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Partial Sums for Normalized Mittag-Leffler-Prabhakar Function and Barnes-Mittag-Leffler Function

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Abstract. Building on recent research that established partial sum and lower bounds for various special functions, this paper extends the scope to investigate the normalized Le Roy-type Mittag-Leffler-Prabhakar and Barnes-Mittag-Leffler functions. We aim to determine lower bounds for these functions and their partial sums. We are also presenting some new consequences, lemmas, and corollaries that highlight the significance of our findings. Our results are novel and enhance existing knowledge in the field.

2020 Mathematics Subject Classifications: 30C45, 30C50, 30C80

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1. Introduction

Let $\mathcal{A}$ be the set of analytic functions (AFs) in the disc $U = \{\xi \in \mathbb{C} : |\xi| < 1\}$. Every $f \in \mathcal{A}$, and f is normalized if $f(0) = 0$ and $f^{'}(0) = 1$ and have the Taylor series representation

$$f(\xi) = \xi + \sum_{n=2}^{+\infty} a_n \xi^n. \tag{1}$$

Additionally, we define $\mathcal{S}$ is subset of $\mathcal{A}$ composed of function f that are univalent (one-to-one) in U. For any function f in $\mathcal{A}$, we can form a partial sum, denoted as $f_m(\xi)$, by its Taylor series expansion to the m^{th} term:

$$f_m(\xi) = \xi + \sum_{n=2}^{m} a_n \xi^n. \tag{2}$$

A function $f \in \mathcal{A}$ is considered starlike in U if its image $f(U)$ is a star-shaped domain and such type of starlike functions [1] denoted by $\mathcal{S}^*$. These functions can be analytically characterized by the condition:

$$\mathrm{Re}\left(\frac{\xi f^{'}(\xi)}{f(\xi)}\right) > 0,\ \xi \in U.$$

Similarly, a function f in $\mathcal{A}$ is considered convex if its image $f(U)$ is a convex [1] domain and denoted by $\mathcal{K}$. These functions can be analytically characterized by the condition:

$$\mathrm{Re}\left(1 + \frac{\xi f^{''}(\xi)}{f^{'}(\xi)}\right) > 0,\ \xi \in U$$

if and only if

$$f \in \mathcal{K}.$$

It is shown by Alexander in [2] that

$$\xi f^{'} \in \mathcal{S}^*$$

if and only if

$$f \in \mathcal{K}.$$

For a function f defined in U, the integral transformation $\mathcal{I}[f]$ is given by the expression $\mathcal{I}[f] = \int_0^z \frac{f(t)}{t} dt$ which can be expanded as:

$$\mathcal{I}[f] = \xi + \sum_{n=2}^{+\infty} \frac{a_n}{n} \xi^n. \tag{3}$$

This transformation is known as the Alexander Transformation, named after Alexander [2], who first introduced it. Alexander made a significant discovery, proving that this

integral transformation $\mathcal{I}[f]$ provides a one-to-one correspondence between the class $\mathcal{S}^*$ and $\mathcal{K}$.
The generalized Pochhammer symbol for $x > 0$ is given by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)\cdots(x+n-1), & n \in \mathbb{N}. \end{cases}$$

and for $x > 0$, the Gamma function is defined as:

$$\Gamma(x+1) = x\,\Gamma(x), \quad \text{and} \quad \Gamma(1) = 1.$$

The theory of special functions (SFs) is the useful area of mathematics that has evolved over the last three centuries, driven by the need to solve problems in classical mechanics, hydrodynamics, and control theory. This development has led to extensive applications in both pure and applied mathematics, including geometric function theory, applied mathematics, physics, and statistics (see literature [3–11] for further reading). One notable example of a SFs is the Mittag-Leffler function, which emerges naturally in solving fractional-order integral and differential equations [12–14]. Its relevance extends to studying fractional generalizations of kinetic equations, random walks, Lévy flights, and superdiffusive transport, particularly in complex systems. The Mittag-Leffler function bridges the gap between exponential and power-law behaviors, characteristic of phenomena governed by classical and fractional kinetic equations, as explored in the works of Lang [15], Hilfer [16], and Saxena [17]. Building on these findings, the study of geometric properties in analytic functions involving special functions remains a vibrant and ongoing field of research, with notable contributions from researchers such as Aktas [18], Aktas and Orhan [19], and Bansal and Prajapat [20].

The Swedish mathematician Mittag-Leffler, who originally introduced the one-parameter version of Mittag-Leffler function (MLF) [21] as:

$$E_d(\xi) = \sum_{n=0}^{+\infty} \frac{1}{\Gamma(dn+1)} \xi^n, \;\; d, \xi \in \mathbb{C}, \;\; \mathrm{Re}(d) > 0.$$

The two-parameter MLF, denoted as $E_{d,v}(\xi)$, is a mathematical function that is defined by Wiman in [22]. The series form of two-parameter Mittag-Leffler function is given as:

$$E_{d,v}(\xi) = \sum_{n=0}^{+\infty} \frac{1}{\Gamma(dn+v)} \xi^n, \;\; d, v, \xi \in \mathbb{C}, \;\; \mathrm{Re}(d) > 0.$$

The Le Roy function denoted by $E^{\sigma}(\xi)$, is defined by French mathematician Édouard Le Roy (see [23]) as follows:

$$E^{\sigma}(\xi) = \sum_{n=0}^{+\infty} \frac{\xi^n}{(\Gamma(n+1))^{\sigma}} = \sum_{n=0}^{+\infty} \frac{1}{(n!)^{\sigma}} \xi^n, \quad \xi \in \mathbb{C}, \tag{4}$$

where σ is a positive real number. In recent work, Gerhold [24] and Garra and Polito [25] separately developed the Le Roy-type MLF, which is given by the following definition:

$$E_{d,v}^{\sigma}(\xi)=\sum_{n=0}^{+\infty}\frac{1}{(\Gamma(dn+v))^{\sigma}}\xi^{n},\ d,v,\sigma>0,\xi\in\mathbb{C}.$$

The Barnes–Mittag-Leffler function $B_{d,v}^{b,s}(\xi)$ is defined by [26] as follows:

$$B_{d,v}^{b,s}(\xi)=\sum_{n=0}^{+\infty}\frac{1}{(n+b)^{s}\,\Gamma(dn+v)}\xi^{n}.$$

In 2017, Tomovski, Mehrez [27] considered the Mittag-Leffler Prabhakar functions (MLPF) of Le Roy-type defined as:

$$E_{d,v}^{\sigma,\chi}(\xi)=\sum_{n=0}^{+\infty}\frac{\Gamma(\chi+n)}{n!\Gamma(\chi)(\Gamma(dn+v))^{\sigma}}\xi^{n},\ d,v,\sigma,\chi>0,\xi\in\mathbb{C}. \tag{5}$$

A special case of the function $E_{d,v}^{\sigma,\chi}(\xi)$ arises when $\chi=d=v=1$, which simplifies to the Le Roy-type function (LRFs) defined in [23] and further investigated by Mehrez and Das in their work [28]. This function, denoted as $E^{\sigma}(\xi)$, and defined in (4).

The MLPF of Le Roy type and their generalizations have found applications in fractional calculus, as discussed in Pane's work [29]. In geometric function theory, Mehraz and Raza [30] determined the conditions under which Mittag-Leffler-Prabhakar functions of Le Roy type possess key geometric properties, including starlikeness, convexity, and pre-starlikeness, by imposing specific constraints on their parameters. For a deeper understanding of the advanced properties and applications of these functions, including their fractional integration and differentiation formulas, solutions to differential equations, integral transforms, and other uses, we refer to recent publications [6, 7].

Now we demonstrate that Le Roy-type Mittag-Leffler-Prabhakar functions (denoted by $\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)$) belong to class $\mathcal{A}$.

$$\begin{aligned}\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)&=\xi(\Gamma(v))^{\sigma}E_{d,v}^{\sigma,\chi}(\xi)\\&=\sum_{n=0}^{+\infty}\frac{\Gamma(\chi+n)(\Gamma(v))^{\sigma}}{n!\Gamma(\chi)(\Gamma(dn+v))^{\sigma}}\xi^{n+1},\quad\xi\in\mathbb{C},\end{aligned} \tag{6}$$

where $d,v,\sigma,\chi>0$. Now $\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)$ satisfies the normalization conditions, namely $\mathcal{F}_{d,v}^{\chi,\sigma}(0)=0$ and $\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)'(0)=1$. Using the Stirling asymptotic formula for the gamma function, which applies to large values of ξ, we find that the series in equation (6) represents an entire function (meaning it converges absolutely for all complex numbers ξ if $d\sigma>0$.

The functions given in equations (5) and (6) provide a extended version that encompasses many well-known special functions found in the mathematical literature. Interestingly, these functions overlap with several well-known functions when specific parameter values are applied. For instance, when $\chi=1$ in (6), then $\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)$ reduces to the normalized version of Le Roy-type Mittag-Leffler function $E_{d,v}^{1,\sigma}(\xi)$, which was first defined by

Garra and Polito in [25]. Further research on the related function $E_{d,v}^{1,\sigma}(\xi)$ was conducted by authors in [28].

Additionally, when $\sigma = 1$ in (6), then, the function $\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)$ becomes the normalized version of three-parameter Mittag-Leffler function $E_{d,v}^{\chi,1}(\xi)$, introduced by Prabhakar [8]. Recently, Garro and Gorropo [31] investigated the applications of the Prabhakar or three-parameter Mittag-Leffler function, focusing on nonlinear heat conduction equations with memory that involve Prabhakar derivatives. They derived exact solutions and analyzed the asymptotic behavior of these equations.

Moreover, when $\chi = \sigma = 1$ in (6), then, the function $\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)$ reduces to the normalized version of two-parameter Mittag-Leffler function $E_{d,v}(\xi)$ defined by Wiman [22], and extensively studied in [9] (see [32]). Finally, using $\chi = v = \sigma = 1$, in (6), then the function $\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)$ becomes the normalized version of Mittag-Leffler function $E_d(\xi)$, defined and studied by Mittag-Leffler [21] in 1903. We also observed that the function $\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)$ contains many well-known functions as its special case, for example

$$\mathcal{F}_{3,1}^{1,1}(\xi) = \frac{e^{\xi^{\frac{1}{3}}}}{2} + e^{-\frac{1}{2}\xi^{\frac{1}{3}}}\cos\frac{\sqrt{3}}{2}\xi^{\frac{1}{3}},\ \mathcal{F}_{4,1}^{1,1}(\xi) = \frac{\cos\xi^{\frac{1}{4}}}{2} + \frac{\cosh\xi^{\frac{1}{4}}}{2},$$
$$\mathcal{F}_{2,1}^{1,1}\left(\xi^2\right) = \cosh\xi,\ \mathcal{F}_{2,1}^{1,1}\left(-\xi^2\right) = \cos\xi,\ \mathcal{F}_{1,1}^{1,1}(\xi) = e^{\xi},\ \mathcal{F}_{1,2}^{1,1}(\xi) = \frac{e^{\xi}-1}{\xi},$$
$$\mathcal{F}_{2,2}^{1,1}\left(\xi^2\right) = \frac{\sinh\xi}{\xi},\ \mathcal{F}_{2,2}^{1,1}\left(-\xi^2\right) = \frac{\sin\xi}{\xi}.$$

Similarly, we perform the normalized Barnes–Mittag-Leffler function, denoted as $BM_{d,v}^{b,s}(\xi)$, which is defined as follows:

$$\begin{aligned} BM_{d,v}^{b,s}(\xi) &= \xi b^s \Gamma(v)\, B_{d,v}^{b,s}(\xi) \\ &= \sum_{n=0}^{+\infty} \frac{b^s\Gamma(v)}{(n+b)^s\,\Gamma(dn+v)}\xi^{n+1}, \end{aligned} \tag{7}$$

where $(\ d, v, b, s > 0,\ \xi \in \mathbb{C})$.

Partial sums of analytic functions play a significant role in Geometric Function Theory, particularly in finding the largest disk $U_r = \{\xi \in \mathbb{C} : |\xi| < r\}$ where the partial sum remains one-to-one. In 1928, Szegö [33] proved that for functions in the class $\mathcal{S}$, each partial sum, $f_m(\xi)$, is one-to-one within the disk $U_{\frac{1}{4}} = \left\{\xi \in \mathbb{C} : |\xi| < \frac{1}{4}\right\}$. However, this does not mean that partial sums of functions in $\mathcal{S}$ are always one-to-one in U. For example, the convex univalent function $f(\xi) = \xi/1-\xi$ shows that this is not the case. Furthermore, the second partial sum $f_2(\xi) = \xi + 2\xi^2$ of the Koebe function

$$k(\xi) = \frac{\xi}{(1-\xi)^2}$$

is one-to-one within $U_{\frac{1}{4}}$, and radius $\frac{1}{4}$ is the best possible.

The radius of starlikeness of the partial sum $(f_m(\xi))$ of functions in the class $\mathcal{S}^*$ was established by Robertson [34]. Moreover, several researchers have investigated the lower bound of the real part of the ratio of the partial sum of analytic functions to their infinite series sum. This concept was first introduced by Silvia in [35]. Silverman [36] later developed more useful techniques to find the partial sum of starlike and convex functions. Subsequent studies have extended these results to various subclasses of analytic functions, as seen in references [37–40]. Recently, researchers have explored partial sum of special functions, such as the normalized Struve functions [41], Dini functions [42], and Wright functions [43] while Kazımoğlu [44] have made significant contributions to this area.

The sequence of partial sums of $\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)$ is defined as:

$$\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)_m(\xi)=\xi+\sum_{n=1}^{m}\frac{\Gamma(\chi+n)(\Gamma(v))^{\sigma}}{n!\Gamma(\chi)(\Gamma(dn+v))^{\sigma}}\xi^{n+1},\quad m\in\mathbb{N}. \tag{8}$$

Similarly, the sequence of partial sums of $BM_{d,v}^{b,s}(\xi)$ is defined as:

$$\left(BM_{d,v}^{b,s}(\xi)\right)_m(\xi)=\xi+\sum_{n=1}^{m}\frac{b^s\Gamma(v)}{(n+b)^s\Gamma(dn+v)}\xi^{n+1},\quad m\in\mathbb{N}. \tag{9}$$

If $m=0$, we have

$$\sum_{n=1}^{0}\frac{\Gamma(\chi+n)(\Gamma(v))^{\sigma}}{n!\Gamma(\chi)(\Gamma(dn+v))^{\sigma}}\xi^{n+1}=0$$

and

$$\sum_{n=1}^{0}\frac{b^s\Gamma(v)}{(n+b)^s\Gamma(dn+v)}\xi^{n+1}=0.$$

In this paper, we investigate the ratio of a function, defined by (6) and (7), to its sequence of partial sums, given by (8) and (9), and establish lower bounds for

$$\operatorname{Re}\left(\frac{\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)_m(\xi)}\right),\ \operatorname{Re}\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)_m(\xi)}{\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)}\right), \tag{10}$$

$$\operatorname{Re}\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)'_m(\xi)}\right),\ \operatorname{Re}\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)'_m(\xi)}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'}\right), \tag{11}$$

$$\operatorname{Re}\left(\frac{\mathcal{I}\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)(\xi)}{\left(\mathcal{I}\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)\right)_m(\xi)}\right),\quad \operatorname{Re}\left(\frac{\left(\mathcal{I}\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)\right)_m(\xi)}{\mathcal{I}\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)(\xi)}\right) \tag{12}$$

and

$$\operatorname{Re}\left(\frac{BM_{d,v}^{b,s}(\xi)}{\left(BM_{d,v}^{b,s}\right)_m(\xi)}\right),\ \operatorname{Re}\left(\frac{\left(BM_{d,v}^{b,s}\right)_m(\xi)}{BM_{d,v}^{b,s}(\xi)}\right), \tag{13}$$

$$\operatorname{Re}\left(\frac{\left(BM_{d,v}^{b,s}(\xi)\right)'}{\left(BM_{d,v}^{b,s}\right)'_{m}(\xi)}\right), \quad \operatorname{Re}\left(\frac{\left(BM_{d,v}^{b,s}\right)'_{m}(\xi)}{\left(BM_{d,v}^{b,s}(\xi)\right)'}\right), \tag{14}$$

$$\operatorname{Re}\left(\frac{\mathcal{I}\left(BM_{d,v}^{b,s}\right)(\xi)}{\left(\mathcal{I}\left(BM_{d,v}^{b,s}\right)\right)_{m}(\xi)}\right), \quad \operatorname{Re}\left(\frac{\left(\mathcal{I}\left(BM_{d,v}^{b,s}\right)\right)_{m}(\xi)}{\mathcal{I}\left(BM_{d,v}^{b,s}\right)(\xi)}\right). \tag{15}$$

The paper is organized as follows. Section 1 provides a comprehensive introduction, covering the historical background, preliminary concepts, Mittag-Leffler-Prabhakar functions of Le Roy type and Barnes-Mittag-Leffler function. It also presents the normalization of these functions within the unit disk and discusses some special cases in this section. Section 2 introduces a set of known and new lemmas essential for proving the main results. Section 3 is divided into two parts: the first part investigates Theorems 1-3 related to the normalized Le Roy-type Mittag-Leffler-Prabhakar function defined in (6), along with several specific cases, while the second part examines Theorems 4-6 related to the normalized Barnes-Mittag-Leffler function defined in (7). Finally, the last section discusses the conclusions and future directions of the main work.

2. Basic concepts or Preliminaries

The following lemmas are necessary to investigate the main results for $F_{d,v}^{\chi,\sigma}(\xi)$ and $BM_{d,v}^{b,s}(\xi)$ defined in (6) and (7).

Lemma 1. *([30], proof of Theorem 3.1 on page 747). Assume that d, v, χ, and σ are arbitrary positive numbers.*
(i) If $d \geq 1$, $d\sigma \geq 1$ and $v \geq \chi$, then the sequence $(b_n)_{n\geq 1}$ defined by

$$b_n(d, v, \chi, \sigma) = \frac{\Gamma(\chi + n)}{\Gamma(\chi)} \left(\frac{\Gamma(v)}{\Gamma(dn + v)}\right)^{\sigma}$$

is decreasing.
(ii)Also,

$$c_n(d, v, \chi, \sigma) = (n + 1)\, b_n(d, v, \chi, \sigma)$$

is decreasing.

Lemma 2. *Assume that d, v, χ, and σ are arbitrary positive numbers.*
(i) If $d \geq 1$, $d\sigma \geq 1$ and $v \geq \chi$, then

$$\left|\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right| \leq 1 + B_1(e - 1), \ \xi \in U.$$

(ii)

$$\left|\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'\right| \leq 1 + 2B_1(e - 1), \ \xi \in U.$$

(iii)

$$\left|\mathcal{I}\left[\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)\right](\xi)\right| \leq 1 + B_1(e-2), \ \xi \in U,$$

where

$$B_1 = \chi\left(\frac{\Gamma(v)}{\Gamma(d+v)}\right)^{\sigma}. \tag{16}$$

Proof. (i) If $d \geq 1, d\sigma \geq 1$ and $v \geq \chi$ and let

$$B_n = \frac{b_n(d,v,\chi,\sigma)}{n!} \text{ and } B_0 = 1. \tag{17}$$

Thus, according to the Lemma 1, we see that

$$\begin{aligned}\left|\sum_{n=0}^{+\infty} B_n \xi^{n+1}\right| &= \left|\xi + \sum_{n=1}^{+\infty} \frac{b_n(d,v,\chi,\sigma)}{n!}\xi^{n+1}\right| \\ &\leq 1 + b_1(d,v,\chi,\sigma)\sum_{n=1}^{+\infty}\frac{1}{n!} \\ &= 1 + (e-1)B_1. \end{aligned} \tag{18}$$

Now from (6) and using (18) we have

$$\begin{aligned}\left|\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right| &= \left|\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{\Gamma(\chi+n)}{\Gamma(\chi)}\left(\frac{\Gamma(v)}{\Gamma(dn+v)}\right)^{\sigma}\right| = \left|\sum_{n=0}^{+\infty} B_n\xi^{n+1}\right| \\ &\leq 1 + (e-1)B_1.\end{aligned}$$

(ii) If $d \geq 1$, $d\sigma \geq 1$ and $v \geq \chi$, and let

$$C_n = \frac{c_n(d,v,\chi,\sigma)}{n!} \text{ and } C_0 = 1.$$

Thus, according to the Lemma 1, we see that

$$\begin{aligned}\left|\sum_{n=0}^{+\infty} C_n \xi^{n+1}\right| &= \left|\xi + \sum_{n=1}^{+\infty} \frac{c_n(d,v,\chi,\sigma)}{n!}\xi^{n+1}\right| \\ &\leq 1 + C_1\sum_{n=1}^{+\infty}\frac{1}{n!} \\ &= 1 + 2B_1(e-1). \end{aligned} \tag{19}$$

Now from (6) and using (19) we have

$$\begin{aligned}\left|\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'\right| &= \left|\sum_{n=0}^{+\infty}(n+1)\frac{1}{n!}\frac{\Gamma(\chi+n)}{\Gamma(\chi)}\frac{(\Gamma(v))^{\sigma}}{(\Gamma(dn+v))^{\sigma}}\xi^n\right| \\ &= \left|\sum_{n=0}^{+\infty} C_n\xi^n\right| \leq 1 + 2(e-1)B_1.\end{aligned}$$

(iii) If $d \geq 1$, $d\sigma \geq 1$ and $v \geq \chi$, then from (6) and using (18) we have

$$\begin{aligned}\left|\mathcal{I}\left[\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right](\xi)\right| &= \left|\xi + \sum_{n=1}^{+\infty} \frac{1}{(n+1)} \frac{1}{n!} \frac{\Gamma(\chi+n)}{\Gamma(\chi)} \left(\frac{\Gamma(v)}{\Gamma(dn+v)}\right)^{\sigma} \xi^{n+1}\right| \\ &\leq 1 + b_1(d,v,\chi,\sigma) \sum_{n=1}^{+\infty} \frac{1}{(n+1)!} \\ &= 1 + B_1(e-2).\end{aligned}$$

Hence, proof of Lemma 2 have completed.

Lemma 3. *([45], proof of Theorem 4 on page 8). Assume that d, v, b, and s are arbitrary positive numbers.*
(i): If $\min(b,d,v) > 0$ and $s \geq 0$. Then the sequence $(\Psi_n(d,v,b,s))_{n\geq 1}$ defined by

$$\Psi_n(d,v,b,s) = n! \left(\frac{b^s \Gamma(v)}{(n+b)^s \Gamma(dn+v)}\right)$$

is decreasing.
(ii) If $\min(b,d,v) > 0$ and $s \geq 0$. Then the sequence $\left(\widehat{\Psi}_n(d,v,b,s)\right)_{n\geq 1}$ defined by

$$\widehat{\Psi}_n(d,v,b,s) = (n+1)! \left(\frac{b^s \Gamma(v)}{(n+b)^s \Gamma(dn+v)}\right)$$

is also decreasing.

Lemma 4. *Assume that d, v, b, and s are arbitrary numbers.*
(i) If $\min(b,d,v) > 0$ and $s \geq 0$, then

$$\left|BM_{d,v}^{b,s}(\xi)\right| \leq 1 + D_1(e-1), \ \xi \in U.$$

(ii)

$$\left|\left(BM_{d,v}^{b,s}(\xi)\right)'\right| \leq 1 + 2D_1(e-1), \ \xi \in U.$$

(iii)

$$\left|\mathcal{I}\left[BM_{d,v}^{b,s}(\xi)\right](\xi)\right| \leq 1 + D_1(e-2), \ \xi \in U,$$

where

$$D_1 = \frac{b^s \Gamma(v)}{(1+b)^s \Gamma(d+v)}. \tag{20}$$

Proof. Arguments used in the proof of Lemma 2 work also in the frame of Lemma 4 with using the unified Lemma 3 instead of Lemma 1.

3. Main Results

In this section, we investigate Theorems related to the normalized Le Roy-type Mittag-Leffler-Prabhakar function defined in (6).

Theorem 1. *If $d \geq 1$, $d\sigma \geq 1$ and $v \geq \chi$ and $B_1(e-1) \leq 1$, then*

$$\operatorname{Re}\left(\frac{\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)_m(\xi)}\right) \geq 1 - B_1(e-1), \ \xi \in U \tag{21}$$

and

$$\operatorname{Re}\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)_m(\xi)}{\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)}\right) \geq \frac{1}{1+B_1(e-1)}, \ \xi \in U, \tag{22}$$

where B_1 is given by (16).

Proof. First, we recall the inequality (i) of Lemma 2, that is

$$\left|\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right| \leq 1 + B_1(e-1), \ \xi \in U. \tag{23}$$

Using (6) in (23), we get

$$\left|\xi + \sum_{n=1}^{+\infty} \frac{\Gamma(\chi+n)(\Gamma(v))^{\sigma}}{n!\Gamma(\chi)(\Gamma(dn+v))^{\sigma}}\xi^{n+1}\right| \leq 1 + B_1(e-1), \ \xi \in U.$$

Further, we have

$$1 + \sum_{n=1}^{+\infty} |B_n| \leq 1 + B_1(e-1), \ \xi \in U.$$

Or equivalently

$$\frac{1}{B_1(e-1)} \sum_{n=1}^{+\infty} |B_n| \leq 1,$$

where B_n is given by (17).

To establish the inequality (21), we set

$$\frac{1}{B_1(e-1)}\left(\frac{\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)_m(\xi)} - (1 - B_1(e-1))\right)$$
$$= \frac{1 + \sum_{n=1}^{m} B_n\xi^n + \frac{1}{B_1(e-1)}\sum_{n=m+1}^{+\infty} B_n\xi^n}{1 + \sum_{n=1}^{m} B_n\xi^n} = \frac{1+h_1(\xi)}{1+h_2(\xi)}. \tag{24}$$

where

$$h_1(\xi) = \sum_{n=1}^{m} B_n\xi^n + \frac{1}{B_1(e-1)} \sum_{n=m+1}^{+\infty} B_n\xi^n. \tag{25}$$

and

$$h_2(\xi) = \sum_{n=1}^{m} B_n \xi^n. \tag{26}$$

Now we consider

$$\frac{1 + h_1(\xi)}{1 + h_2(\xi)} = \frac{1 + u(\xi)}{1 - u(\xi)}.$$

After some simplification, we have

$$u(\xi) = \frac{h_1(\xi) - h_2(\xi)}{2 + h_1(\xi) + h_2(\xi)}.$$

Thus, clearly, we have

$$u(\xi) = \frac{\frac{1}{B_1(e-1)} \sum_{n=m+1}^{+\infty} B_n \xi^n}{2 + 2\sum_{n=1}^{m} B_n \xi^n + \frac{1}{B_1(e-1)} \sum_{n=m+1}^{+\infty} B_n \xi^n}.$$

Thus, we have

$$|u(\xi)| \leq \frac{\frac{1}{B_1(e-1)} \sum_{n=m+1}^{+\infty} |B_n|}{2 - 2\sum_{n=1}^{m} |B_n| - \frac{1}{B_1(e-1)} \sum_{n=m+1}^{+\infty} |B_n|}.$$

A well-known fact states that the following equivalence is hold:

$$\operatorname{Re}\left(\frac{1 + u(\xi)}{1 - u(\xi)}\right) \geq 0,\ \xi \in U \Leftrightarrow |u(\xi)| \leq 1,\ \xi \in U.$$

We can now see that $|u(\xi)| \leq 1$ follows once we prove

$$\frac{1}{B_1(e-1)} \sum_{n=m+1}^{+\infty} |B_n| \leq 1 - \sum_{n=1}^{m} |B_n|.$$

This is equivalent to the inequality

$$\sum_{n=1}^{m} |B_n| + \frac{1}{B_1(e-1)} \sum_{n=m+1}^{+\infty} |B_n| \leq 1. \tag{27}$$

Our goal is to prove that the left-hand side of inequality (27) is bounded above by

$$\frac{1}{B_1(e-1)} \sum_{n=1}^{+\infty} |B_n|.$$

After some simple calculations, we have

$$\left(\frac{1}{B_1(e-1)} - 1\right) \sum_{n=1}^{m} |B_n| \geq 0. \tag{28}$$

Thus, by virtue of (28), the proof of the inequality in (21) is now complete.

Next, to prove (22), we set

$$\left(1+\frac{1}{B_1\left(e-1\right)}\right)\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)_m\left(\xi\right)}{\mathcal{F}_{d,v}^{\chi,\sigma}\left(\xi\right)}-\frac{1}{1+B_1\left(e-1\right)}\right)$$
$$=\frac{1+\sum_{n=1}^{m}B_n\xi^n+\frac{1}{B_1(e-1)}\sum_{n=m+1}^{+\infty}B_n\xi^n}{1+\sum_{n=1}^{+\infty}B_n\xi^n}$$
$$=\frac{1+h_1\left(\xi\right)}{1+h_2\left(\xi\right)}.$$

After some simplification, we have

$$u(\xi)=\frac{h_1\left(\xi\right)-h_2\left(\xi\right)}{2+h_1\left(\xi\right)+h_2\left(\xi\right)}.$$

Thus, clearly, we have

$$|u(\xi)|\leq\frac{\left(1+\frac{1}{B_1(e-1)}\right)\sum_{n=m+1}^{+\infty}|B_n|}{2-2\sum_{n=1}^{m}|B_n|-\left(\frac{1}{B_1(e-1)}-1\right)\sum_{n=m+1}^{+\infty}|B_n|}.$$

We can now see that $|u(\xi)|\leq 1$ follows once we prove

$$\frac{1}{B_1\left(e-1\right)}\sum_{n=m+1}^{+\infty}|B_n|\leq 1-\sum_{n=1}^{m}|B_n|.$$

This is equivalent to the inequality

$$\sum_{n=1}^{m}|B_n|+\frac{1}{B_1\left(e-1\right)}\sum_{n=m+1}^{+\infty}|B_n|\leq 1. \tag{29}$$

Our goal is to prove that the left-hand side of inequality (29) is bounded above by

$$\frac{1}{B_1\left(e-1\right)}\sum_{n=1}^{+\infty}|B_n|.$$

Alternatively,

$$\left(\frac{1}{B_1\left(e-1\right)}-1\right)\sum_{n=1}^{+\infty}|B_n|\geq 0. \tag{30}$$

Thus, by virtue of (30), the proof of the inequality in (22) is now complete. Hence, this completes the proof of the Theorem 1.

Theorem 2. *If* $d \geq 1$, $d\sigma \geq 1$ *and* $v \geq \chi$ *and* $1 \geq 2B_1(e-1)$, *then*

$$\operatorname{Re}\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)'_m(\xi)}\right) \geq 1 - 2B_1(e-1), \ \xi \in U \tag{31}$$

and

$$\operatorname{Re}\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)'_m(\xi)}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'}\right) \geq \frac{1}{1+2B_1(e-1)}, \ \xi \in U, \tag{32}$$

where B_1 *is defined by (16).*

Proof. First, we recall the inequality (ii) of Lemma 2, that is

$$\left|\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'\right| \leq 1 + 2B_1(e-1), \ \xi \in U. \tag{33}$$

Using (6) in (33), we have

$$\left|1 + \sum_{n=1}^{+\infty}(n+1)\frac{\Gamma(\chi+n)(\Gamma(v))^{\sigma}}{n!\Gamma(\chi)(\Gamma(dn+v))^{\sigma}}\xi^n\right|$$
$$\leq 1 + \sum_{n=1}^{+\infty}(n+1)|B_n| \leq 1 + 2B_1(e-1), \ \xi \in U.$$

Or equivalently

$$\frac{1}{2B_1(e-1)}\sum_{n=1}^{+\infty}(n+1)|B_n| \leq 1,$$

where B_n is given (17). To establish the inequality (31), we set

$$\frac{1}{2B_1(e-1)}\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)'_m(\xi)} - (1-2B_1(e-1))\right)$$
$$= \frac{1 + \sum_{n=1}^{m}(n+1)B_n\xi^n + \frac{1}{2B_1(e-1)}\sum_{n=m+1}^{+\infty}(n+1)B_n\xi^n}{1 + \sum_{n=1}^{m}(n+1)B_n\xi^n}$$
$$= \frac{1+h_1(\xi)}{1+h_2(\xi)}.$$

We can write

$$u(\xi) = \frac{\frac{1}{2B_1(e-1)}\sum_{n=m+1}^{+\infty}(n+1)B_n\xi^n}{2 + 2\sum_{n=1}^{m}(n+1)B_n\xi^n + \frac{1}{2B_1(e-1)}\sum_{n=m+1}^{+\infty}(n+1)B_n\xi^n}.$$

Thus, we get

$$|u(\xi)| \leq \frac{\frac{1}{2B_1(e-1)}\sum_{n=m+1}^{+\infty}(n+1)|B_n|}{2-2\sum_{n=1}^{m}(n+1)|B_n| - \frac{1}{2B_1(e-1)}\sum_{n=m+1}^{+\infty}(n+1)|B_n|}.$$

We can now see that $|u(\xi)| \leq 1$ follows once we prove

$$\frac{1}{2B_1(e-1)}\sum_{n=m+1}^{+\infty}(n+1)|B_n| \leq 1-\sum_{n=1}^{m}(n+1)|B_n|.$$

This is equivalent to the inequality

$$\sum_{n=1}^{m}(n+1)|B_n| + \frac{1}{2B_1(e-1)}\sum_{n=m+1}^{+\infty}(n+1)|B_n| \leq 1. \tag{34}$$

Our goal is to prove that the left-hand side of inequality (34) is bounded above by

$$\frac{1}{2B_1(e-1)}\sum_{n=1}^{+\infty}(n+1)|B_n|.$$

Alternatively,

$$\left(\frac{1}{2B_1(e-1)}-1\right)\sum_{n=1}^{m}(n+1)|B_n| \geq 0. \tag{35}$$

Thus, by virtue of (35), the proof of the inequality in (31) is now complete. To establish inequality (32), consider the expression

$$\begin{aligned}
&\left(1+\frac{1}{2B_1(e-1)}\right)\left(\frac{\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)_m'(\xi)}{\left(\mathcal{F}_{d,v}^{\chi,\sigma}(\xi)\right)'} - \frac{1}{1+2B_1(e-1)}\right)\\
&= \frac{1+\sum_{n=1}^{m}B_n(n+1)\xi^n + \left(1+\frac{1}{2B_1(e-1)}\right)\sum_{n=m+1}^{+\infty}(n+1)B_n\xi^n}{1+\sum_{n=1}^{m}(n+1)B_n\xi^n}\\
&= \frac{1+h_1(\xi)}{1+h_2(\xi)}.
\end{aligned}$$

We can write

$$u(\xi) = \frac{\left(1+\frac{1}{2B_1(e-1)}\right)\sum_{n=m+1}^{+\infty}|B_n|(n+1)}{2+2\sum_{n=1}^{m}(n+1)|B_n| + \left(\frac{1}{2B_1(e-1)}-1\right)\sum_{n=m+1}^{+\infty}|B_n|(n+1)}.$$

Thus, we have

$$|u(\xi)| \leq \frac{\left(1+\frac{1}{2B_1(e-1)}\right)\sum_{n=m+1}^{+\infty}(n+1)|B_n|}{2-2\sum_{n=1}^{m}(n+1)|B_n| - \left(\frac{1}{2B_1(e-1)}-1\right)\sum_{n=m+1}^{+\infty}(n+1)|B_n|}.$$

We can now see that $|u(\xi)| \leq 1$ follows once we prove

$$\sum_{n=1}^{m} (n+1) |B_n| + \frac{1}{2B_1 (e-1)} \sum_{n=m+1}^{+\infty} (n+1) |B_n| \leq 1. \tag{36}$$

Therefore, we can see that the left hand side of (36) is bounded above by

$$\frac{1}{2B_1 (e-1)} \sum_{n=1}^{+\infty} (n+1) |B_n| .$$

Alternatively,

$$\left(\frac{1}{2B_1 (e-1)} - 1 \right) \sum_{n=1}^{m} (n+1) |B_n| \geq 0. \tag{37}$$

Thus, by virtue of (37), the proof of the inequality in (32) is now complete.

Remark 1. *If $m = 0$ in (31), we find that $\operatorname{Re} \left(\mathcal{F}_{d,v}^{\chi,\sigma} (\xi) \right)' > 0$. So by Noshiro-Warschawski Theorem (see [46]), we conclude that the normalized Le Roy-type MLF is univalent in the unit disk U for*

$$1 \geq 2B_1 (e-1) ,$$

where $d \geq 1$, $d\sigma \geq 1$ and $v \geq \chi$.

Theorem 3. *If $d \geq 1$, $d\sigma \geq 1$ and $v \geq \chi$ and $1 \geq B_1 (e-2)$, then*

$$\operatorname{Re} \left(\frac{\mathcal{I} \left(\mathcal{F}_{d,v}^{\chi,\sigma} \right) (\xi)}{\left(\mathcal{I} \left(\mathcal{F}_{d,v}^{\chi,\sigma} \right) \right)_m (\xi)} \right) \geq 1 - B_1 (e-2) , \ \xi \in U \tag{38}$$

and

$$\operatorname{Re} \left(\frac{\left(\mathcal{I} \left(\mathcal{F}_{d,v}^{\chi,\sigma} \right) \right)_m (\xi)}{\mathcal{I} \left(\mathcal{F}_{d,v}^{\chi,\sigma} \right) (\xi)} \right) \geq \frac{1}{1 + B_1 (e-2)}, \ \xi \in U. \tag{39}$$

Proof. To establish equation (38), first, we recall the inequality (iii) of Lemma 2, that is

$$\left| \mathcal{I} \left[\left(\mathcal{F}_{d,v}^{\chi,\sigma} (\xi) \right) \right] (\xi) \right| \leq 1 + B_1 (e-2) , \ \xi \in U. \tag{40}$$

Using (6) in (40), we get

$$\begin{aligned} & \left| \mathcal{I} \left[\xi + \sum_{n=1}^{+\infty} \frac{\Gamma (\chi + n) (\Gamma (v))^{\sigma}}{n! \Gamma (\chi) (\Gamma (dn + v))^{\sigma}} \xi^{n+1} \right] (\xi) \right| \\ & = \left| \xi + \sum_{n=1}^{+\infty} \frac{1}{(n+1)} \frac{\Gamma (\chi + n) (\Gamma (v))^{\sigma}}{n! \Gamma (\chi) (\Gamma (dn + v))^{\sigma}} \xi^{n+1} \right| \end{aligned}$$

$$\leq 1+\sum_{n=1}^{+\infty}\frac{1}{n+1}\left|B_n\right| \leq 1+B_1\left(e-2\right),\ \xi \in U.$$

Alternatively

$$\frac{1}{B_1\left(e-2\right)}\sum_{n=1}^{+\infty}\frac{1}{n+1}\left|B_n\right| \leq 1,$$

where B_n is given (17). Now, we set

$$\begin{aligned}&\frac{1}{B_1\left(e-2\right)}\left(\frac{\mathcal{I}\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)(\xi)}{\left(\mathcal{I}\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)\right)_m(\xi)}-\left(1-B_1\left(e-2\right)\right)\right)\\ &=\frac{1+\sum_{n=1}^{m}\frac{1}{n+1}B_n\xi^n+\frac{1}{B_1(e-2)}\sum_{n=m+1}^{+\infty}\frac{1}{n+1}B_n\xi^n}{1+\sum_{n=1}^{m}\frac{1}{n+1}B_n\xi^n}\\ &=\frac{1+h_1\left(\xi\right)}{1+h_2\left(\xi\right)}.\end{aligned}$$

We can write

$$u(\xi)=\frac{\frac{1}{B_1(e-2)}\sum_{n=m+1}^{+\infty}\frac{1}{n+1}B_n\xi^n}{2+2\sum_{n=1}^{m}\frac{1}{n+1}B_n\xi^n+\frac{1}{B_1(e-2)}\sum_{n=m+1}^{+\infty}\frac{1}{n+1}B_n\xi^n},$$

thus, we have

$$|u(\xi)| \leq \frac{\frac{1}{B_1(e-2)}\sum_{n=m+1}^{+\infty}\frac{1}{n+1}\left|B_n\right|}{2-2\sum_{n=1}^{m}\frac{1}{n+1}\left|B_n\right|-\frac{1}{B_1(e-2)}\sum_{n=m+1}^{+\infty}\frac{1}{n+1}\left|B_n\right|}.$$

We can now see that $|u(\xi)| \leq 1$ follows once we prove

$$\frac{1}{B_1\left(e-2\right)}\sum_{n=m+1}^{+\infty}\frac{1}{n+1}\left|B_n\right| \leq 1-\sum_{n=1}^{m}\frac{1}{n+1}\left|B_n\right|.$$

This gives us

$$\sum_{n=1}^{m}\frac{1}{n+1}\left|B_n\right|+\frac{1}{B_1\left(e-2\right)}\sum_{n=m+1}^{+\infty}\frac{1}{n+1}\left|B_n\right| \leq 1. \tag{41}$$

It is enough to demonstrate that the inequality (41) is bounded above by

$$\frac{1}{B_1\left(e-2\right)}\sum_{n=1}^{+\infty}\frac{1}{n+1}\left|B_n\right|.$$

Alternatively,

$$\left(\frac{1}{B_1\left(e-2\right)}-1\right)\sum_{n=1}^{m}\frac{1}{n+1}\left|B_n\right| \geq 0. \tag{42}$$

Thus, by virtue of (42), the proof of the inequality in (38) is now complete.
To prove (39), we set

$$\left(1+\frac{1}{B_1\left(e-2\right)}\right)\left(\frac{\left(\mathcal{I}\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)\right)_m\left(\xi\right)}{\mathcal{I}\left(\mathcal{F}_{d,v}^{\chi,\sigma}\right)\left(\xi\right)}-\frac{1}{1+B_1\left(e-2\right)}\right)$$

$$=\frac{1+\sum_{n=1}^{m}\frac{1}{n+1}B_n\xi^n+\left(1+\frac{1}{B_1(e-2)}\right)\sum_{n=m+1}^{+\infty}\frac{1}{n+1}B_n\xi^n}{1+\sum_{n=1}^{m}\frac{1}{n+1}B_n\xi^n}=\frac{1+h_1\left(\xi\right)}{1+h_1\left(\xi\right)}.$$

We can write

$$u(\xi)=\frac{\left(1+\frac{1}{B_1(e-2)}\right)\sum_{n=m+1}^{+\infty}\frac{1}{n+1}\left|B_n\right|}{2+2\sum_{n=1}^{m}\frac{1}{n+1}\left|B_n\right|+\left(\frac{1}{B_1(e-2)}-1\right)\sum_{n=m+1}^{+\infty}\frac{1}{n+1}\left|B_n\right|}.$$

Thus, we have

$$|u(\xi)|\leq\frac{\left(1+\frac{1}{B_1(e-2)}\right)\sum_{n=m+1}^{+\infty}\left|B_n\right|\frac{1}{n+1}}{2-2\sum_{n=1}^{m}\frac{1}{n+1}\left|B_n\right|-\left(\frac{1}{B_1(e-2)}-1\right)\sum_{n=m+1}^{+\infty}\left|B_n\right|\frac{1}{n+1}}.$$

We can now see that $|u(\xi)|\leq 1$ follows once we prove

$$\sum_{n=1}^{m}\frac{1}{n+1}\left|B_n\right|+\frac{1}{B_1\left(e-2\right)}\sum_{n=m+1}^{+\infty}\frac{1}{n+1}\left|B_n\right|\leq 1. \tag{43}$$

It is enough to demonstrate that the inequality (43) is bounded above by

$$\frac{1}{B_1\left(e-2\right)}\sum_{n=1}^{+\infty}\frac{1}{n+1}\left|B_n\right|.$$

Alternatively,

$$\left(\frac{1}{B_1\left(e-2\right)}-1\right)\sum_{n=1}^{m}\frac{1}{n+1}\left|B_n\right|\geq 0. \tag{44}$$

Thus, by virtue of (44), the proof of the inequality in (39) is now complete.

By substituting the values $m=0,\gamma=1,\sigma=1,d=3$ and $v=1$ into Theorem 1, we arrive at the following consequence:

Example 1. *We have the following inequalities:*

$$\operatorname{Re}\left(\frac{e^{\xi^{\frac{1}{3}}}+2e^{-\frac{1}{2}\xi^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}\xi^{\frac{1}{3}}\right)}{2}\right)\geq\frac{7-e}{6}$$

and

$$\operatorname{Re}\left(\frac{2}{e^{\xi^{\frac{1}{3}}}+2e^{-\frac{1}{2}\xi^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}\xi^{\frac{1}{3}}\right)}\right)\geq\frac{6}{5+e}.$$

By substituting the values $m=0,\gamma=1,\sigma=1,d=3$ and $v=1$ in Theorem 2, we arrive at the following consequence:

Example 2. *We have the following inequalities:*

$$\operatorname{Re}\left(\frac{e^{\xi^{\frac{1}{3}}}-\sqrt{3}e^{-\frac{1}{2}\xi^{\frac{1}{3}}}\sin\left(\frac{\sqrt{3}}{2}\xi^{\frac{1}{3}}\right)-e^{-\frac{1}{2}\xi^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}\xi^{\frac{1}{3}}\right)}{6\tau^{\frac{2}{3}}}\right)\geq\frac{4-e}{3}$$

and

$$\operatorname{Re}\left(\frac{6\tau^{\frac{2}{3}}}{e^{\xi^{\frac{1}{3}}}-\sqrt{3}e^{-\frac{1}{2}\xi^{\frac{1}{3}}}\sin\left(\frac{\sqrt{3}}{2}\xi^{\frac{1}{3}}\right)-e^{-\frac{1}{2}\xi^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}\xi^{\frac{1}{3}}\right)}\right)\geq\frac{3}{2+e}.$$

By substituting the values $m=0,\gamma=1,\sigma=1,d=1$ and $v=1$ in Theorem 3, we arrive at the following consequence:

Example 3. *For $\gamma=1,\sigma=1,d=1$ and $v=1$, then*

$$\mathcal{F}_{1,1}^{1,1}(\xi)=\tau e^{\xi}$$

and for $m=0,\gamma=1,\sigma=1,d=1$ and $v=1$, then $\left(\mathcal{F}_{1,1}^{1,1}\right)_0(\xi)=\xi$. Thus

$$\mathcal{I}\left[\mathcal{F}_{1,1}^{1,1}(\xi)\right]=\int_0^{\xi}e^t dt=e^{\xi}-1$$

and

$$\mathcal{I}\left[\left(\mathcal{F}_{1,1}^{1,1}\right)_0\right]=\int_0^{\xi}dt=\xi.$$

Therefore by Theorem 3, we have

$$\operatorname{Re}\left(\frac{e^{\xi}-1}{\xi}\right)\geq 3-e$$

and

$$\operatorname{Re}\left(\frac{\xi}{e^{\xi}-1}\right)\geq\frac{1}{e-1}.$$

Remark 2. *For $\chi=1$, then our results reduces to the known results proved in [47].*

3.1. Normalized Barnes–Mittag-Leffler function

This section explores theorems involving the normalized Barnes-Mittag-Leffler function defined in (7).

Theorem 4. *If* $\min(b,d,v) > 0$, $s \geq 0$ *and* $1 \geq D_1(e-1)$, *then*

$$\mathrm{Re}\left(\frac{BM_{d,v}^{b,s}(\xi)}{\left(BM_{d,v}^{b,s}\right)_m(\xi)}\right) \geq 1 - D_1(e-1) \tag{45}$$

and

$$\mathrm{Re}\left(\frac{\left(BM_{d,v}^{b,s}\right)_m(\xi)}{BM_{d,v}^{b,s}(\xi)}\right) \geq \frac{1}{1+D_1(e-1)}, \tag{46}$$

where D_1 *is given by (20).*

Proof. The proof of (45) and (46) are analogous to the proof of Theorem 1, so omitted.

Theorem 5. *If* $\min(b,d,v) > 0$, $s \geq 0$ *and* $1 \geq 2D_1(e-1)$, *then*

$$\mathrm{Re}\left(\frac{\left(BM_{d,v}^{b,s}(\xi)\right)'}{\left(BM_{d,v}^{b,s}\right)_m'(\xi)}\right) \geq 1 - 2D_1(e-1), \ \xi \in U \tag{47}$$

and

$$\mathrm{Re}\left(\frac{\left(BM_{d,v}^{b,s}\right)_m'(\xi)}{\left(BM_{d,v}^{b,s}(\xi)\right)'}\right) \geq \frac{1}{1+2D_1(e-1)}, \ \xi \in U, \tag{48}$$

where D_1 *is given by (20).*

Proof. The proof of (47) and (48) are analogous to the proof of Theorem 2, so we omitted.

Theorem 6. *If* $\min(b,d,v) > 0$, $s \geq 0$ *and* $1 \geq D_1(e-2)$, *then*

$$\mathrm{Re}\left(\frac{\mathcal{I}\left(BM_{d,v}^{b,s}\right)(\xi)}{\left(\mathcal{I}\left(BM_{d,v}^{b,s}\right)\right)_m(\xi)}\right) \geq 1 - D_1(e-2), \ \xi \in U \tag{49}$$

and

$$\mathrm{Re}\left(\frac{\left(\mathcal{I}\left(BM_{d,v}^{b,s}\right)\right)_m(\xi)}{\mathcal{I}\left(BM_{d,v}^{b,s}\right)(\xi)}\right) \geq \frac{1}{1+D_1(e-2)}, \ \xi \in U, \tag{50}$$

where D_1 *is given by (20).*

Proof. The proof of (49) and (50) are analogous to the proof of Theorem 3, so we omitted.

4. Conclusion

This study investigated the normalized Le Roy-type Mittag-Leffler-Prabhakar function and the normalized Barnes-Mittag-Leffler function and determined the lower bounds for our theorems. We also derived bounds for the real parts of specific quotient expressions involving their Alexander transforms. Several examples are given to demonstrate the main findings.

The functions defined in (6) and (7) may inspire researchers to explore new subclasses of analytic functions, investigating properties like coefficients, distortion theorems, and extreme points. This could also lead to introducing new subclasses of bi-univalent functions and p-valent functions, and estimating their second and third Taylor-Maclaurin coefficients, as well as solving Fekete-Szegő problems in future work.

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