \alpha - i\beta \quad (\Re\lambda = \alpha > 0). \tag{25}$$

For $A > 0$ the Gamma gamma function (by the substitution $t = \lambda u$) gives,

$$\int_0^\infty u^{A-1}e^{-\lambda u}\,du = \int_0^\infty \left(\frac{t}{\lambda}\right)^{A-1} e^{-t}\,\frac{dt}{\lambda} = \lambda^{-A}\int_0^\infty t^{A-1}e^{-t}\,dt = \frac{\Gamma(A)}{\lambda^A}. \tag{26}$$

Applying (26) with (25) in (24) yields

$$I_{m,q}(k,\alpha,\beta) = m\,\Im\!\left(\frac{\Gamma(A)}{(\alpha - i\beta)^A}\right). \tag{27}$$

We write $\alpha - i\beta$ in polar form, with

$$r = \sqrt{\alpha^2+\beta^2}, \qquad \theta = \arctan\frac{\beta}{\alpha} \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right), \tag{28}$$

we have $\alpha - i\beta = r\,e^{-i\theta}$ and hence

$$(\alpha - i\beta)^{-A} = r^{-A}\,e^{iA\theta}. \tag{29}$$

Substituting (29) into (27) and using $\Im(e^{iA\theta}) = \sin(A\theta)$ gives

$$\begin{aligned} I_{m,q}(k,\alpha,\beta) &= m\,\Gamma(A)\,\Im\!\big(r^{-A}e^{iA\theta}\big) = m\,\Gamma(A)\,r^{-A}\,\sin(A\theta) \\ &= m\,\Gamma(A)\,(\alpha^2+\beta^2)^{-A/2}\,\sin\!\Big(A\,\arctan\frac{\beta}{\alpha}\Big), \end{aligned} \tag{30}$$

which is (17).
Since $m > 0$, $\Gamma(A) > 0$ for $A > 0$, and $(\alpha^2+\beta^2)^{-A/2} > 0$, the vanishing $I_{m,q}(k,\alpha,\beta) = 0$ (when $\beta \neq 0$) is equivalent to $\sin(A\theta) = 0$, i.e. (18). If $\beta = 0$ then $\sin(\beta x^{1/m}) \equiv 0$ and (15) gives $I_{m,q}(k,\alpha,0) = 0$.

The following corollary recovers Theorem 4 from Theorem 5.

Corollary 1. *Let $m = 4$, $\alpha = 1$, $\beta = 1$, $q = 0$. For $k > -1$,*

$$I_{4,0}(k,1,1) = \int_0^\infty e^{-x^{1/4}}\sin\!\big(x^{1/4}\big)\,x^k\,dx = 2^{-2k}\,\Gamma(4k+4)\,\sin\!\big((k+1)\pi\big).$$

Proof. Here $A = mk+q+m = 4k+4$, $r = \sqrt{\alpha^2+\beta^2} = \sqrt{2}$, and $\theta = \arctan(1) = \pi/4$. Substituting these into (17) gives

$$I_{4,0}(k,1,1) = 4\,\Gamma(4k+4)\,(\sqrt{2})^{-(4k+4)}\,\sin\!\big((4k+4)\tfrac{\pi}{4}\big) = 2^{-2k}\,\Gamma(4k+4)\,\sin\!\big((k+1)\pi\big).$$

We also have the following special cases.

Corollary 2 (Equal parameters $\alpha = \beta > 0$)**.** *For any $m > 0$ and any k, q with $A > 0$,*

$$I_{m,q}(k,\alpha,\alpha) = m\,\Gamma(A)\,2^{-A/2}\,\alpha^{-A}\,\sin\!\big(A\,\tfrac{\pi}{4}\big).$$

In particular, $I_{m,q}(k,\alpha,\alpha) = 0$ if and only if $A \in 4\mathbb{Z}$.

Proof. When $\alpha = \beta$, we have $r = \sqrt{2}\,\alpha$ and $\theta = \pi/4$, so (17) reduces to the displayed formula. The zero condition follows from $\sin(A\pi/4) = 0$.

Corollary 3 ($q = 0$)**.** *If $q = 0$ and $k > -1$, then $A = m(k+1)$ and*

$$I_{m,0}(k, \alpha, \beta) = m\,\Gamma\big(m(k+1)\big)\,(\alpha^2 + \beta^2)^{-m(k+1)/2}\,\sin\Big(m(k+1)\arctan\frac{\beta}{\alpha}\Big).$$

Proof. Put $q = 0$ in (17).

Next, we give the cosine variant of Theorem 4. The proof is similar as in Theorem 5 but for the completeness, we write it here.

Theorem 6. *Let $m > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, $q \in \mathbb{R}$, and $k \in \mathbb{R}$. Define*

$$J_{m,q}(k, \alpha, \beta) = \int_0^\infty e^{-\alpha x^{1/m}}\,\cos\big(\beta x^{1/m}\big)\,x^k\,(x^{1/m})^q\,dx, \tag{31}$$

and set

$$A = mk + q + m. \tag{32}$$

Then the integral in (31) converges for every $A > -1$. Moreover, if $A > 0$ (which we assume for the evaluation below), one has the closed form

$$J_{m,q}(k, \alpha, \beta) = m\,\Gamma(A)\,(\alpha^2 + \beta^2)^{-A/2}\,\cos\Big(A\,\arctan\frac{\beta}{\alpha}\Big). \tag{33}$$

Consequently, if $\beta \neq 0$ then $J_{m,q}(k, \alpha, \beta) = 0$ if and only if

$$A\,\arctan\frac{\beta}{\alpha} \in \frac{\pi}{2} + \pi\mathbb{Z}; \tag{34}$$

if $\beta = 0$ then

$$J_{m,q}(k, \alpha, 0) = m\,\Gamma(A)\,\alpha^{-A}. \tag{35}$$

Proof. We write the integrand as $e^{-\alpha x^{1/m}}\cos(\beta x^{1/m})\,x^{k+q/m}$. For $x \downarrow 0$ we use $|\cos t| \leq 1$ to obtain

$$\big|e^{-\alpha x^{1/m}}\cos(\beta x^{1/m})\,x^{k+q/m}\big| \ll x^{k+q/m} \qquad (x \downarrow 0), \tag{36}$$

so $\int_0^1 |e^{-\alpha x^{1/m}}\cos(\beta x^{1/m})\,x^{k+q/m}|\,dx < \infty$ provided $k + q/m > -1$, i.e.

$$mk + q > -1 \iff A - m > -1 \iff A > -1. \tag{37}$$

As $x \to \infty$ we have $|\cos(\beta x^{1/m})| \leq 1$ and the factor $e^{-\alpha x^{1/m}}$ yields super–polynomial decay, hence

$$\int_1^\infty \big|e^{-\alpha x^{1/m}}\cos(\beta x^{1/m})\,x^{k+q/m}\big|\,dx < \infty \quad \text{for every } k, q \text{ when } \alpha > 0. \tag{38}$$

Therefore (31) converges for all $A > -1$.

Assume $A > 0$ for evaluation. We make the substitution

$$u = x^{1/m}, \qquad x = u^m, \qquad dx = m\, u^{m-1}\, du, \tag{39}$$

and note $x^k (x^{1/m})^q = u^{mk+q}$, to obtain

$$\begin{aligned} J_{m,q}(k,\alpha,\beta) &= \int_0^\infty e^{-\alpha u} \cos(\beta u)\, u^{mk+q}\, (m\, u^{m-1})\, du \\ &= m \int_0^\infty e^{-\alpha u} \cos(\beta u)\, u^{A-1}\, du. \end{aligned} \tag{40}$$

Using $\cos(\beta u) = \Re\big(e^{i\beta u}\big)$ and absolute integrability for $A > -1$, we may pass the real part inside the integral:

$$e^{-\alpha u} \cos(\beta u) = \Re\big(e^{-(\alpha - i\beta)u}\big) \implies J_{m,q}(k,\alpha,\beta) = m\, \Re \int_0^\infty u^{A-1}\, e^{-(\alpha - i\beta)u}\, du. \tag{41}$$

Let

$$\lambda = \alpha - i\beta \qquad (\Re\lambda = \alpha > 0). \tag{42}$$

For $A > 0$, the Gamma function, obtained by the substitution $t = \lambda u$, gives

$$\int_0^\infty u^{A-1} e^{-\lambda u}\, du = \int_0^\infty \Big(\frac{t}{\lambda}\Big)^{A-1} e^{-t}\, \frac{dt}{\lambda} = \frac{1}{\lambda^A} \int_0^\infty t^{A-1} e^{-t}\, dt = \frac{\Gamma(A)}{\lambda^A}. \tag{43}$$

Applying (43) in (41) yields

$$J_{m,q}(k,\alpha,\beta) = m\, \Re\bigg(\frac{\Gamma(A)}{(\alpha - i\beta)^A} \bigg). \tag{44}$$

We write $\alpha - i\beta$ in polar form. With

$$r = \sqrt{\alpha^2 + \beta^2}, \qquad \theta = \arctan\frac{\beta}{\alpha} \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big), \tag{45}$$

we have $\alpha - i\beta = r\, e^{-i\theta}$, hence

$$(\alpha - i\beta)^{-A} = r^{-A}\, e^{iA\theta}. \tag{46}$$

Substituting (46) into (44) and using $\Re(e^{iA\theta}) = \cos(A\theta)$ gives

$$\begin{aligned} J_{m,q}(k,\alpha,\beta) &= m\, \Gamma(A)\, \Re\big(r^{-A} e^{iA\theta}\big) = m\, \Gamma(A)\, r^{-A} \cos(A\theta) \\ &= m\, \Gamma(A)\, (\alpha^2 + \beta^2)^{-A/2} \cos\Big(A \arctan\frac{\beta}{\alpha}\Big), \end{aligned} \tag{47}$$

which proves (33).
The zero condition (34) follows since $m > 0$, $\Gamma(A) > 0$ for $A > 0$, and $(\alpha^2 + \beta^2)^{-A/2} > 0$, so $J_{m,q}(k,\alpha,\beta) = 0$ if and only if $\cos(A\theta) = 0$, i.e. $A\theta \in \frac{\pi}{2} + \pi\mathbb{Z}$. When $\beta = 0$, we have $\theta = 0$ and $\cos(A\theta) = \cos 0 = 1$, and (47) reduces to (35).

We have the following cosine analog of the Steiltjes example.

Corollary 4. *Let $m = 4$, $\alpha = 1$, $\beta = 1$, $q = 0$. For $k > -1$,*

$$G(k) = \int_0^\infty e^{-x^{1/4}} \cos\big(x^{1/4}\big)\, x^k\, dx = 2^{-2k}\, \Gamma(4k+4)\, \cos\big((k+1)\pi\big).$$

Furthermore, $G(k) = 0$ if and only if $k = n - \frac{1}{2}$, $\quad n \in \{0, 1, 2, 3....\}$.

Proof. Here $A = 4k + 4$, $r = \sqrt{2}$, and $\theta = \arctan(1) = \pi/4$. Substituting into (33),

$$J_{4,0}(k, 1, 1) = 4\,\Gamma(4k+4)\,(\sqrt{2})^{-(4k+4)}\, \cos\big((4k+4)\tfrac{\pi}{4}\big) = 2^{-2k}\,\Gamma(4k+4)\,\cos\big((k+1)\pi\big).$$

Note that in Corollary 4, for the vanishing of $G(k)$, the values of k are not non-negative fractions, whereas the Steiltjes type moment problem requires k to be non-negative integers. To have k non-negative integers and a cosine variant of Steiltjes , we have the following result.

Corollary 5. *Let $m > 2$ and $\alpha > 0$, and set*

$$\theta = \frac{\pi}{m}, \qquad \beta = \alpha \tan\theta = \alpha \tan\Big(\frac{\pi}{m}\Big), \qquad q = -\frac{m}{2}. \tag{48}$$

For $k \in \mathbb{R}$ define

$$J_{m,q}(k, \alpha, \beta) = \int_0^\infty e^{-\alpha x^{1/m}}\, \cos\big(\beta x^{1/m}\big)\, x^k\, (x^{1/m})^q\, dx. \tag{49}$$

Then $J_{m,q}(k, \alpha, \beta)$ converges for $k > -\frac{1}{2}$ and

$$J_{m,q}(k, \alpha, \beta) = m\,\Gamma\big(m(k + \tfrac{1}{2})\big)\,(\alpha^2 + \beta^2)^{-\frac{m(k+1/2)}{2}}\, \cos\Big(\pi\big(k + \tfrac{1}{2}\big)\Big). \tag{50}$$

In particular,

$$J_{m,q}(k, \alpha, \beta) = 0 \quad \iff \quad k \in \mathbb{Z}_{\geq 0}. \tag{51}$$

Proof. By Theorem 6, for $A = mk + q + m$ and $\theta = \arctan(\beta/\alpha) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ one has

$$J_{m,q}(k, \alpha, \beta) = m\,\Gamma(A)\,(\alpha^2 + \beta^2)^{-A/2}\, \cos\big(A\theta\big), \tag{52}$$

whenever $A > 0$ (and convergence holds for $A > -1$). With the parameter choice (48) we have $\theta = \pi/m \in (0, \frac{\pi}{2})$, $A = mk + q + m = m\big(k + \frac{1}{2}\big)$, and hence

$$A\theta = m\Big(k + \tfrac{1}{2}\Big) \cdot \frac{\pi}{m} = \pi\Big(k + \tfrac{1}{2}\Big). \tag{53}$$

Substituting (53) and $A = m(k + \frac{1}{2})$ into (52) yields (50). Since $A > 0$ is equivalent to $k > -\frac{1}{2}$, convergence and evaluation hold precisely on $k > -\frac{1}{2}$. In (50) the prefactor $m\,\Gamma(m(k + \frac{1}{2}))\,(\alpha^2 + \beta^2)^{-m(k+1/2)/2}$ is strictly positive for $k > -\frac{1}{2}$, so

$$J_{m,q}(k, \alpha, \beta) = 0 \quad \iff \quad \cos\big(\pi(k + \tfrac{1}{2})\big) = 0 \quad \iff \quad k + \tfrac{1}{2} \in \mathbb{Z} \quad \iff \quad k \in \mathbb{Z}.$$

Intersecting with $k > -\frac{1}{2}$ gives $k \in \{0, 1, 2, \ldots\}$, which is (51).

Corollary 5 gives the following explicit examples,

Example 3. *Set*

$$m = 4, \qquad \alpha = 1, \qquad \beta = \alpha \tan\Big(\frac{\pi}{4}\Big) = 1, \qquad q = -\frac{m}{2} = -2,$$

and define,

$$f_1(x) = e^{-x^{1/4}} \cos\big(x^{1/4}\big)\,(x^{1/4})^{-2} = e^{-x^{1/4}} \cos\big(x^{1/4}\big)\,x^{-1/2}.$$

Then for $k > -\frac{1}{2}$,

$$J_{4,-2}(k,1,1) = \int_0^\infty f_1(x)\,x^k\,dx = 4\,\Gamma(4k+2)\,2^{-(2k+1)}\,\cos\big(\pi(k+\tfrac{1}{2})\big),$$

and therefore

$$J_{4,-2}(k,1,1) = 0 \quad \Longleftrightarrow \quad k \in \mathbb{Z}_{\geq 0}.$$

Example 4. *Choose*

$$m = 6, \qquad \alpha = \sqrt{3}, \qquad \beta = \alpha \tan\Big(\frac{\pi}{6}\Big) = 1, \qquad q = -\frac{m}{2} = -3,$$

and define

$$f_2(x) = e^{-\sqrt{3}\,x^{1/6}} \cos\big(x^{1/6}\big)\,(x^{1/6})^{-3} = e^{-\sqrt{3}\,x^{1/6}} \cos\big(x^{1/6}\big)\,x^{-1/2}.$$

Then for $k > -\frac{1}{2}$,

$$J_{6,-3}\big(k,\sqrt{3},1\big) = \int_0^\infty f_2(x)\,x^k\,dx = 6\,\Gamma(6k+3)\,2^{-(6k+3)}\,\cos\big(\pi(k+\tfrac{1}{2})\big),$$

which gives

$$J_{6,-3}\big(k,\sqrt{3},1\big) = 0 \quad \Longleftrightarrow \quad k \in \mathbb{Z}_{\geq 0}.$$

We also have the following integral representations of the important constants,

Corollary 6. *In Theorem 5, choose*

$$m = 2, \quad \alpha = 1, \quad \beta = 1, \quad k = 0, \quad q = 0.$$

Then $A = mk + q + m = 2$ *and* $\theta = \arctan(\beta/\alpha) = \arctan(1)$ *(so* $\sin(2\theta) = 1$*), which gives*

$$\int_0^\infty e^{-x^{1/2}} \sin\big(x^{1/2}\big)\,dx \;=\; 1 \qquad (\textit{converges since } A = 2 > 0). \tag{54}$$

Proof. By (17), $I = m\Gamma(2)r^{-1}\sin(2\theta)$ with $m = 2$, $r = \alpha^2 + \beta^2 = 2$. Thus $I = 2 \cdot 1 \cdot (1/2) \cdot 1 = 1$.

The following theorem gives integral representation of specific values of the gamma function.

Theorem 7. *Let $A > 0$, $m > 0$, and choose any angle $\theta \in \big(0, \frac{\pi}{2}\big)$. Define*

$$r = \big(m \sin(A\theta)\big)^{\frac{2}{A}}, \qquad \alpha = \sqrt{r}\, \cos\theta > 0, \qquad \beta = \sqrt{r}\, \sin\theta, \tag{55}$$

and set

$$k = 0, \qquad q = A - m. \tag{56}$$

Then $A = mk + q + m$ and, with

$$I_{m,q}(k, \alpha, \beta) = \int_0^\infty e^{-\alpha x^{1/m}} \sin\big(\beta x^{1/m}\big)\, x^k\, (x^{1/m})^q\, dx, \tag{57}$$

we have $A > -1$ (indeed $A > 0$) so the integral converges, and moreover

$$I_{m,q}(k, \alpha, \beta) = \Gamma(A). \tag{58}$$

Proof. By (56), $A = mk + q + m = 0 + (A - m) + m = A$, so the parameter A in Theorem 5 agrees with the present A. Using $\theta = \arctan(\beta/\alpha) \in (0, \frac{\pi}{2})$ and $r = \alpha^2 + \beta^2$ from (55), Theorem 5 gives (for $A > 0$)

$$I_{m,q}(k, \alpha, \beta) = m\, \Gamma(A)\, r^{-A/2}\, \sin(A\theta). \tag{59}$$

By construction, $r^{A/2} = \big(m \sin(A\theta)\big)$, hence

$$r^{-A/2} \sin(A\theta) = \frac{\sin(A\theta)}{m \sin(A\theta)} = \frac{1}{m}. \tag{60}$$

Substituting (60) into (59) yields $I_{m,q} = \Gamma(A)$, i.e. (58). Convergence is ensured because $A > 0$ implies the hypothesis $A > -1$ in Theorem 5.

Remark 2. *The choice of $\theta \in (0, \frac{\pi}{2})$ guarantees $\alpha > 0$ and is compatible with $\theta = \arctan(\beta/\alpha)$. The representation (58) uses only the freely chosen numerical parameters (m, θ) and the target parameter A, the constants α, β are then determined by (55) and do not involve $\Gamma(A)$.*

Corollary 7 (Integer case $A = n \in \mathbb{N}$ with $m = 1$)**.** *Let $n \in \mathbb{N}$ and $m = 1$. Choose any $\theta \in (0, \frac{\pi}{2})$ and define*

$$\alpha = \big(\sin(n\theta)\big)^{2/n} \cos\theta, \qquad \beta = \big(\sin(n\theta)\big)^{2/n} \sin\theta, \qquad k = 0, \quad q = n - 1. \tag{61}$$

Then

$$\int_0^\infty e^{-\alpha x} \sin(\beta x)\, x^{n-1}\, dx = \Gamma(n) = (n-1)!. \tag{62}$$

Proof. This is Theorem 7 with $A = n$ and $m = 1$, note $r = (\sin(n\theta))^{2/n}$ so that $r^{n/2} = \sin(n\theta)$ and (60) holds.

Corollary 8 (Half–integer case $A = \ell + \frac{1}{2}$)**.** *Let $\ell > 0$ and $A = \ell + \frac{1}{2}$, $m = 1$. Choosing any $\theta \in (0, \frac{\pi}{2})$ and*

$$\alpha = \big(\sin(A\theta)\big)^{2/A} \cos\theta, \qquad \beta = \big(\sin(A\theta)\big)^{2/A} \sin\theta, \qquad k = 0, \quad q = A - 1, \tag{63}$$

one obtains

$$\int_0^\infty e^{-\alpha x} \sin(\beta x)\, x^{A-1}\, dx = \Gamma\Big(\ell + \tfrac{1}{2}\Big) = \frac{(2\ell)!}{4^\ell\, \ell!} \sqrt{\pi} \quad (\ell \in \tfrac{1}{2}\mathbb{N}). \tag{64}$$

Proof. Immediate from Theorem 7 with $m = 1$, $A = \ell + \frac{1}{2}$.

Corollary 9. *Let $A > 1$ and $\theta = \dfrac{\pi}{2A} \in (0, \frac{\pi}{2})$. For any $m > 0$, set*

$$\alpha = m^{1/A} \cos\Big(\frac{\pi}{2A}\Big), \qquad \beta = m^{1/A} \sin\Big(\frac{\pi}{2A}\Big), \qquad k = 0, \quad q = A - m. \tag{65}$$

Then $\sin(A\theta) = 1$ and $r = \alpha^2 + \beta^2 = m^{2/A}$, so

$$\int_0^\infty e^{-\alpha x^{1/m}} \sin\big(\beta x^{1/m}\big)\, x^{\frac{A}{m}-1}\, dx = \Gamma(A). \tag{66}$$

Proof. Here $r^{A/2} = m$ and $\sin(A\theta) = 1$, (59) yields $I_{m,q} = m\,\Gamma(A)\, m^{-1} \cdot 1 = \Gamma(A)$.

3. Conclusion

Motivated by Stieltjes' classical counterexample in the (indeterminate) Stieltjes moment problem, we revisited the kernel

$$f(x) = e^{-x^{1/4}} \sin\big(x^{1/4}\big)$$

and evaluated its moments against continuous powers. Theorem 4 gives the exact formula

$$I(k) = \int_0^\infty e^{-x^{1/4}} \sin\big(x^{1/4}\big)\, x^k\, dx = 2^{-2k}\Gamma(4k+4) \sin\big((k+1)\pi\big),$$

from which it follows that $I(k) = 0$ if and only if $k \in \mathbb{Z}_{\geq 0}$. This settles, in the affirmative and with a complete characterization, the question of which non-negative real exponents yield vanishing moments for the Stieltjes kernel.

We then proved a generalization (Theorem 5) to the family

$$I_{m,q}(k, \alpha, \beta) = \int_0^\infty e^{-\alpha x^{1/m}} \sin\big(\beta x^{1/m}\big)\, x^k (x^{1/m})^q\, dx, \qquad A = mk + q + m,$$

establishing: (i) sharp convergence for $A > -1$, (ii) the closed form $I_{m,q} = m\,\Gamma(A)\,(\alpha^2 + \beta^2)^{-A/2} \sin\big(A \arctan(\beta/\alpha)\big)$; and (iii) a necessary and sufficient vanishing criterion $A \arctan(\beta/\alpha) \in$

$\pi\mathbb{Z}$. A parallel cosine variant (Theorem 6) yields $J_{m,q} = m\,\Gamma(A)\,(\alpha^2+\beta^2)^{-A/2}\cos\bigl(A\arctan(\beta/\alpha)\bigr)$ with the expected zero set $\cos\bigl(A\arctan(\beta/\alpha)\bigr) = 0$. A particularly effective parameterization (Corollary 5), taking $\theta = \pi/m$, $\beta = \alpha\tan\theta$, and $q = -m/2$, forces $J_{m,q}(k,\alpha,\beta) = 0 \iff k \in \mathbb{Z}_{\geq 0}$, thus providing a cosine analogue that vanishes exactly at the nonnegative integers.

Beyond vanishing criteria, we showed that this framework generates Gamma values. Theorem 7 prescribes (α,β) via $r = (m\sin(A\theta))^{2/A}$ to give the exact representation.

$$\int_0^\infty e^{-\alpha x^{1/m}} \sin\bigl(\beta x^{1/m}\bigr)\, x^{A/m-1}\, dx = \Gamma(A),$$

from which the integer and half-integer cases follow transparently, recovering factorials and $\sqrt{\pi}$ factors in a unified way. This paper also includes a brief discussion of Salem's equivalence of the Riemann hypothesis, which is based on the vanishing of an integral dependent on a continuous parameter.

Acknowledgements

The author would like to thank the Prince Sultan University for paying the publication fees for this work through TAS LAB.

References

[1] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Academic Press, New York, 7th edition, 2007.

[2] Christophe Chesneau. New integral formulas inspired by an old integral result. *International Journal of Open Problems in Computer Science and Mathematics*, 18(2):53–71, 2025.

[3] Robert Reynolds and Andrew Stauffer. A definite integral involving the logarithmic function in terms of the lerch function. *Mathematics*, 7(1):1–5, 2019.

[4] Robert Reynolds and Andrew Stauffer. Definite integral of arctangent and polylogarithmic functions expressed as a series. *Mathematics*, 7(7):1–7, 2019.

[5] Robert Reynolds and Andrew Stauffer. Derivation of logarithmic and logarithmic hyperbolic tangent integrals expressed in terms of special functions. *Mathematics*, 8:1–6, 2020.

[6] Robert Reynolds and Andrew Stauffer. A quadruple definite integral expressed in terms of the lerch function. *Symmetry*, 13(1):1–8, 2021.

[7] Irshad Ayoob. On the evaluation of certain unsolved definite integrals. *European Journal of Pure and Applied Mathematics*, 18(3):65–75, 2025.

[8] Christophe Chesneau. New two-parameter integral formulas proved via the digamma function. *International Journal of Open Problems in Computer Science and Mathematics*, 18(4):32–38, December 2025.

[9] Christophe Chesneau. On some trigonometric and inverse trigonometric integral formulas. *Annals of Communications in Mathematics*, 8(3):386–392, September 2025.

[10] Daniel Felipe Martinez Barreto and Christophe Chesneau. Study of a particular class of integrals. *International Journal of Open Problems in Computer Science and Mathematics*, 18(4):119–127, December 2025.

[11] Konrad Schmüdgen. *The Moment Problem*, volume 277 of *Graduate Texts in Mathematics*. Springer International Publishing, Cham, 2017.

[12] B. Sury. *Weierstrass's Theorem – Leaving No Stone Unturned*. 2003. Available at: https://www.isibang.ac.in/ sury/hyderstone.pdf.

[13] Thomas Joannes Stieltjes. *Œuvres complètes, Vol. 2*. Wiskundig Genootschap / P. Noordhoff, Groningen, 1914–1918. Available online: https://archive.org/details/oeuvresthomasja02stierich.

[14] Raphaël Salem. Sur une proposition équivalente à l'hypothèse de riemann. *Comptes Rendus de l'Académie des Sciences de Paris*, 236:127–128, 1953.

[15] S. Yakubovich. Integral and series transformations via ramanujan's identities and salem's type equivalences to the riemann hypothesis. *Integral Transforms and Special Functions*, 25(4):255–271, 2014.

[16] B. J. González and E. R. Negrín. Inversion formulae for a lambert-type transform and the salem's equivalence to the riemann hypothesis. *Integral Transforms and Special Functions*, 34(8):614–618, 2023.

[17] B. J. González and E. R. Negrín. Approaching the riemann hypothesis using salem's equivalence and inversion formulae of a widder–lambert-type transform. *Integral Transforms and Special Functions*, 35(4):291–297, 2024.

[18] E. R. Negrín and J. Maan. New inversion formulae for the widder–lambert and stieltjes–poisson transforms. *Axioms*, 14(4):291, 2025.