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Equitable k-Fair Domination Under Some Binary Operations

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Abstract. A subset S of the vertex set $V(G)$ of a graph G is called an equitable fair dominating set of G if S is an equitable dominating set of G and for any $v, w \in V(G) \setminus S$, $|N_G(v) \cap S| = |N_G(w) \cap S| \geq 1$. The equitable fair domination number of G, denoted by $\gamma_{efd}(G)$, is the minimum cardinality of an EFD-set of G. The set S is called an equitable k-fair dominating set (abbreviated EkFD-set) of G if $|N_G(v) \cap S| = k$ for any $v \in V(G) \setminus S$, where k is a positive integer. The equitable k-fair domination number of G, denoted by $\gamma^e_{kfd}(G)$, is the minimum cardinality of an EkFD-set. An equitable k-fair dominating set of cardinality $\gamma^e_{kfd}(G)$ is called a γ^e_{kfd}-set of G. In this paper, we characterize the notions of equitable k-fair domination in graphs, study the EkFD-sets under some binary operations of graphs, and determine exact values or bounds for this domination variant.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: k-fair domination, equitable domination, equitable k-fair domination, join, corona

1. Introduction

The rigorous study of domination set in graph theory started around 1960 with Claude Berge [1]. He wrote a book on graph theory in which he defined the concept of the domination number in 1958. He called this number the coefficient of external stability. He used the notation $d(G)$ for the domination number of a graph. The notation $\gamma(G)$ was first used by E.J. Cockayne and S.T. Hedetniemi [2] for the domination number of a graph which subsequently became the accepted notation. The concepts were studied in more detail by brothers A.M. Yaglom and I.M. Yaglom [3] around 1964. A decade later, Cockayne and Hedetniemi [2] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G.

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In 2010, Bresar and Rall [4] defined fair domination and used it to prove the Vizing's conjecture which appeared in the paper of J.M Tarr [5]. Vizing's conjecture states that the domination number of the Cartesian product of graphs G and H is at least as large as the product of their domination numbers. The concept of fair domination and k-fair domination was introduced by Yair Caro et al [6].

In 2014, Maravilla, Isla and Canoy [7],[8] characterized the fair dominating and k-fair dominating sets in the join, corona, lexicographic product and cartesian product of graphs and determined the bounds or exact values of the fair and k-fair domination numbers, respectively, of these graphs. Swaminathan et al [9], studied equitable fair domination in graphs in 2021.

Inspired by two concepts, this paper comes into existence. One is the degree equitability in graphs conceived by E. Sampathkumar [10] and the other is k-fair domination in graphs [6].

2. Preliminary Results

Definition 2.1. [11] A subset S of $V(G)$ is called an *equitable dominating set* if for every $v \in V(G)\backslash S$, there exists a vertex $u \in S$ such that $uv \in E(G)$ and $|d_G(v) - d_G(u)| \leq 1$. The minimum cardinality among such equitable dominating sets is called the *equitable domination number* of G which is denoted by $\gamma^e(G)$.

Definition 2.2. [9] A subset S of the vertex set $V(G)$ of a graph G is called an *equitable fair dominating set* of G if S is an equitable dominating set of G and for any $v, w \in V(G)\backslash S$, $|N_G(v) \cap S| = |N_G(w) \cap S| \geq 1$. The *equitable fair domination number* of G denoted by $\gamma_{efd}(G)$ is the minimum cardinality of an EFD-set of G. S is called an *equitable k-fair dominating set* (abbreviated EkFD-set) of G if $|N_G(v) \cap S| = k$ for any $v \in V(G)\backslash S$ where k is a positive integer. The *equitable k-fair domination number* of G denoted by $\gamma^e_{kfd}(G)$ is the minimum cardinality of an EkFD-set. An equitable k-fair dominating set of cardinality $\gamma^e_{kfd}(G)$ is called a γ^e_{kfd}-set of G.

Theorem 2.3. [12] *Let G be a graph. Then $\gamma^e(G) = 1$ if and only if $\gamma(G) = 1$.*

Theorem 2.4. [7] *Let G be a connected graph. Then $\gamma_{fd}(G) = 1$ if and only if $\gamma(G) = 1$.*

Lemma 2.5. [8] *Let G be a connected graph with $\gamma_{fd}(G) = k < |V(G)|$. If S is a γ_{fd}-set of G, then S is not an mFD-set for every positive integer m with $m > k$.*

Lemma 2.6. [8] *Let G be a non-trivial connected graph and $k \in \mathbb{N}$. Then $\gamma_{kfd}(G) = 1$ if and only if $k = 1$ and $\gamma(G) = 1$.*

Lemma 2.7. [8] *Let G be a connected graph of order $n \geq 1$ and let k be a positive integer such that $k \leq n$. Then:*

(i) $k \leq \gamma_{kfd}(G) \leq n$.

(ii) $\gamma_{kfd}(G) = k$ *if and only if G has a kFD-set S with $|S| = k$.*

(iii) If $\gamma_{kfd}(G) = n$, then G has no vertex of degree k.

Proposition 2.8. [11]

(i) For the complete graph K_n on n vertices, $\gamma^e(K_n) = 1$.

(ii) For the paths P_n and the cycles C_n on n vertices, $\gamma^e(P_n) = \gamma^e(C_n) = \lceil \frac{n}{3} \rceil$.

(iii) If W_n denotes the wheel on n vertices, then

$$\gamma^e(W_n) = \begin{cases} 1, & \text{if } n = 3, 4; \\ \lceil \frac{n}{3} \rceil + 1, & \text{otherwise.} \end{cases}$$

Theorem 2.9. [13] *Let C_n be a cycle of length n, then*

$$\gamma_{1fd}(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 (\text{mod } 3) \\ \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1 (\text{mod } 3) \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2 (\text{mod } 3) \end{cases}$$

Lemma 2.10. [14]

1. *For complete graph K_n, $\gamma_{2fd}(G) = 2$.*
2. *Let P_n, be a path of length $(n-1)$, then for $n \geq 3$,*

$$\gamma_{_{2fd}}(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1, & \text{if } n \text{ is even.} \end{cases}$$

3. *Let C_n be a cycle of length n, then $\gamma_{_{2fd}}(C_n) = \lceil \frac{n}{2} \rceil$, for $n \geq 3$.*

Theorem 2.11. [13] *Let P_n be a path of length $(n-1)$, then $\gamma_{1fd}(P_n) = \lceil \frac{n}{3} \rceil$, for $n \geq 2$.*

3. Graphs with $\gamma^e_{_{kfd}}(G) = 1$

Remark 3.1. An equitable k-fair dominating set (EkFD-set) in G is an EFD-set in G if $k = 1$.

Remark 3.2. For any connected graph G of order $n \geq 2$ and any positive integer k,

$$1 \leq \gamma_{kfd}(G) \leq \gamma^e_{_{kfd}}(G).$$

Remark 3.3. Let G be connected graph. Then every EkFD-set is an equitable dominating set. Thus, $\gamma^e(G) \leq \gamma^e_{_{kfd}}(G)$.

Proposition 3.4. *Let G be a connected graph and $k \in \mathbb{N}$. Then*

(i) If $\gamma^e(G) = 1$, then $\gamma^e_{_{kfd}}(G) = 1$ for $k = 1$.

(ii) If $\gamma^e_{kfd}(G) = 1$, then $\gamma(G) = 1$ for $k = 1$.

Proof: (i) Suppose $\gamma^e(G) = 1$. By Theorem 2.3, $\gamma(G) = 1$. By Theorem 2.4, $\gamma_{fd}(G) = 1$. Let $\{u\}$ be a γ_{fd}-set of G. Then $\{u\}$ is not a mFD-set for some integer $m > 1$ by Lemma 2.5. It follows that $\gamma_{kfd}(G) = 1$ by Lemma 2.6. Thus, $\{u\}$ is a γ_{kfd}-set. Since $\gamma^e(G) = 1$, for all $v \in V(G)\backslash\{u\}$, $|d_G(u) - d_G(v)| \leq 1$. Therefore, $\{u\}$ is a γ^e_{kfd}-set and so $\gamma^e_{kfd}(G) = 1$.
(ii) Suppose $\gamma^e_{kfd}(G) = 1$. By Remark 3.2, $\gamma_{kfd}(G) = 1 = \gamma(G)$. By Lemma 2.6, $k = 1$ and $\gamma(G) = 1$. □

Proposition 3.5. *Let G be a connected graph and $k \in \mathbb{N}$. Then $\gamma^e_{kfd}(G) = 1$ if and only if $k = 1$, $\gamma(G) = 1$ and $|d_G(u) - d_G(v)| \leq 1$ for all $v \in V(G)\backslash\{u\}$ where $\{u\}$ is a dominating set.*

Proof: Suppose $\gamma^e_{kfd}(G) = 1$. By Proposition 3.4 (ii), $\gamma(G) = 1$ and $k = 1$. Let $\{u\}$ be an EkFD-set of G. Then for all $v \in V(G)\backslash\{u\}$, $|d_G(u) - d_G(v)| \leq 1$.

The converse is obvious. □

4. Realization Problem

Theorem 4.1. *Let a be a positive integer. Then there exists a connected graph G, such that $\gamma^e(G) = \gamma_{1fd}(G) = \gamma^e_{1fd}(G) = a$.*

Proof: Suppose $a = 1$. Let $G = P_3$. Then $\gamma^e(G) = \gamma_{1fd}(G) = \gamma^e_{1fd}(G) = a$. Suppose $a \geq 2$. Consider the graph G as shown below.

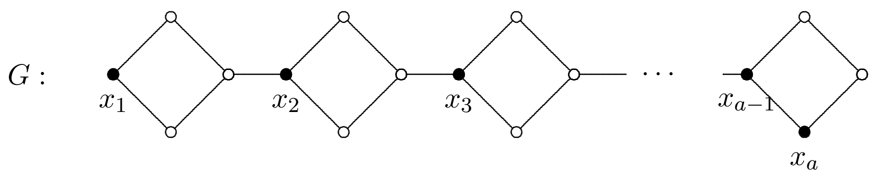

Figure 1: A graph G with $\gamma^e(G) = \gamma_{1fd}(G) = \gamma^e_{1fd}(G) = a$

Clearly, the set $S = \{x_i : i = 1, 2, \cdots, a\}$ is a γ_{1fd}-set of G. It can also be verified that for all $v \in V(G)\backslash S$, $|d_G(x_i) - d_G(v)| \leq 1$ for some $i \geq 1$. Hence, S is γ^e_{1fd}-set of G. Note that S is γ^e-set of G and $|S| = a$. Thus, $\gamma^e(G) = \gamma_{1fd}(G) = \gamma^e_{1fd}(G) = a$. □

Theorem 4.2. *Let a and b be positive integers. Then there exists a connected graph G such that $\gamma^e(G) = a$, $\gamma_{1fd}(G) = \gamma^e_{1fd}(G) = b$ and $a < b$.*

Proof: Consider the graph G as shown below.

Let $X = \{x_i : i = 1, 2, \cdots, c\}$, $Y = \{y_j : j = 1, 2, \cdots, c\}$, and $Z = \{z_k : k = 1, 2, \cdots, c\}$. Clearly, the set $S = X \cup Y \cup Z$ is a γ_{1fd}-set. It can be verified that for all $u \in V(G)\backslash S$, there exists $w \in S$ such that $|d_G(w) - d_G(u)| \leq 1$. Thus, S is an E1FD-set. Moreover, S is a γ^e_{1fd}-set. Hence,

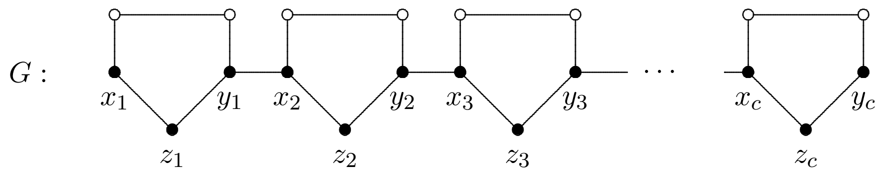

Figure 2: A graph G with $\gamma^e(G) = a < \gamma_{1fd}(G) = \gamma^e_{1fd}(G) = b$

$$\gamma_{1fd}(G) = \gamma^e_{1fd}(G) = |S| = |X| + |Y| + |Z| = c + c + c = 3c.$$

Now, consider the set $S^* = X \cup Y$. Then S^* is a γ-set of G. In addition, for all $s \in V(G)\backslash S^*$, there exists $v \in S^*$ such that $|d_G(v) - d_G(s)| \leq 1$. Thus, S^* is a γ^e-set of G. Thus, $\gamma^e(G) = |S^*| = |X| + |Y| = c + c = 2c$. Clearly, S^* is not a 1FD-set for all $z_k \in V(G)\backslash S^*$ for all $k = 1, 2, \cdots, c$, $|N_G(z_k) \cap S^*| = 2 \neq 1$.

Let $a = 2c$ and $b = 3c$. Then $\gamma^e(G) = 2c = a$, $\gamma_{1fd}(G) = \gamma^e_{1fd}(G) = 3c = b$ and $a < b$. This completes the proof. □

Theorem 4.3. *Let a and b be any positive integers. Then there exists a connected graph G such that $\gamma_{1fd}(G) = a$, $\gamma^e_{1fd}(G) = \gamma^e(G) = b$ and $a < b$.*

Proof: Consider the graph G as shown below.

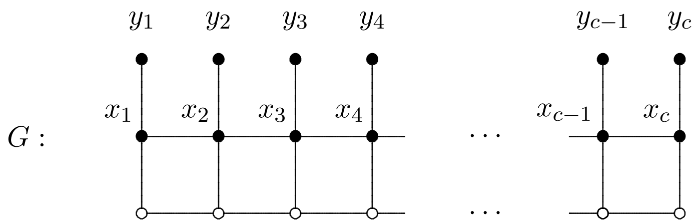

Figure 3: A graph G with $\gamma_{1fd}(G) = a < \gamma^e_{1fd}(G) = \gamma^e(G) = b$

Let $X = \{x_i : i = 1, 2, \cdots, c\}$ and $Y = \{y_j : j = 1, 2, \cdots, c\}$. Consider the set $S = X$. Then S is clearly a 1FD-set of minimum cardinality but not an E1FD-set since for all $y_i \in V(G)\backslash S$,

$$|d_G(x_i) - d_G(y_i)| = |4 - 1| = 3 > 1 \text{ for } i = 2, 3, \cdots, c - 1$$

and

$$|d_G(x_i) - d_G(y_i)| = |3 - 1| = 2 > 1 \text{ for } i = 1, c.$$

Thus $\gamma_{1fd}(G) = |S| = c$. Now, let $S^* = X \cup Y$. Then it can be verified that S^* is a 1FD-set. Moreover, S^* is an equitable dominating set since for all $u \in V(G)\backslash S^*$, there exists $x_i \in S^*$, for some i such that $|d_G(x_i) - d_G(u)| = 1$. Note that S^* is a γ^e-set of G and a γ^e_{1fd}-set of G. Hence, $\gamma^e_{1fd}(G) = \gamma^e(G) = |S^*| = |X| + |Y| = c + c = 2c$.

Let $a = c$ and $b = 2c$. Then $\gamma_{1fd}(G) = a$, $\gamma^e_{1fd}(G) = \gamma^e(G) = b$ and so $a < b$. □

From the previous results, the following remark is obtained.

Remark 4.4. The equitable dominating set and k-fair dominating set are incomparable. In particular, $\gamma^e(G)$ and $\gamma_{kfd}(G)$ are incomparable.

5. Equitable k-Fair Domination in Some Special Graphs

Theorem 5.1. *Let G be a connected graph of order $n \geq 1$ and let k be a positive integer such that $k \leq n$. Then:*

(i) $k \leq \gamma^e_{kfd}(G) \leq n$.

(ii) $\gamma^e_{kfd}(G) = k$ *if and only if G has an EkFD-set S with $|S| = k$.*

(iii) *If $\gamma^e_{kfd}(G) = n$, then G has no vertex of degree k and $|d_G(v) - d_G(w)| \geq 2$ for all $vw \in E(G)$.*

Proof: By Remark 3.2, $\gamma_{kfd}(G) \leq \gamma^e_{kfd}(G)$ and by Lemma 2.7 (i), $k \leq \gamma_{kfd}(G) \leq \gamma^e_{kfd}(G) \leq n$. Next, suppose that $\gamma^e_{kfd}(G) = k$. If S is a γ^e_{kfd}-set, then $|S| = k$. For the converse, suppose that G has an EkFD-set S with $|S| = k$. Then $\gamma^e_{kfd}(G) = |S| \leq k$. Since by (i), $\gamma^e_{kfd}(G) \geq k$, it follows that $\gamma^e_{kfd}(G) = k$. Thus, (ii) holds.

Next, suppose that $\gamma^e_{kfd}(G) = n$. Then by Lemma 2.7 (iii), G has no vertex of degree k. Now, suppose that G has a vertex v with $d_G(v) = k$ and $|d_G(v) - d_G(w)| \leq 1$ for all $vw \in E(G)$. Let $S = V(G)\backslash\{v\}$. Then $|N_G(v) \cap S| = |N_G(v)| = k$ and there exists $u \in S$ such that $|d_G(u) - d_G(v)| \leq 1$. Thus, S is a EkFD-set and $\gamma^e_{kfd}(G) = n-1$, contrary to the assumption that $\gamma^e_{kfd}(G) = n$. Hence, $|d_G(v) - d_G(w)| \geq 2$ for all $vw \in E(G)$. □

Remark 5.2. Consider Theorem 5.1,

(a) The strict (right) inequality in Theorem 5.1 (i) can be attained. However, the given upperbound is sharp.

(b) The converse of Theorem 5.1 (iii) is not always true.

To see (a), consider the graphs shown in Figure 4. The shaded vertices in each graph form a γ^e_{2fd}-set. Thus, $2 < \gamma^e_{2fd}(G) = 3 < 5$, $2 = \gamma^e_{2fd}(H) = 2 < 4$ and $\gamma^e_{4fd}(G) = 5 = n$. To see (b), consider the wheel W_5 shown in Figure 5. Clearly, W_5 has no vertex of degree 2 and $|d_{W_5}(u) - d_{W_5}(v)| = |5 - 3| = 2 \nleq 1$ for all $v \neq u$; however, $\gamma^e_{2fd}(W_5) = 4 \neq 6$.

Figure 4: The graphs G and K_4

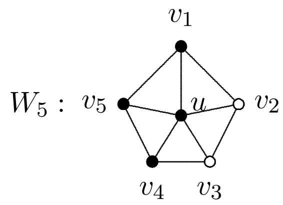

Figure 5: The wheel graph W_5

Theorem 5.3. *If G is a regular graph or $|d_G(v) - d_G(w)| \leq 1$ for all $vw \in E(G)$. Then, $\gamma^e_{kfd}(G) = \gamma_{kfd}(G)$.*

Proof: Let S be a γ_{kfd}-set in G. Suppose that G is a regular graph. Then for all $v \in V(G) \backslash S$, there exists $u \in S$ such that $|d_G(u) - d_G(v)| = 0 \leq 1$. Thus, S is an γ^e_{kfd}-set in G. Hence, $\gamma^e_{kfd}(G) = \gamma_{kfd}(G)$. Similarly, if $|d_G(v) - d_G(w)| \leq 1$ for all $vw \in E(G)$. Then, S is an γ^e_{kfd}-set in G. Hence, $\gamma^e_{kfd}(G) = \gamma_{kfd}(G)$. □

Proposition 5.4. *Let W_n be the wheel graph of order $n \geq 3$ and k a positive integer. Then*

$$\gamma^e_{kfd}(W_n) = \begin{cases} 1, & \text{if and only if } k = 1 \text{ and } n \in \{3,4\}, \\ 2, & \text{if } k = 2 \text{ and } n \in \{3,4\}, \\ \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } k = 2 \text{ and } n = 3m \text{ or } n = 3m+1, m \geq 2, \\ \left\lceil \frac{n}{3} \right\rceil + 2, & \text{if } k = 2 \text{ and } n = 3m+2, m \geq 1, \\ \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } k = 3, \\ 4, & \text{if } k = 4 \text{ and } n \in \{3,4\}, \\ n+1, & \text{otherwise.} \end{cases}$$

Proof: Let $V(W_n) = \{v_1, v_2, \cdots, v_n, u\}$ where u is the apex vertex and $v_i : i = 1, 2, \cdots, n$ are the rim vertices of W_n. Then $d_{W_n}(v_i) = 3$ for all $i = 1, 2, 3, \cdots, n$ and $d_{W_n}(u) = n$. Note that $\gamma(W_n) = 1$. So, $\gamma^e_{kfd}(W_n) \geq \gamma(W_n)$. Now, consider the following cases:

Case 1: For $k = 1$ and $n \in \{3,4\}$

Note that $\gamma(W_n) = 1$ and $\gamma_{1fd}(W_n) = 1$ by Lemma 2.6. Note also that $\gamma^e(W_n) = 1$ for $n \in \{3,4\}$ by Proposition 2.8 (iii). Now, by Propositions 3.4 and 3.5, $\gamma^e_{1fd}(W_n) = 1$ for $n \in \{3,4\}$.

For $n \geq 5$, note that $\{u\}$ is a γ_{1fd}-set but not γ^e-set since $|d_{W_n}(u) - d_{W_n}(v_i)| \geq |5 - 3| = 2$. Suppose $1 < \gamma^e_{1fd}(W_n) = m < n+1$. If $S = \{u\} \cup \{y_1, y_2, \cdots, y_{m-1}\}$ where $\{y_j\} \subseteq \{v_i\}$, then there exists $v_i \in V(W_n) \backslash S$ such that $|N_{W_n}(v_i) \cap S| > 1$ which is a contradiction since $|N_{W_n}(x) \cap S| = 1$ for all $x \in V(W_n) \backslash S$. Thus, $\gamma^e_{1fd}(W_n) = n+1$ for $n \geq 5$.

Case 2: For $k = 2$

Consider the following subcases:

Subcase 2.1: For $n \in \{3,4\}$

Let $S = \{v_1, v_3\}$. Then S is a dominating set and for all $x \in V(W_n)\backslash S$, $|N_{W_n}(x) \cap S| = |S| = 2$ and $|d_{W_n}(x) - d_{W_n}(y)| \leq 1$ for all $y \in S$. Thus S is a γ^e_{2fd}-set of W_n. Thus, $\gamma^e_{2fd}(W_n) = 2$.

Subcase 2.2: For $n = 3m, m \geq 2$

Note that $V(W_n)\backslash\{u\}$ is a cycle of order n. Let $S = \{u\} \cup T$, where T is a γ^e_{1fd}-set of C_n. By Theorem 5.3 and Theorem 2.9, $|T| = \frac{n}{3}$ when $n = 3m, m \geq 1$. Now, for all $x \in V(W_n)\backslash S$,

$$\begin{aligned}
|N_{W_n}(x) \cap S| &= |N_{W_n}(x) \cap (\{u\} \cup T)| \\
&= |(N_{W_n}(x) \cap \{u\}) \cup (N_{W_n}(x) \cap T)| \\
&= |(N_{W_n}(x) \cap \{u\})| + |(N_{W_n}(x) \cap T)| \\
&= 1 + 1 \\
&= 2.
\end{aligned}$$

Also, for all $x \in V(W_n)\backslash S$, $|d_{W_n}(x) - d_{W_n}(y)| = 0$ for some $y \in S$. Thus, S is a γ^e_{2fd}-set of W_n. Hence, $\gamma^e_{2fd}(W_n) = |S| = |\{u\} \cup T| = |\{u\}| + |T| = 1 + \frac{n}{3}$.

A similar proof for $n = 3m + 1$ and $n = 3m + 2, m \geq 2$. Hence, $\gamma^e_{2fd}(W_n) = |S| = 1 + \lceil \frac{n}{3} \rceil$ and $\gamma^e_{2fd}(W_n) = |S| = 2 + \lceil \frac{n}{3} \rceil$, for $n = 3m + 1$ and $n = 3m + 2, m \geq 2$, respectively. Note that $\lceil \frac{n}{3} \rceil = \frac{n}{3}$ for all $n = 3m, m \geq 1$. Thus, $\gamma^e_{2fd}(W_n) = \lceil \frac{n}{3} \rceil + 1$ for $n = 3m$ and $n = 3m + 1, m \geq 2$.

Case 3: For $k = 3$

Let $S = \{u\} \cup T$, where T is a γ^e_{2fd}-set of C_n. By Theorem 5.3 and Lemma 2.10 (3), $|T| = \lceil \frac{n}{2} \rceil$. Now, for all $x \in V(W_n)\backslash S$,

$$\begin{aligned}
|N_{W_n}(x) \cap S| &= |N_{W_n}(x) \cap (\{u\} \cup T)| \\
&= |(N_{W_n}(x) \cap \{u\}) \cup (N_{W_n}(x) \cap T)| \\
&= |(N_{W_n}(x) \cap \{u\})| + |(N_{W_n}(x) \cap T)| \\
&= 1 + 2 \\
&= 3.
\end{aligned}$$

Also, for all $x \in V(W_n)\backslash S$, $|d_{W_n}(x) - d_{W_n}(y)| = 0$ for some $y \in S$. Thus, S is a γ^e_{3fd}-set of W_n. Hence, $\gamma^e_{3fd}(W_n) = |S| = |\{u\} \cup T| = |\{u\}| + |T| = 1 + \lceil \frac{n}{2} \rceil$.

Case 4: For $k = 4$

Note that for all $v_i \in V(W_n)$, $d_{W_n}(v_i) = 3$ and $d_{W_n}(u) = n$. Clearly, for $n = 3$, $\gamma^e_{4fd}(W_3) = 4$. For $n = 4$, $d_{W_4}(u) = 4$. Let $S = \{v_1, v_2, v_3, v_4\}$ and $V(W_4)\backslash S = \{u\}$. Now, $|N_{W_4}(u) \cap S| = |S| = 4$ and there exists $v_1 \in S$ such that $|d_{W_4}(v_1) - d_{W_4}(u)| = 1$. Hence, $\gamma^e_{4fd}(W_4) = |S| = 4$.

For $n \geq 5$, W_n has no vertex of degree 4 so it is impossible to have $|N_{W_n}(x) \cap S| = 4$ for all $x \in V(W_n)\backslash S$. Thus, $\gamma^e_{4fd}(W_n) = n + 1$.

Case 5: For $k \geq 5$

Suppose $\gamma^e_{kfd}(W_n) = m \leq n + 1$. Let S be a γ^e_{kfd}-set of W_n. Then there exists $v \in V(W_n)\backslash S$ such that $|N_{W_n}(v) \cap S| = m > 3$, a contradiction since $|N_{W_n}(v)| = 3$ for all $v \in V(W_n), v \neq u$. Hence, $\gamma^e_{kfd}(W_n) = n + 1$ for $k \geq 5$. □

Proposition 5.5. *Let F_n be the fan graph of order $n \geq 2$ and k a positive integer. Then*

$$\gamma^e_{kfd}(F_n) = \begin{cases} 1, & \text{if and only if } k = 1 \text{ and } n \in \{2,3\}, \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } k = 2, \\ \lceil \frac{n}{2} \rceil + 1, & \text{if } k = 3 \text{ and } n \text{ is odd}, \\ \frac{n}{2} + 2, & \text{if } k = 3 \text{ and } n \text{ is even}, \\ 4, & \text{if } k = 4 \text{ and } n \in \{3,4\}, \\ n+1, & \text{otherwise.} \end{cases}$$

Proof: Let $V(F_n) = \{v_1, v_2, \cdots, v_n, u\}$. Then $d_{F_n}(v_1) = d_{F_n}(v_n) = 2$, $d_{F_n}(v_i) = 3$ for all $i = 2, 3, \cdots, n-1$ and $d_{F_n}(u) = n$. Note that $\gamma(F_n) = 1$. So, $\gamma^e_{kfd}(F_n) \geq \gamma(F_n) = 1$. Now, consider the following cases:

Case 1: For $k = 1$ and $n \in \{2,3\}$

Note that $\gamma(F_n) = 1$ and by Lemma 2.6, $\gamma_{1fd}(F_n) = 1$. Note also that $F_2 \cong C_3$. Then by Theorem 5.3 and Theorem 2.9, $\gamma^e_{1fd}(F_2) = \gamma^e_{1fd}(C_3) = 1$. For $n = 3$, let $S = \{u\}$. Clearly, S is a γ_{1fd}-set of W_3. Now, for all $x \in V(F_3)\backslash S$, $|d_{F_3}(u) - d_{F_3}(x)| \leq 1$. Hence, S is a γ^e_{1fd}-set of F_3. Thus, $\gamma^e_{1fd}(F_n) = 1$ for $n \in \{2,3\}$.

For $n \geq 4$, note that $\{u\}$ is a γ_{1fd}-set but not γ^e-set since $|d_{F_n}(u) - d_{F_n}(v_i)| \geq |4-2| = 2$ for some $v_i \in V(F_n)\backslash\{u\}$. Suppose $1 < \gamma^e_{1fd}(F_n) = m < n+1$. If $S = \{u\} \cup T$ where $T \subseteq \{v_i : 1 \leq i \leq n\}$, then there exists $v_i \in V(F_n)\backslash S$ such that $|N_{F_n}(v_i) \cap S| > 1$ which is a contradiction since $|N_{F_n}(x) \cap S| = 1$ for all $x \in V(F_n)\backslash S$. Thus, $\gamma^e_{1fd}(F_n) = n+1$ for $n \geq 4$.

Case 2: For $k = 2$

Note that $V(F_n)\backslash\{u\}$ is a path of order n. Note also that $F_2 \cong C_3$. Then by Theorem 5.3 and Lemma 2.10 (3), $\gamma^e_{2fd}(F_2) = \gamma^e_{2fd}(C_3) = 2$. Now, for $n \geq 3$, let $S = \{u\} \cup T$, where T is a γ^e_{1fd}-set of P_n. By Theorem 5.3 and Theorem 2.11, $|T| = \lceil \frac{n}{3} \rceil$. Now, for all $x \in V(F_n)\backslash S$,

$$\begin{aligned} |N_{F_n}(x) \cap S| &= |N_{F_n}(x) \cap (\{u\} \cup T)\,| \\ &= |(N_{F_n}(x) \cap \{u\}) \cup (N_{F_n}(x) \cap T)| \\ &= |(N_{F_n}(x) \cap \{u\})| + |(N_{F_n}(x) \cap T)| \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

Also, for all $x \in V(F_n)\backslash S$, $|d_{F_n}(x) - d_{F_n}(y)| \leq 1$ for some $y \in S$. Thus, S is a γ^e_{2fd}-set of F_n. Hence, $\gamma^e_{2fd}(F_n) = |S| = |\{u\} \cup T| = |\{u\}| + |T| = 1 + \lceil \frac{n}{3} \rceil$.

Case 3: For $k = 3$

Clearly, for $n = 2$, $\gamma^e_{3fd}(F_2) = 3$. Suppose $n \geq 3$ and n is odd, let $S = \{u\} \cup T$, where T is a γ^e_{2fd}-set of P_n. By Theorem 5.3 and Lemma 2.10 (2), $|T| = \lceil \frac{n}{2} \rceil$ when n is odd. Now, for all $x \in V(F_n)\backslash S$,

$$|N_{F_n}(x) \cap S| = |N_{F_n}(x) \cap (\{u\} \cup T)\,|$$

$$\begin{aligned} &= |(N_{F_n}(x) \cap \{u\}) \cup (N_{F_n}(x) \cap T)| \\ &= |(N_{F_n}(x) \cap \{u\})| + |(N_{F_n}(x) \cap T)| \\ &= 1 + 2 \\ &= 3. \end{aligned}$$

Also, for all $x \in V(F_n)\backslash S$, $|d_{F_n}(x) - d_{F_n}(y)| = 0$ for some $y \in S$. Thus, S is a γ^e_{3fd}-set of F_n. Hence, $\gamma^e_{3fd}(F_n) = |S| = |\{u\} \cup T| = |\{u\}| + |T| = 1 + \lceil \frac{n}{2} \rceil$ when n is odd.

A similar proof when n is even. Hence, by Theorem 5.3 and Lemma 2.10 (2), $\gamma^e_{3fd}(F_n) = |S| = 1 + (1 + \frac{n}{2}) = 2 + \frac{n}{2}$, when n is even.

Case 4: For $k = 4$

Note that for all $v_i \in V(F_n)$, $d_{F_n}(v_i) = 2, 3$ or n. Clearly, for $n = 3$, $\gamma^e_{4fd}(F_3) = 4$. Now, for $n = 4$, $d_{F_4}(u) = 4$. Let $S = \{v_1, v_2, v_3, v_4\}$ and $V(F_4)\backslash S = \{u\}$. Now, $|N_{F_4}(u) \cap S| = |S| = 4$ and there exists $v_2 \in S$ such that $|d_{F_4}(v_2) - d_{F_4}(u)| = 1$. Hence, $\gamma^e_{4fd}(F_4) = |S| = 4$.

For $n \geq 5$, F_n has no vertex of degree 4 so it is impossible to have $|N_{F_n}(x) \cap S| = 4$ for all $x \in V(F_n)\backslash S$. Thus, $\gamma^e_{4fd}(F_n) = n + 1$.

Case 5: For $k \geq 5$

Suppose $\gamma^e_{kfd}(F_n) = m \leq n + 1$. Let S be a γ^e_{kfd}-set of F_n. Then there exists $v \in V(F_n)\backslash S$ such that $|N_{F_n}(v) \cap S| = m > 3$, a contradiction since $|N_{F_n}(v)| = 2$ or 3 for all $v \in V(F_n), v \neq u$. Hence, $\gamma^e_{kfd}(F_n) = n + 1$ for $k \geq 5$. □

Proposition 5.6. *Let $f_{3,n}$ be the friendship graph of order $2n+1$ and k a positive integer. Then*

$$\gamma^e_{kfd}(f_{3,n}) = \begin{cases} 1, & \text{if } k = 1 \text{ and } n = 1, \\ n + 1, & \text{if } k = 2, \\ 2n + 1, & \text{otherwise.} \end{cases}$$

Proof: Let $V(f_{3,n}) = \{v_1, v_2, \cdots, v_n, v_{n+1}, \cdots, v_{2n-1}, v_{2n}, u\}$. Then $d_{f_{3,n}}(u) = 2n$ and $d_{f_{3,n}}(v_i) = 2$ for all $i = 1, 2, 3, \cdots, 2n$. Note that $\gamma(f_{3,n}) = 1$. So, $\gamma^e_{kfd}(f_{3,n}) \geq \gamma(f_{3,n}) = 1$. Now, consider the following cases:

Case 1: For $k = 1$

By Lemma 2.6, $\gamma_{1fd}(f_{3,n}) = \gamma(f_{3,n}) = 1$. So, $\gamma^e_{1fd}(f_{3,n}) \geq 1$. If $n = 1$, then $f_{3,1} \cong C_3$. By Theorem 5.3 and Theorem 2.9, $\gamma^e_{1fd}(f_{3,1}) = \gamma^e_{1fd}(C_3) = 1$.

Suppose $n \geq 2$. Then $d_{f_{3,n}}(u) = 2n \geq 4$ and $d_{f_{3,n}}(v_i) = 2$ for all $i = 1, 2, \cdots, 2n - 1, 2n$. Note that $\{u\}$ is a γ_{1fd}-set but not γ^e-set since $|d_{f_{3,n}}(u) - d_{f_{3,n}}(v_i)| = |2n - 2| \geq |4 - 2| = 2$ for all $v_i \neq u$. Suppose $1 < \gamma^e_{1fd}(f_{3,n}) = m < 2n + 1$. If $S = \{u\} \cup T$ where $T \subseteq \{v_i : 1 \leq i \leq 2n\}$, then there exists $v_i \in V(f_{3,n})\backslash S$ such that $|N_{f_{3,n}}(v_i) \cap S| = 2 > 1$ which is a contradiction since $|N_{f_{3,n}}(x) \cap S| = 1$ for all $x \in V(f_{3,n})\backslash S$. Thus, $\gamma^e_{1fd}(f_{3,n}) = 2n + 1$ for $n \geq 2$.

Case 2: For $k = 2$

Note that $V(f_{3,n})\backslash\{u\}$ are disconnected P_2 of n copies. Now, let $S = \{u\} \cup T_n$, where T_n are γ^e_{1fd}-sets of nP_2. Note also that $\gamma^e_{1fd}(P_2) = 1$. Thus, $\gamma^e_{1fd}(T_n) = n(1) = n$. Now, for all $x \in V(f_{3,n})\backslash S$,

$$\begin{aligned} |N_{f_{3,n}}(x) \cap S| &= |N_{f_{3,n}}(x) \cap (\{u\} \cup T_n)| \\ &= |(N_{f_{3,n}}(x) \cap \{u\}) \cup (N_{f_{3,n}}(x) \cap T_n)| \\ &= |(N_{f_{3,n}}(x) \cap \{u\})| + |(N_{f_{3,n}}(x) \cap T_n)| \\ &= 1+1 \\ &= 2. \end{aligned}$$

Also, for all $x \in V(f_{3,n})\backslash S$, $|d_{f_{3,n}}(x) - d_{F_n}(y)| = 0$ for all $y \in S, y \neq u$. Thus, S is a γ^e_{2fd}-set of $f_{3,n}$. Hence, $\gamma^e_{2fd}(f_{3,n}) = |S| = |\{u\} \cup T_n| = |\{u\}| + |T_n| = 1 + n$.

Case 3: For $k \geq 3$

Suppose $\gamma^e_{kfd}(f_{3,n}) = m \leq 2n+1$. Let S be a γ^e_{kfd}-set of $f_{3,n}$. Then there exists $v \in V(f_{3,n})\backslash S$ such that $|N_{f_{3,n}}(v) \cap S| = m \geq 3$, a contradiction since $|N_{f_{3,n}}(v)| = 2$ for all $v \in V(f_{3,n}), v \neq u$. Hence, $\gamma^e_{kfd}(f_{3,n}) = |V(f_{3,n})| = 2n+1$ for $k \geq 3$. □

Proposition 5.7. *Let $K_{1,n}$ be the star graph of order $n+1$ and k a positive integer. Then*

$$\gamma^e_{kfd}(K_{1,n}) = \begin{cases} 1, & \text{if } k = 1 \text{ and } n \in \{1,2\}, \\ 2, & \text{if } k = 2 \text{ and } n \in \{1,2\}, \\ n+1, & \text{otherwise.} \end{cases}$$

Proof: Let $V(K_{1,n}) = \{v_1, v_2, \cdots, v_n, u\}$. Then $d_{K_{1,n}}(u) = n$ and $d_{K_{1,n}}(v_i) = 1$ for all $i = 1, 2, 3, \cdots, n$, that is, every $v_i \in V(K_{1,n}), v_i \neq u$ is a pendant vertex of $K_{1,n}$. Note that $\gamma(K_{1,n}) = 1$. So, $\gamma^e_{kfd}(K_{1,n}) \geq \gamma(K_{1,n}) = 1$. Now, consider the following cases:

Case 1: For $k = 1$

Note that by Lemma 2.6, $\gamma_{1fd}(K_{1,n}) = 1$. So, $\gamma^e_{kfd}(K_{1,n}) \geq \gamma_{1fd}(K_{1,n}) = 1$.

For $n \in \{1,2\}$, clearly, $(K_{1,n}) \cong P_{n+1}$. By Theorem 5.3 and Theorem 2.11, $\gamma^e_{1fd}(K_{1,1}) = \gamma^e_{1fd}(P_2) = 1$ and $\gamma^e_{1fd}(K_{1,2}) = \gamma^e_{1fd}(P_3) = 1$, respectively.

Suppose $n \geq 3$. Then $d_{K_{1,n}}(u) = n \geq 3$. Note that $\{u\}$ is a γ_{1fd}-set but not γ^e-set since $|d_{K_{1,n}}(u) - d_{K_{1,n}}(v_i)| = |n-1| \geq |3-1| = 2$ for all $v_i \neq u$. Suppose $1 < \gamma^e_{1fd}(K_{1,n}) = m < n+1$. If $S = \{u\} \cup T$ where $T \subseteq V(K_{1,n})\backslash\{u\}$, then there exists $v_i \in V(K_{1,n})\backslash S$ such that $|d_{K_{1,n}}(u) - d_{K_{1,n}}(v_i)| \leq 1$, which is a contradiction since $|d_{K_{1,n}}(u) - d_{K_{1,n}}(v_i)| \geq 2$ for all $v_i \in V(K_{1,n})\backslash S$. Thus, $\gamma^e_{1fd}(K_{1,n}) = n+1$ for $n \geq 2$.

Case 2: For $k = 2$

Clearly, for $n = 1$, $\gamma^e_{2fd}(K_{1,1}) = 2$. Now, since $(K_{1,2}) \cong P_3$, then by Theorem 5.3 and Lemma 2.10 (2) , $\gamma^e_{2fd}(K_{1,2}) = \gamma^e_{2fd}(P_3) = \lceil \frac{3}{2} \rceil = 2$.

For $n \geq 3$, $K_{1,n}$ has no vertex of degree 2. So it is impossible to have $|N_{K_{1,n}}(x) \cap S| = 2$ for all $x \in V(K_{1,n})\backslash S$. Thus, $\gamma^e_{2fd}(K_{1,n}) = |V(K_{1,n})| = n+1$.

Case 3: For $k \geq 3$

Suppose $\gamma^e_{kfd}(K_{1,n}) = m \leq n+1$. Let S be a γ^e_{kfd}-set of $K_{1,n}$. Then there exists $v \in V(K_{1,n})\backslash S$ such that $|N_{K_{1,n}}(v) \cap S| = m \geq 3$, a contradiction since $|N_{K_{1,n}}(v)| = 1$ for all $v \in V(K_{1,n}), v \neq u$. Hence, $\gamma^e_{kfd}(K_{1,n}) = n+1$ for $k \geq 3$.

6. Equitable k-Fair Domination in the Join of Graphs

Theorem 6.1. [8] *Let G and H be non-trivial connected graphs of orders m and n, respectively, and k a positive integer with $1 \leq k \leq max\{m,n\}$. Then $S \subseteq V(G+H)$ is a kFD-set of $G+H$ if and only if one of the following holds:*

(a) $S = V(G+H)$.

(b) $S \subseteq V(G), |S| = k$ *and* S *is a* kFD*-set in* G.

(c) $S \subseteq V(H), |S| = k$ *and* S *is a* kFD*-set in* H.

(d) $S = S_G \cup S_H$, *where* S_G *is a* $(k-|S_H|)FD$*-set of* G *and* S_H *is a* $(k-|S_G|)FD$*-set of* H.

(e) $S = V(G) \cup T$, *where* $|V(G)| = m < k$ *and* T *is a* $(k-m)FD$*-set of* H.

(f) $S = D \cup V(H)$, *where* $|V(H)| = n < k$ *and* D *is a* $(k-n)FD$*-set of* G.

Theorem 6.2. *Let G and H be non-trivial connected graphs of orders m and n, respectively, and k a positive integer with $1 \leq k \leq max\{m,n\}$. Then $S \subseteq V(G+H)$ is an EkFD-set of $G+H$ if and only if one of the following holds:*

(a) $S = V(G+H)$.

(b) $S \subseteq V(G), |S| = k$ *and* S *is an EkFD-set in* G *and for every* $v \in V(H)$, *there exists* $u \in S$ *such that* $|d_{G+H}(u) - d_{G+H}(v)| \leq 1$.

(c) $S \subseteq V(H), |S| = k$ *and* S *is an EkFD-set in* H *and for every* $v \in V(G)$, *there exists* $u \in S$ *such that* $|d_{G+H}(u) - d_{G+H}(v)| \leq 1$.

(d) $S = V(G) \cup S_H$, *where* $|V(G)| = m < k$ *and* S_H *is an E*$(k-m)$*FD-set of* H.

(e) $S = S_G \cup V(H)$, *where* $|V(H)| = n < k$ *and* S_G *is an E*$(k-n)$*FD-set of* G.

(f) $S = S_G \cup S_H$, *where* S_G *is an E*$(k-|S_H|)$*FD-set of* G *and* S_H *is an E*$(k-|S_G|)$*FD-set of* H.

Proof: Let $S \subseteq V(G+H)$ be an EkFD-set in $G+H$ where $k \geq 1$. Then by Definition 2.2, S is a kFD-set in $G+H$. Suppose further that $S \neq V(G+H)$. Consider the following cases:

Case 1: $S \subseteq V(G)$ or $S \subseteq V(H)$

If $S \subseteq V(G)$, then $|S| = k$ and S is a kFD-set in G by Theorem 6.1 (b). Since S is an EkFD-set in $G+H$, then for all $v \in V(H)$, there exists $u \in S$ such that $|d_{G+H}(u) - d_{G+H}(v)| \leq 1$. Similarly, if $S \subseteq V(H)$, the same conclusion follows.

Case 2: $S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$

If $S_G = V(G)$, then $S_H \neq V(H)$ and $m < k$. It follows from Theorem 6.1 (e) that S_H is a $(k-m)$FD-set of H. Now, since S is an EkFD-set in $G+H$, then for all $y \in V(H) \backslash S_H$, there exists $u \in S_H$ such that $|d_H(u) - d_H(y)| \leq 1$. Thus, S_H is an E$(k-m)$FD-set of H. Similarly, if $S_H = V(H)$, then $S_G \neq V(G)$ and $n < k$. It follows that S_G is a $(k-n)$FD-set of G and for all $x \in V(G) \backslash S_G$, there exists $u \in S_G$ such that $|d_G(u) - d_G(x)| \leq 1$. Thus, S_G is an E$(k-n)$FD-set of G.

If $S_G \neq V(G)$ and $S_H \neq V(H)$, then $S = S_G \cup S_H$, where S_G is a $(k - |S_H|)$FD-set of G and S_H is a $(k - |S_G|)$FD-set of H by Theorem 6.1 (d). Now, for all $v \in V(G) \backslash S_G, w \in V(H) \backslash S_H$, there exists $u_1 \in S_G$ and $u_2 \in S_H$, respectively, such that $|d_G(u_1) - d_G(v)| \leq 1$ and $|d_H(u_2) - d_H(w)| \leq 1$, respectively. Thus, S_G and S_H are E$(k - |S_H|)$FD-set of G and E$(k - |S_G|)$FD-set of H, respectively.

Conversely, suppose one of the Statements (a) to (f) holds. Then S is a kFD-set in $G + H$ by Theorem 6.1. Moreover, since for all $v \in V(G+H) \backslash S$, there exists $u \in S$ such that $|d_{G+H}(u) - d_{G+H}(v)| \leq 1$. Then S is an EkFD-set in $G + H$ by Definition 2.2. □

Corollary 6.3. *Let G and H be non-trivial connected graphs of orders m and n, respectively, and k a positive integer with $1 \leq k \leq max\{m, n\}$. If G or H has a kFD-set S with $|S| = k$, and for all $v \in V(G + H) \backslash S$, there exists $u \in S$ such that $|d_{G+H}(u) - d_{G+H}(v)| \leq 1$, then $\gamma^e_{kfd}(G + H) = k$. Moreover, if one of the Statements (d),(e), or (f) of Theorem 6.2 holds, respectively, then $\gamma^e_{kfd}(G + H) = m + \gamma^e_{(k-m)fd}(H)$, $\gamma^e_{(k-n)fd}(G) + n$, or $\gamma^e_{(k-|S_H|)fd}(G) + \gamma^e_{(k-|S_G|)fd}(H)$, respectively. Otherwise, $\gamma^e_{kfd}(G + H) = |V(G + H)| = m + n$.*

Proof: Suppose G or H has a kFD-set S with $|S| = k$, and for all $v \in V(G+H) \backslash S$, there exists $u \in S$ such that $|d_{G+H}(u) - d_{G+H}(v)| \leq 1$. Then, by Theorem 6.2 (b) and (c), S is an EkFD-set in $G + H$. Thus, $\gamma^e_{kfd}(G + H) \leq |S| = k$. Note that by Remark 3.2, $\gamma^e_{kfd}(G + H) \geq \gamma_{kfd}(G + H) = k$. Hence, $\gamma^e_{kfd}(G + H) = |S| = k$. Suppose Statement (d), (e) or (f) of Theorem 6.2 holds, respectively, then S is clearly an EkFD-set. Thus,

$$\gamma^e_{kfd}(G + H) = |S_G| + |S_H| = \begin{cases} m + \gamma^e_{(k-m)fd}(H), & \text{if } S_G = V(G) \\ & \text{and } S_H \subset V(H). \\ \gamma^e_{(k-n)fd}(G) + n, & \text{if } S_G \subset V(G) \\ & \text{and } S_H = V(H). \\ \gamma^e_{(k-|S_H|)fd}(G) + \gamma^e_{(k-|S_G|)fd}(H), & \text{if } S_G \subset V(G) \\ & \text{and } S_H \subset V(H). \end{cases}$$

Otherwise, $\gamma^e_{kfd}(G + H) = |V(G + H)| = m + n$. □

7. Equitable k-Fair Domination in the Corona of Graphs

Theorem 7.1. [8] *Let G and H be non-trivial connected graphs and let k be a positive integer with $k \leq |V(H)|$. Then $C \subseteq V(G \circ H)$ is a kFD-set in $G \circ H$ if and only if one of the following holds:*

(a) $C = V(G) \cup B$ where $B = \varnothing$ or $B = \displaystyle\bigcup_{v \in V(G)} S_v$, where each S_v is a $(k-1)FD$-set of

H^v.

(b) $C = \bigcup_{v \in V(G)} S_v$, *where each S_v is a kFD-set of H^v and $|S_v| = k$.*

Theorem 7.2. *Let G and H be non-trivial connected graphs and let k be a positive integer with $k \leq |V(H)|$. Then $C \subseteq V(G \circ H)$ is an EkFD-set in $G \circ H$ if and only if one of the following holds:*

(a) $C = V(G \circ H)$

(b) $C = V(G) \cup B$ *where*

(i) $B = \varnothing$ *and* $\{v\} \subseteq V(G)$ *is an equitable dominating set of* $H^v + v$ *for all* $v \in V(G)$ *or*

(ii) $B = \bigcup_{v \in V(G)} S_v$, *where each S_v is an $E(k-1)$FD-set of H^v or*

(iii) $B = \bigcup_{v \in V(G)} H^{v_i}$, *where $\{v_i\}$ is not an equitable dominating set of $H^{v_i} + v_i$ for some $v_i \in V(G)$ and $k = 1$.*

(c) $C = \bigcup_{v \in V(G)} S_v$, *where each S_v is an EkFD-set of $H^v + v$ and $|S_v| = k$.*

Proof: Let C be an EkFD-set in $G \circ H$ where $k \leq |V(H)|$. Then by Definition 2.2, C is a kFD-set in $G \circ H$. Now, consider the following cases:
Case 1: $V(G) \subseteq C$

By Theorem 7.1 (a), $C = V(G) \cup B$ where $B = \varnothing$ or $B = \bigcup_{v \in V(G)} S_v$, where each S_v is a $(k-1)$FD-set of H^v.

Subcase 1: $B = \varnothing$

Clearly, C is a 1FD-set in $G \circ H$. Since C is an EkFD-set in $G \circ H$, then for all $x \in V(H^v)$, $|d_{G \circ H}(v) - d_{G \circ H}(x)| \leq 1$ where $v \in V(G)$. That is, $V(H^v + v) \cap C = \{v\}$ is an equitable dominating set in $H^v + v$ for all $v \in V(G)$.

Subcase 2: $B = \bigcup_{v \in V(G)} S_v$

Suppose $k \geq 2$. Then by Theorem 7.1 (a), $B = \bigcup_{v \in V(G)} S_v$, where each S_v is a $(k-1)$FD-set of H^v. Note that $d_{G \circ H}(x) < d_{G \circ H}(v)$ where $x \in V(H^v)$. Since C is an equitable dominating set in $G \circ H$, then for all $x \in V(H^v)$, there exists $w \in S_v$ such that $|d_{G \circ H}(w) - d_{G \circ H}(x)| \leq 1$. Hence, $V(H^v + v) \cap C = \bigcup_{v \in V(G)} (S_v \cup \{v\})$ is an equitable dominating set in $H^v + v$.

Subcase 3: $B = \bigcup_{v \in V(G)} H^{v_i}$

Suppose $\{v_i\}$ is not an equitable dominating set of $H^{v_i}+v_i$ for some $v_i \in V(G)$ and $k=1$. Then $B = \bigcup_{v\in V(G)} H^{v_i}$ where $H^{v_i}+v_i$ is a γ^e_{1fd}-set in $H^{v_i}+v_i$.

Case 2: $V(G)\cap C = \varnothing$

By Theorem 7.1 (b), $C = \bigcup_{v\in V(G)} S_v$, where each S_v is a kFD-set of H^v and $|S_v| = k$. Suppose there exists $v_i \in V(H^{v_i}+v_i)\backslash S_v$ such that $|d_{G\circ H}(w) - d_{G\circ H}(v_i)| > 1$ where $w \in S_v$. Then, S_v is not equitable dominating set in $H^{v_i}+v_i$ and C is not an EkFD-set in $G\circ H$, which is a contradiction. Thus, S_v is an EkFD-set in H^v+v for all $v\in V(G)$ and $|S_v| = k$.

Conversely, if $C = V(G)\cup B$ and (i) or (iii) holds, then by Theorem 7.1 (a) and Definition 2.1, C is an EkFD-set in $G\circ H$ where $k=1$. Now, if $k\geq 2$ and (ii) holds, that is, for all $x\in V(G\circ H)\backslash C$, there exists $w\in S_v$ such that $|d_{G\circ H}(w)-d_{G\circ H}(x)|\leq 1$. Then, C is an EkFD-set in $G\circ H$. Similarly, if (c) holds, then by Theorem 7.1 (b), C is a kFD-set in $G\circ H$. Since each S_v is an equitable dominating set in H^v+v for all $v\in V(G)$, it follows that C an EkFD-set in $G\circ H$. □

Corollary 7.3. *Let G and H be non-trivial connected graphs of orders m and n, respectively, and let k be a positive integer with $1\leq k\leq n$. If $V(G)$ is an E1FD-set in $G\circ H$, then $\gamma^e_{kfd}(G\circ H) = m$. If one of Statement ($ii$) or ($iii$) of Theorem 7.2 holds, respectively, then $\gamma^e_{kfd}(G\circ H) = m\left(1+\gamma^e_{(k-1)fd}(H)\right)$ or $m+\sum_{v_i\in V(G)}|H^{v_i}|$, respectively. If H^v+v has an EkFD-set S, $|S|=k$, then $\gamma^e_{kfd}(G\circ H) = mk$. Otherwise, $\gamma^e_{kfd}(G\circ H) = m+mn$.*

Proof: Clearly, $\gamma^e_{kfd}(G\circ H) = m$ if $k=1$ and $\{v\}$ is an equitable dominating set of H^v+v for all $v\in V(G)$. Suppose that $k\geq 2$ and H^v+v has no EkFD-set S with $|S|=k$. Then, if C is an γ^e_{kfd}-set of $G\circ H$, then $C = V(G)\cup B$ where $B = \bigcup_{v\in V(G)} S_v$ and each S_v is an $\gamma^e_{(k-1)fd}$-set of $G\circ H$ by Theorem 7.2. Thus,

$$\begin{aligned}\gamma^e_{kfd}(G\circ H) \leq |C|| &= m+\sum_{v\in V(G)}|S_v| \\ &= m+m\left(\gamma^e_{(k-1)fd}(H)\right) \\ &= m\left(1+\gamma^e_{(k-1)fd}(H)\right).\end{aligned}$$

Suppose $\{v_i\}$ is not an equitable dominating set for some $H^{v_i}+v_i$, where $v_i\in V(G)$ and $k=1$. Then,

$$\gamma^e_{kfd}(G\circ H) = m+\sum_{v_i\in V(G)}|H^{v_i}|, \text{ for some } \{v_i\}\subseteq V(G).$$

Now, suppose that H^v+v has an EkFD-set S with $|S|=k$, for all $v\in V(G)$. Let $S_v\subseteq H^v+v\subseteq V(G\circ H)$ such that $\langle S_v\rangle \approx \langle S\rangle$. Then by Theorem 7.2 ($c$), $C = \bigcup_{v\in V(G)} S_v$ is an EkFD-set of $G\circ H$. Thus,

$$\gamma^{e}_{kfd}(G \circ H) \leq |C| = \sum_{v \in V(G)} |S_v| = m|S| = mk.$$

If $C^* = V(G) \cup B$ where $B = \bigcup_{v \in V(G)} S_v$ and each S_v is an E$(k-1)$FD-set of H^v is a γ^{e}_{kfd}-set of $G \circ H$, then $|S_v| \geq k-1$ for all $v \in V(G)$. Thus,

$$\gamma^{e}_{kfd}(G \circ H) = |C^*| = m + \sum_{v \in V(G)} |S_v| \geq m + m(k-1) = mk.$$

Therefore, $\gamma^{e}_{kfd}(G \circ H) = mk$. Finally, suppose that none of Statements (b) or (c) of Theorem 7.2 holds, then $\gamma^{e}_{kfd}(G \circ H) = m + mn$. □

References

[1] C. Berge. *The Theory of Graphs and Applications*. Methuen, London, 1962.

[2] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi. Total domination in graphs. *Networks*, 10:211–219, 1980.

[3] A. M. Yaglom and I. M. Yaglom. *Challenging Mathematical Problems with Elementary Solutions*, volume 1. Dover Publications, 1964.

[4] B. Brešar and D. Rall. Fair reception and vizing's conjecture. *Journal of Graph Theory*, pages 45–54, 2009.

[5] J. M. Tarr. *Domination in Graphs*. PhD thesis, University of South Florida, 2010.

[6] Y. Caro, A. Hansberg, and M. Henning. Fair domination in graphs. *Discrete Mathematics*, 312(19):2905–2914, 2011.

[7] E. Maravilla, R. Isla, and S. Canoy Jr. Fair domination in the join, corona and composition of graphs. *Applied Mathematical Sciences*, 8(93):4609–4620, 2014.

[8] E. Maravilla, R. Isla, and S. Canoy Jr. k-fair domination in the join, corona, composition and cartesian product of graphs. *Applied Mathematical Sciences*, 8(178):8863–8874, 2014.

[9] V. Swaminathan, R. Sundareswaran, D. Lakshmanaraj, P. Nataraj, and L. Muthusubramanian. Equitable fair domination in graphs. *Discrete Mathematics, Algorithms and Applications*, 13(6):2150083, 2021.

[10] A. Anitha, S. Arumugam, and E. Sampathkumar. Degree equitable sets in a graph. *International Journal of Mathematical Combinatorics*, 3:32–47, 2009.

[11] V. Swaminathan and K. M. Dharmalingam. Degree equitable domination in graphs. *Kragujevac Journal of Mathematics*, 35(1):191–197, 2011.

[12] E. P. Sandueta. Equitable domination in some graphs. *Applied Mathematical Sciences*, 13(7):309–314, 2019.

[13] T. G. Jayasree and R. R. Iyer. 1-fair dominating sets in some class of graphs. *International Journal of Advanced Trends in Computer Science and Engineering*, 8(3):610–614, 2019.

[14] T. G. Jayasree and R. R. Iyer. 2-fair dominating sets in some class of graphs. *International Journal of Advanced Trends in Computer Science and Engineering*, 8(6):2900–2903, 2019.