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The Ordered Implicit Relations: Fixed Point Problems and Applications

Anam Arif[1], Muhammad Nazam[2], Hijaz Ahmad[3,4,5,9,*], Ibrahim Alraddadi[3,*], Changjin Xu[6], Waleed Mohammed Abdelfattah[7,8]

[1] *Department of Mathematics, Government College University, Lahore, Pakistan*
[2] *Department of Mathematics, Allama Iqbal Open University, H-8, Islamabad, Pakistan*
[3] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Saudi Arabia*
[4] *Operational Research Center in Healthcare, Near East University, TRNC Mersin 10, Nicosia, 99138, Turkey*
[5] *Department of Mathematics, College of Science, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, South Korea*
[6] *Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang, P.R. China*
[7] *College of Engineering, University of Business and Technology, Jeddah 23435, Saudi Arabia*
[8] *Department of Engineering Mathematics and Physics, Faculty of Engineering, Zagazig University, P.O. 44519, Egypt*
[9] *Jadara University Research Center, Jadara University, Irbid, Jordan*

Abstract. We introduce an ordered implicit relation and obtain fixed point theorems in rectangular cone b-metric space, as an extension of the results on cone metric, rectangular cone metric and cone b-metric space. We provide some examples as an explanation of established outcomes. Our theorems universalize many fixed point outcomes in literature([1], [2]). A homotopy result as an application of main theorem is given, which further applied to the human aging process. The obtained fixed point results is an extension of [3], as the present article deals with nonlinear contractions. The convergence of the sequence generated by Urysohn integral operator is also shown by using fixed point technique.

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*Corresponding author.
*Corresponding author.
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Email addresses: amanarif970@gmail.com (A. Arif), muhammad.nazam@aiou.edu.pk (M. Nazam), hijazahmad@korea.ac.kr (H. Ahmad), ialraddadi@iu.edu.sa (I. Alraddadi), xcj403@126.com (C. Xu), w.abdelfattah@ubt.edu.sa (W. M. Abdelfattah)

1. Introduction

Ran [4] applied the concept of partial order to obtain a new generalization of BCP [5]. The appearance of this idea set a new direction in metric fixed point theory, and many authors contributed with remarkable research articles (see [6, 7]). Popa [1] introduced the concept of an implicit relation to present generalization of BCP. Beg *et al.* [8, 9], Berinde *et al.* [10, 11] and Sedghi *et al.* [12] contributed with several new fixed point results of self-operators satisfying implicit relations. Altun and Simsek applied the concept of implicit relation (see[13]), and obtained a generalization of [4, 14]. Fixed point theory deals with metric structures and their generalizations. Recently, Jachymski [15] generalized the BCP subject to a graphic metric space, Nadler [16] used Hausdorff metric space to manifest the fixed point results for multivalued mappings. Rashem at al. [17] presented some fixed point outcomes in a modular like metric structure with graph. Hammad at al. [18], contributed some tripled fixed point techniques in partially ordered metric spaces(see [19, 20]). Huang and Zhang [21] proposed the idea of cone metric, and hence proved the BCP and Kannan fixed point theorem in this setting. They used the idea of normal cones in their work (see [21]), later on, Rezapour *et al.* [22] improved these results. Motivated by b-metric space [23], Hussain and Shah [24] inaugurated the theory of cone b-metric structure. Huang and Xu [25] presented some fixed point results in the cone b-metric space. Recently, Anam *et al.* investigated some necessary conditions for the convergence of a sequence generated by implicit contraction in cone b-metric space [26]. For more information on this direction, we suggest reading of ([23, 27–35]). George [2] defined the rectangular cone b-metric space that universalizes the idea of [36], and established some well-known consequences in this setting.

- The fixed point theorem and the variational iteration method (VIM) are related being fundamental tools for resolving a variety of mathematical issues, including differential and integral equations.
    - (a) Iterative procedures are the foundation of both VIM and the fixed point theorem. Similar to finding a fixed point, the correction process in VIM can be thought of as a mapping that seeks to move successive approximations closer to the genuine solution.
    - (b) It is feasible to understand the convergence of the VIM iterations as the discovery of a fixed point in the space of potential solutions. In particular, if VIM's iterative process is successful, it indicates that, much like when a fixed point is found, the series of approximations converges to a point (the solution) where the correction functional no longer changes.
    - (c) In some VIM analysis, by proving that the iterative process is, in fact, a contraction mapping, the fixed point theorem (more precisely, Banach's Fixed Point Theorem) can be used to establish the existence and uniqueness of the solution to the problem.

We observe that the implicit relation defined by Popa, can be generalized to vector spaces ([37, 38]). In this research paper, motivated by Beg *et al.* [8, 9], Berinde *et al.* [10, 11] and Sedghi [12], we define an ordered implicit relation in a rectangular cone b-metric space. We contribute with a fixed point problem subject to monotone mappings, satisfying an implicit contraction. We aslo solve a homotopy problem and show existence of solution to a Urysohn Integral Equation as applications of the obtained fixed point theorems. The obtained fixed point theorems are independent of the observations presented by Ercan [3]. Ercan worked on linear contractions only, while in this paper, we considered nonlinear contractions. So, the obtained results are real generalizations and could not be followed from known ones in literature.

This paper has been organized in eight sections. Section 1, contains the introduction of the topic and motivation for this research work. Section 2, consists of related basic notions and results. Section 3, shed light on the notion of ordered implicit relation and its associated properties. Section 4, explains the methodology for the existence of fixed points of the self mapping satisfying an implicit contractive condition under ordered implicit relation. Section 5, contains the examples that explain the hypotheses of the fixed point theorems stated in Section 4. It also contains the corollaries of the main theorem. Section 6, consists of a homotopy result as an application of the main theorem and a result about human aging process that uses homotopy result. Section 7, shed light on the process of application of main theorem to show the existence of solution to Urysohn integral equation. Section 8, contains the conclusion of the research work done in this paper.

2. Preliminaries

This section consists of some basic notions and related axioms of cone, cone metric space, cone b-metric space, rectangular cone metric space and rectangular cone b-metric space. Throughout in this article, we will take $\mathcal{E}$ as a real Banach space.

Definition 1. *[21] Let $\mathcal{C} \subseteq \mathcal{E}$, then $\mathcal{C}$ is said to be a cone if it admits the following axioms:*

(1) $\mathcal{C}$ is non empty and closed set, and $\mathcal{C} \neq \{\mathbf{0}\}$;

(2) $px + qy \in \mathcal{C}$, $\forall\ x, y \in \mathcal{C}$ where $p, q \in \mathbb{R}$ and $a, b \geq 0$;

(3) $\mathcal{C} \cap (-\mathcal{C}) = \{\mathbf{0}\}$.

For $\mathcal{C} \subset \mathcal{E}$, the partial order $\preceq$ in $\mathcal{C}$ is taken as:

$$x \preceq y \Leftrightarrow y - x \in \mathcal{C} for\ all x, y \in \mathcal{E}.$$

The expression $x \ll y$ shows that $y - x \in \mathcal{C}^\circ$(the interior of $\mathcal{C}$).

The cone $\mathcal{C} \subseteq \mathcal{E}$ is called normal, if there exists $\mathcal{K} > 0$ such that

$$0 \preceq x \preceq y \Rightarrow \|x\| \leq \mathcal{K}\|y\|.$$

Let $\Re$ denotes the partial order in an ordinary set X and $\preceq$ be a partial order in the cone $\mathcal{C} \subseteq \mathcal{E}$. For $X \subseteq \mathcal{E}$, $\Re$ and $\preceq$ assumed to be same.

Definition 2. *[21] Let* $X \neq \emptyset$ *and* $\mathrm{d}_c : X \times X \mapsto \mathcal{E}$ *satisfies the following axioms:*

(d1) $\mathrm{d}_c(\ell, r) \succeq \mathbf{0}$, $\forall\ \ell, r \in X$ *and* $\mathrm{d}_c(\ell, r) = \mathbf{0} \Leftrightarrow \ell = r$;

(d2) $\mathrm{d}_c(\ell, r) = \mathrm{d}_c(r, \ell)$;

(d3) $\mathrm{d}_c(\ell, r_1) \preceq \mathrm{d}_c(\ell, r) + \mathrm{d}_c(r, r_1)$, $\forall\ \ell, r, r_1 \in X$.

Then d_c *is called cone metric and the pair* (X, d_c) *represents a cone metric space.*

Example 1. *[36] Consider* $X = R$, $\mathcal{E} = R^2$ *with cone* $\mathcal{C} = \{(\ell, y) \in \mathcal{E} : \ell, y \geq 0\}$. *Take* $\mathrm{d}_c : X \times X \to \mathcal{E}$, *in a way*

$$\mathrm{d}_c(\ell, y) = (|\ \ell - y\ |, \delta\ |\ \ell - y\ |),$$

here $\delta \geq 0$ *(scalar). Subsequently* d_c *represents a cone metric, and* (X, d_c) *a cone metric space.*

Proposition 3. *[36] Let* $\mathcal{C}$ *be a cone with cone metric space* (X, d_c). *Then for* $x, \ell, \varrho \in \mathcal{E}$, *we have*

(1) If $x \preceq \alpha x$ *and* $\alpha \in [0, 1)$, *then* $x = 0$.

(2) If $0 \preceq x \ll \ell$ *for each* $0 \ll \ell$, *then* $x = 0$.

(3) If $x \preceq \ell$ *and* $\ell \ll \varrho$, *then* $x \ll \varrho$.

Definition 4. *[34] Let* $d_b : X \times X \mapsto \mathcal{E}$ *fulfills the following axioms:*

(db1) $\mathbf{0} \preceq d_b(p, y)$ *along with* $d_b(p, y) = \mathbf{0} \Leftrightarrow p = y$;

(db2) $d_b(p, y) = d_b(y, p)$;

(db3) $d_b(p, r) \preceq \alpha[d_b(p, x) + d_b(x, r)]$ *for some* $\alpha \geq 1$, $\forall\ p, x, r \in X$.

Then d_b *is called a cone b-metric, and* (X, d_b) *represents a cone b-metric space. It is observed that every* d_c *is a* d_b *metric space, but converse may not be true.*

Example 2. *[36] Let* $\mathcal{E} = R^2$, $\mathcal{C} = \{(\ell, r) \in \mathcal{E} : \ell, r \geq 0\} \subset R^2$, *and* $X = \{1, 2, 3, 4\}$. *Define*

$$d_b(\ell, r) = \begin{cases} (|\ell - r|^{-1}, |\ell - r|^{-1}) & \text{if } \ell \neq r \\ 0 & \text{if } \ell = r. \end{cases}$$

Here (X, d_b) *represents a cone b-metric space for* $s = \frac{6}{5}$, *and since* $d_b(1, 2) \succ d_b(1, 4) + d_b(4, 2)$, d_b *is not a cone metric.*

Definition 5. *[34] Let* $d_r : X \times X \mapsto \mathcal{E}$, $\forall\ f_1, f_2, f_3, f_4 \in X$, *satisfies the following axioms:*

(dR1) $\mathbf{0} \preceq d_r(f_1, f_2)$ *and* $d_r(f_2, f_1) = \mathbf{0}$ *if and only if* $f_1 = f_2$;

(dR2) $d_r(f_1, f_2) = d_r(f_2, f_1)$;

(dR3) $d_r(f_1, f_4) \preceq d_r(f_1, f_2) + d_r(f_2, f_3) + d_r(f_3, f_4)$ *for all distinct* $f_3, f_4 \in X \setminus \{f_1, f_2\}$.

Then d_r *is known as rectangular cone metric and the pair* (X, d_r) *represents a rectangular cone metric space.*

Remark 6. *A cone metric space always represents a rectangular cone metric space, but converse may not hold.*

Definition 7. *[34] Let* $R : X \times X \mapsto \mathcal{E}$, $\forall\ \ell, y, x, \upsilon \in X$, *satisfies the following axioms:*

(dRb1) $\mathbf{0} \preceq R(\ell, y)$ *and* $R(\ell, y) = \mathbf{0} \Leftrightarrow \ell = y$;

(dRb2) $R(\ell, y) = R(y, \ell)$;

(dRb3) $R(\ell, \upsilon) \preceq s[R(\ell, y) + R(y, x) + R(x, \upsilon)]$ *for all distinct* $y, x \in X \setminus \{\ell, \upsilon\}$, *where* $s \geq 1$.

Then R *stands for rectangular cone b-metric, and* (X, R) *represents a rectangular cone b-metric space (RCBMS).*

Example 3. *[36] Consider* $X = [0, 2]$ *and* $\mathcal{E} = C_{R^2}(X)$. *For* $x = (\ell, y)$ *and* $\wp = (u, v)$ *in* $\mathcal{E}$, *we define* $x.\wp = (\ell u, yv)$ *and* $\|x\| = \max(\|\ell\|, \|y\|)$, *where* $\|\wp\| = \sup_{x\in X} \mid \wp(x) \mid$. *Then* $\mathcal{E}$ *is a Banach algebra with unit element* $e = (1, 1)$ *and zero element* $\omega = (0, 0)$. *Suppose that* $\mathcal{C} = \{(\ell, y) \in \mathcal{E} : \ell(x), y(x) \geq 0, x \in X\} \subset C_{R^2}(X)$, *clearly* $\mathcal{C}$ *represents a cone in* $\mathcal{E}$. *Define* $R : X \times X \to \mathcal{E}$, *such that*

$$R(\ell, y)(x) = \begin{cases} (0,0) & \text{if } \ell = y \\ (a + bx, c + dx^2) & \text{if } \ell, y \in [0, \frac{1}{2}) \\ & \text{and a,b,c,d are some fixed real numbers.} \\ (\frac{a+bx}{n^3}, \frac{c+dx^2}{n^3}) & \text{if } \ell = \frac{1}{n}(n \geq 2) \in [0, \frac{1}{2}) \text{ and } y \in \{1, 2\} \\ (|\ell - y|^2(a + bx), |\ell - y|^2(c + dx^2)) & \text{if } \ell, y \in [0, \frac{1}{2}) \text{ otherwise.} \end{cases}$$

Here (X, R) *represents rectangular cone b-metric space with* $s = 2$ *over* $\mathcal{E}$, *but is not cone b-metric space.*

Remark 8. *A cone b-metric space always represents a rectangular cone b-metric space, converse may not hold, as illustrated in the above mentioned examples.*

Definition 9. *[36] Let* $\mathcal{E}$ *be a real Banach space,* (X, R) *is taken as a rectangular cone b-metric space and* $\epsilon \in \mathcal{E}$ *with* $0 \ll \epsilon$*(arbitrary).*

(1) Any sequence $\{\ell_n\}$ *is referred as a Cauchy sequence (CS), whenever we get* $K \in \mathbb{N}$ *with* $R(\ell_n, \ell_m) \ll c\ \forall\ n, m > K$.

(2) Any sequence $\{\ell_n\}$ *is referred to as convergent if we have an* $K \in \mathbb{N}$ *with* $R(\ell_n, \ell) \ll \epsilon$ $\forall\ n \geq K$ *and* $\ell \in X$.

(3) Any (RCBMS) referred as complete, whenever every (CS) *has a convergent point in* X.

The main theorem of Simsek [13] paper is as follows:

Theorem 10. [13] Let $(Y, d, \preceq)$ be a partially ordered metric space, and $L : Y \to Y$ satisfies the following inequality:

$$\mathcal{T}\left(d(Lx, Ly), d(x, y), d(x, Lx), d(y, Ly), d(x, Ly), d(y, Lx)\right) \leq 0, \tag{1}$$

$\forall\ x, y \in Y$ with $x \preceq y$, where $\mathcal{T} : [0, \infty)^6 \to (-\infty, \infty)$. Then L admits a fixed point in Y provided L is continuous or $(Y, d, \preceq)$ is as a regular space.

A number of contractive conditions were obtained by using (1), for instance, define $\mathcal{T} : [0, \infty)^6 \to (-\infty, \infty)$ such that

$$\mathcal{T}(f_1, f_2, f_3, f_4, f_5, f_6) = f_1 - \psi\left(\max\left\{f_2, f_3, f_4, \frac{1}{2}(f_5 + f_6)\right\}\right),$$

which leads to main theorems proved in [14]. Equivalently by choosing

$$\mathcal{T}(f_1, f_2, f_3, f_4, f_5, f_6) = f_1 - kf_2;\ k \in [0, 1),$$

in (1), we get results proved in [4]. So, we get different contractive conditions for different definitions of $\mathcal{T} : [0, \infty)^6 \to (-\infty, \infty)$.

Note: Throughout in this article we will take $(\mathcal{E}, \|.\|)$ as a real Banach space and $B(\mathcal{E}, \mathcal{E})$ taken as collection of all bounded linear mappings.

3. Ordered Implicit Relations

Motivated by [8–11, 39, 40], we construct an ordered implicit relation as follows:

Definition 11. *Assume $(\mathcal{E}, \|.\|)$ is taken as a real Banach space, along with $B(\mathcal{E}, \mathcal{E})$ (the collection of all bounded linear operators $\mathcal{T}$), $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ where $\|\mathcal{T}\|_1 < \frac{1}{s}$ for $s \geq 1$, we take $\|.\|_1$ as the usual norm. The mapping $\mathcal{J} : \mathcal{E}^6 \to \mathcal{E}$ will define an ordered implicit relation, if $\mathcal{J}$ taken as continuous on $\mathcal{E}^6$ and also it satisfy the given conditions:*

($\mathcal{J}_1$) *$r_1 \preceq \upsilon_1$, $r_5 \preceq \upsilon_5$ and $r_6 \preceq \upsilon_6$*
$\Rightarrow \mathcal{J}(\upsilon_1, r_2, r_3, r_4, \upsilon_5, \upsilon_6) \preceq \mathcal{J}(r_1, r_2, r_3, r_4, r_5, r_6)$.

($\mathcal{J}_2$) *if $\mathcal{J}(r_1, r_2, r_2, r_1, \alpha[r_1 + r_2 + r_3], r_1) \preceq \mathbf{0}_\mathcal{E}$*
or
if $\mathcal{J}(r_1, r_2, r_1, r_2, r_1, \alpha[r_1 + r_2 + r_3]) \preceq \mathbf{0}_\mathcal{E}$, then $\exists\ \mathcal{T} \in B(\mathcal{E}, \mathcal{E})$ so that $r_1 \preceq \frac{1}{\alpha}\mathcal{T}(r_2)$ and $r_3 \preceq \frac{1}{\alpha}\mathcal{T}(r_1)$ ($\forall\ r_1, r_2 \in \mathcal{E}$) for some $R \ni \alpha \geq 1$.

($\mathcal{J}_3$) *$\mathcal{J}(\alpha r, \mathbf{0}_\mathcal{E}, \mathbf{0}_\mathcal{E}, r, \alpha r, \mathbf{0}_\mathcal{E}) \succ \mathbf{0}_\mathcal{E}$ whenever $\|r\| > 0$ and $\alpha \geq 1$.*

Let $\mathcal{F} = \{\mathcal{J} : \mathcal{E}^6 \to \mathcal{E}\,|\,\mathcal{J}$ owns $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$.

Example 4. *Consider the partial order $\preceq$ in a cone $\mathcal{C}$. For all $r_i \in \mathcal{E}($ for all $i = 1$ to $6)$ and for some $\alpha < 1$, define $\mathcal{J} : \mathcal{E}^6 \to \mathcal{E}$ such that*

$$\mathcal{J}(r_1, r_2, r_3, r_4, r_5, r_6) = r_5 - \alpha\{r_1 + r_2\} - r_1.$$

Then the operator $\mathcal{J} \in \mathcal{F}$. Indeed
$(\mathcal{J}_1)$. Let $r_1 \preceq \gamma_1$, $r_5 \preceq \gamma_5$ and $r_6 \preceq \gamma_6$, then $\gamma_1 - r_1 \in \mathcal{C}$, $\gamma_5 - r_5 \in \mathcal{C}$ and $\gamma_6 - r_6 \in \mathcal{C}$. Now we show that $\mathcal{J}(r_1, r_2, r_3, r_4, r_5, r_6) - \mathcal{J}(\gamma_1, r_2, r_3, r_4, \gamma_5, \gamma_6) \in \mathcal{C}$. Consider,

$$\begin{aligned} & \mathcal{J}(r_1, r_2, r_3, r_4, r_5, r_6) - \mathcal{J}(\gamma_1, r_2, r_3, r_4, \gamma_5, \gamma_6) \\ = \; & r_5 - \alpha\{r_1 + r_2\} - r_1 - (\gamma_5 - \alpha\{\gamma_1 + r_2\} + r_1) \\ = \; & -(\gamma_5 - r_5) + \alpha(\gamma_1 - r_1) \in \mathcal{C}. \end{aligned}$$

Thus, $\mathcal{J}(\gamma_1, r_2, r_3, r_4, \gamma_5, \gamma_6) \preceq \mathcal{J}(r_1, r_2, r_3, r_4, r_5, r_6)$.
$(\mathcal{J}_2)$. Let $r_1, r_2, r_3 \in \mathcal{E}$ be such that $\mathbf{0}_{\mathcal{E}} \preceq r_1$, $\mathbf{0}_{\mathcal{E}} \preceq r_2$, $\mathbf{0}_{\mathcal{E}} \preceq r_3$. If $\mathcal{J}(r_1, r_2, r_1, r_2, \mathbf{s}[r_1 + r_2 + r_3], r_1) \preceq \mathbf{0}_{\mathcal{E}}$ then by definition of $\mathcal{J}$, we have $-\mathbf{s}[r_1 + r_2 + r_3] + \alpha(r_1 + r_2) + r_1 \in \mathcal{C}$. So we get two equations,

$$\alpha r_2 - (\mathbf{s} - \alpha - 1)r_1 \in \mathcal{C}, \tag{2}$$

$$(\alpha + 1 - \mathbf{s})r_1 - \mathbf{s}r_3 \in \mathcal{C}. \tag{3}$$

For (2) if $r_1 = \mathbf{0}_{\mathcal{E}}$, then $\alpha r_2 \in \mathcal{C}$. So, we get a $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ such that $\mathcal{T}(r_2) = \alpha r_2$ ($\alpha < 1$ is fixed) and $\| \mathcal{T} \| = \alpha < 1$. If $r_1 \neq \mathbf{0}_{\mathcal{E}}$, then, (2) implies $r_1 \preceq \frac{\alpha}{(\mathbf{s}-\alpha-1)} r_2$. So $\exists$, $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ such that $\mathcal{T}(r_2) = y r_2$ ($y = \frac{\alpha}{(\mathbf{s}-\alpha-1)}$ is a scalar) such that $r_1 \preceq \mathcal{T}(r_2)$, for $\alpha < 1$.
For (3) if $r_3 = \mathbf{0}_{\mathcal{E}}$, then $(\alpha + 1 - \mathbf{s})r_1 \in \mathcal{C}$. So, we have $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ taken as $\mathcal{T}(r_1) = y r_1$ (where $y = (\alpha + 1 - \mathbf{s})$ is a scalar with $\alpha < 1 \leq \mathbf{s}$ implies $\alpha - \mathbf{s} < 0$ and $\alpha - \mathbf{s} + 1 < 1$) such that $r_3 \preceq \mathcal{T}(r_1)$. Now if $r_3 \neq \mathbf{0}_{\mathcal{E}}$, then, $r_3 \preceq \frac{(1+\alpha-\mathbf{s})}{\mathbf{s}} r_1$. So, we get $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ taken as $\mathcal{T}(r_2) = y r_2$ ($y = \frac{(1+\alpha-\mathbf{s})}{\mathbf{s}}$ is fixed) with $r_3 \preceq (r_1)$, for $\alpha < 1$.
$(\mathcal{J}_3)$. Take $r \in \mathcal{E}$ and $\|r\| > 0$ with, $\mathbf{0}_{\mathcal{E}} \preceq \mathcal{J}(\mathbf{sr}, \mathbf{0}_{\mathcal{E}}, \mathbf{0}_{\mathcal{E}}, \mathbf{r}, \mathbf{sr}, \mathbf{0}_{\mathcal{E}})$ then $(\mathbf{s} - \alpha\mathbf{s} - \mathbf{1})\mathbf{r} \in \mathcal{C}$, which is true for $\|r\| > 0$.

Example 5. *Similarly, the operator $\mathcal{J} : \mathcal{E}^6 \to \mathcal{E}$ defined by*

(i). $\mathcal{J}_1^*(r_1, r_2, r_3, r_4, r_5, r_6) = r_1 + r_5 - \alpha(r_2 + r_4); \alpha < \frac{1}{2}$.

(ii). $\mathcal{J}_2^*(r_1, r_2, r_3, r_4, r_5, r_6) = (1 - \beta)r_5 - \beta(r_3 + r_4) - r_3; \beta \in (-\infty, \frac{1}{4})$.

(iii). $\mathcal{J}_3^*(r_1, r_2, r_3, r_4, r_5, r_6) = \gamma r_1 - r_2; \gamma > 1$.

(iv). $\mathcal{J}_4^*(r_1, r_2, r_3, r_4, r_5, r_6) = r_1 + r_5 - \varrho\{r_2 + r_4\}; \varrho < \frac{3}{2}$.

Then $\mathcal{J}_i^* \in \mathcal{F}$ for each $i = 1, 2, 3, 4$.

We will apply this implicit relation along with some additional conditions to construct a sequence, and to solve the following problem **fixed-point problem:**
"find $z^* \in (X, R)$ along $f(z^*) = z^*$", when $\mathcal{T} \in B(\mathcal{E}, \mathcal{E})$, $I : \mathcal{E} \to \mathcal{E}$ considered as

an identity operator, $\mathcal{J} \in \mathcal{F}$ and $F : X \to X$ admits (4), for each pair of comparable elements $r, x \in X$ when $s \geq 1$.

$$(I - \mathcal{T})^2 (I + \mathcal{T})(R(r, F(r))) \preceq sR(r, x) \text{ further implies}$$

$$\mathcal{F}\left(R(F(r), F(x)), R(r, x), R(r, F(r)), R(x, F(x)), R(r, F(x)), R(x, F(r))\right) \preceq \mathbf{0}_{\mathcal{E}}. \quad (4)$$

Remark 12. *For $H \in B(\mathcal{E}, \mathcal{E})$, $I + H + H^2 + \cdots + H^n + \cdots$ converges for $\|H\|_1 < 1$, otherwise diverge. In addition if $\|H\|_1 < 1$, then we get a $y > 0$, so that $\|H\|_1 < y < 1$ along with $\|H^n\|_1 \leq y^n < 1$.*

4. Results on Ordered Implicit Contractions

Popa[1] considered a self-mapping satisfying an implicit contractive conditions and established well-known fixed point results. Ran *et al.* [4] used monotone functions to generalize the famous Banach fixed point theorem in partially ordered metric space. Haung *et al.*[21] introduced cone metric space, Hussain and Shah [24] introduced cone b-metric space, Huang and Xu [25] practiced with cone b-metric structure to establish new fixed point results. Azam *et al.* [34] introduced the concept of rectangular cone metric space and generalized BCP. Anam *et al.*[26] investigated necessary convergence axioms for an implicit contraction in a cone b-metric space. Motivated by [34, 36], George [2] defined the rectangular cone b-metric space, that generalizes the rectangular b-metric space and established some well-known theorems in this setting. In this section, we utilize the ideas of Popa[1] and George [2] to develop the notion of ordered implicit relation, and prove some fixed point results in rectangular cone b-metric spaces.

Theorem 13. *Let (W, R) be a complete rectangular cone b-metric space along with a cone $\mathcal{C} \subset \mathcal{E}$ and $\hbar : W \to W$. Let $\mathcal{T} \in B(\mathcal{E}, \mathcal{E})$ such that $\|\mathcal{T}\|_1 < \frac{1}{s}(s \geq 1)$, $I : \mathcal{E} \to \mathcal{E}$ and $\mathcal{J} \in \mathcal{F}$ such that, for each pair of comparable elements $f, x \in W$ and for some $s \geq 1$, the following holds*

$$(I - \mathcal{T})^2 (I + \mathcal{T})(R(f, \hbar(f))) \preceq sR(f, x) \text{ implies}$$

$$\mathcal{J}\left(R(\hbar(f), \hbar(x)), R(f, x), R(f, \hbar(f)), R(x, \hbar(x)), R(f, \hbar^2(x)), R(x, \hbar^2(f))\right) \preceq \mathbf{0}_{\mathcal{E}}, \quad (5)$$

and

(1) there exists $f_0 \in W$ satisfying $f_0 \Re \hbar(f_0)$;

(2) for every pair $f, x \in W$, $f \Re x \Rightarrow \hbar(f) \Re \hbar(x)$;

(3) for any sequence $\{f_n\}$ such that $f_n \Re f_{n+1}$ and $f_n \to z^*$, we have $f_n \Re z^*$ $\forall$ $n \in \mathbb{N}$ and $R(z^*, \hbar(z^*)) \preceq R(z^*, \hbar^2(z^*))$.

Then, there exists $z^* \in W$ such that $z^* = \hbar(z^*)$.

Proof. Let $f_0 \in W$ satisfying (1), define $\{f_n\}$ by $\hbar(f_{n-1}) = f_n$, then $f_0 \Re f_1$. By assumption (2), we get $f_1\Re f_2, f_2\Re f_3, \cdots, f_{n-1}\Re f_n$. In (5), put $f = f_0$ and $x = f_1$,

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(f_0,\hbar(f_0))) = (I-\mathcal{T})^2(I+\mathcal{T})(R(f_0,f_1)) \preceq sR(f_0,f_1) \text{ implies}$$

$$\mathcal{J}\left(R(\hbar(f_0),\hbar(f_1)), R(f_0,f_1), R(f_0,\hbar(f_0)), R(f_1,\hbar(f_1)), R(f_0,\hbar^2(f_1)), R(f_1,\hbar^2(f_0))\right) \preceq \mathbf{0}_{\mathcal{E}},$$

that is,

$$\mathcal{J}\left(R(f_1,f_2), R(f_0,f_1), R(f_0,f_1), R(f_1,f_2), R(f_0,f_3), R(f_1,f_2)\right) \preceq \mathbf{0}_{\mathcal{E}}. \tag{6}$$

By (dR3), we have

$$R(f_0,f_3) \preceq s[R(f_0,f_1) + R(f_1,f_2) + R(f_2,f_3)].$$

By $(\mathcal{J}_1)$, and rewriting (6) we get:

$$\mathcal{J}\left(R(f_1,f_2), R(f_0,f_1), R(f_0,f_1), R(f_1,f_2), s[R(f_0,f_1)+R(f_1,f_2)+R(f_2,f_3)], R(f_1,f_2)\right) \preceq \mathbf{0}_{\mathcal{E}}.$$

By $(\mathcal{J}_2)$, there exists $\mathcal{T} \in B(\mathcal{E},\mathcal{E})$ along with $\|\mathcal{T}\|_1 < 1$, $s \geq 1$, so that

$$R(f_1,f_2) \preceq \frac{1}{s}\mathcal{T}(R(f_0,f_1)) \preceq \mathcal{T}(R(f_0,f_1)) \text{ and } R(f_2,f_3) \preceq \frac{1}{s}\mathcal{T}(R(f_1,f_2)) \preceq \mathcal{T}(R(f_1,f_2)).$$

Now put $f = f_1$ and $x = f_2$ in (5) to have

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(f_1,\hbar(f_1))) = (I-\mathcal{T})^2(I+\mathcal{T})(R(f_1,f_2)) \preceq sR(f_1,f_2) \text{ implies}$$

$$\mathcal{J}\left(R(\hbar(f_1),\hbar(f_2)), R(f_1,f_2), R(f_1,\hbar(f_1)), R(f_2,\hbar(f_2)), R(f_1,\hbar(f_3)), R(f_2,\hbar(f_2))\right) \preceq \mathbf{0}_{\mathcal{E}},$$

that is,

$$\mathcal{J}\left(R(f_2,f_3), R(f_1,f_2), R(f_1,f_2), R(f_2,f_3), R(f_1,f_4), R(f_2,f_3)\right) \preceq \mathbf{0}_{\mathcal{E}}.$$

By (dRb3), we get

$$R(f_1,f_4) \preceq s[R(f_1,f_2) + R(f_2,f_3) + R(f_3,f_4)],$$

$(\mathcal{J}_1)$ implies

$$\mathcal{J}\left(R(f_2,f_3), R(f_1,f_2), R(f_1,f_2), R(f_2,f_3), s[R(f_1,f_2)+R(f_2,f_3)+R(f_3,f_4)], R(f_2,f_3)\right) \preceq \mathbf{0}_{\mathcal{E}}.$$

Now $(\mathcal{J}_2)$ guarantees the existence of $\mathcal{T} \in B(\mathcal{E},\mathcal{E})$ along with $\|\mathcal{T}\|_1 < 1$ so that

$$R(f_3,f_4) \preceq \frac{1}{s}\mathcal{T}(R(f_2,f_3)) \preceq \frac{1}{s^2}\mathcal{T}^2(R(f_2,f_3)) \preceq \frac{1}{s^3}\mathcal{T}^3(R(f_0,f_1)) \preceq \mathcal{T}^3(R(f_0,f_1)).$$

Continuing, we construct a sequence $\{f_n\}$ so that $f_n\Re f_{n+1}$ with $f_{n+1} = \hbar(f_n)$, hence for

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(f_{n-1},\hbar(f_{n-1}))) = (I-\mathcal{T})^2(I+\mathcal{T})(R(f_{n-1},f_n)) \preceq sR(f_{n-1},f_n),$$

we get

$$R(f_n, f_{n+1}) \preceq \frac{1}{s}\mathcal{T}(R(f_{n-1}, f_n)) \preceq \frac{1}{s^2}\mathcal{T}^2(R(f_{n-2}, f_{n-1})) \preceq \cdots \preceq \frac{1}{s^n}\mathcal{T}^n(R(f_0, f_1)) \preceq \mathcal{T}^n(R(f_0, f_1)).$$

Assume two situations for $R(f_n, f_{n+p})$, for if p is odd, that is $p = 2m + 1$ and $R^*_n = R(f_n, f_{n+1})$

$$\begin{aligned}
R(f_n, f_{n+2m+1}) &\preceq s[R^*_n + R^*_{n+1} + R(f_{n+2}, f_{n+2m+1})]\\
&\preceq s[R^*_n + R^*_{n+1}] + s^2[R^*_{n+2} + R^*_{n+3} + R(f_{n+3}, f_{n+2m+1})]\\
&\preceq s[R^*_n + R^*_{n+1}] + s^2[R^*_{n+2} + R^*_{n+3}] + \cdots + s^m R(f_{n+2m}, f_{n+2m+1})\\
&\preceq s[\mathcal{T}^n R^*_0 + \mathcal{T}^{n+1} R^*_1] + s^2[\mathcal{T}^{n+2} R^*_0 + \mathcal{T}^{n+3} R^*_0] + \cdots + s^m \mathcal{T}^{n+2m} R^*_0\\
&\prec s\mathcal{T}^n(I + s\mathcal{T}^2 + s^2\mathcal{T}^4 + \cdots)R^*_0 + s\mathcal{T}^{n+1}(I + s\mathcal{T}^2 + (s\mathcal{T}^2)^2 + \cdots)R^*_0\\
&= s\mathcal{T}^n(I - s\mathcal{T}^2)^{-1}R^*_0 + s\mathcal{T}^{n+1}(I - s\mathcal{T}^2)^{-1}R^*_0\\
&= s\mathcal{T}^n(I - s\mathcal{T}^2)^{-1}R^*_0(I + \mathcal{T}).(\text{ By Remark 12})
\end{aligned}$$

For if p is even, that is $p = 2m$ and $R^*_n = R(f_n, f_{n+1})$.

$$\begin{aligned}
R(f_n, f_{n+2m}) &\preceq s[R^*_n + R^*_{n+1} + R(f_{n+2}, f_{n+2m})]\\
&\preceq s[R^*_n + R^*_{n+1}] + s^2[R^*_{n+2} + R^*_{n+3} + R(f_{n+3}, f_{n+2m+1})]\\
&\preceq s[R^*_n + R^*_{n+1}] + s^2[R^*_{n+2} + R^*_{n+3}] + \cdots + s^{m-1}[R^*_{2m-4} + R^*_{2m-3}]\\
&+ s^{m-1}R(f_{n+2m-2}, f_{n+2m})\\
&\preceq s[\mathcal{T}^n R^*_0 + \mathcal{T}^{n+1} R^*_1] + s^2[\mathcal{T}^{n+2} R^*_0 + \mathcal{T}^{n+3} R^*_0] + \cdots + s^{m-1}\mathcal{T}^{n+2m-2} R^*_0\\
&\prec (s\mathcal{T}^n + s\mathcal{T}^{n+1})(I + s\mathcal{T}^2 + s^2\mathcal{T}^4 + \cdots)R^*_0 + s^{m-1}\mathcal{T}^{n+2m-2}R^*_0\\
&= (s\mathcal{T}^n + s\mathcal{T}^{n+1})(I - s\mathcal{T}^2)^{-1}R^*_0 + s^{m-1}\mathcal{T}^{2m-2}\mathcal{T}^n R^*_0\\
&= \frac{1}{s}(s\mathcal{T}^n + s\mathcal{T}^{n+1})(\frac{1}{s} - \mathcal{T}^2)^{-1}R^*_0 + s^{m-1}\mathcal{T}^{2m-2}\mathcal{T}^n R^*_0.
\end{aligned}$$

Here $\|\mathcal{T}\|_1 < \frac{1}{s}$, which further implies that $\lim_{n\to\infty} \mathcal{T}^n = \mathbf{0}$. Hence, $\lim_{n\to\infty} R(f_n, f_{n+p}) = \mathbf{0}_{\mathcal{E}}$ and $\{f_n\}$ is a Cauchy sequence in W. Now (W, R) is a complete rectangular cone b-metric space, so $f_n \to z^*$ for some $z^* \in W$ if n being very large. So

$$R(f_n, z^*) \ll \epsilon \text{ for all } n \geq N_2 \in N \text{ and } 0 \ll \epsilon.$$

Let

$$(I - \mathcal{T})^2(I + \mathcal{T})(R(f_n, \hbar(f_n))) \succ sR(f_n, z^*) \text{ and}$$

$$(I - \mathcal{T})^2(I + \mathcal{T})(R(f_{n+2}, \hbar(f_{n+2}))) \succ sR(f_{n+2}, z^*) \text{ for some } n \in \mathbb{N} \text{ and } s \geq 1.$$

Using (5) and (dRb3), implies

$$\begin{aligned}
R(f_n, \hbar(f_n)) &\preceq s[R(f_n, z^*) + R(z^*, f_{n+2}) + R(f_{n+2}, f_{n+1})]\\
&\prec \frac{s}{s}(I - \mathcal{T})^2(I + \mathcal{T})(R(f_n, \hbar(f_n))) + \frac{s}{s}(I - \mathcal{T})^2(I + \mathcal{T})R(f_{n+2}, \hbar(f_{n+2})))
\end{aligned}$$

$$\begin{aligned} &+ \quad s\frac{1}{s}\mathcal{T}(R(f_n,\hbar(f_n))) \\ &\prec \quad (I-\mathcal{T})^2(I+\mathcal{T})(R(f_n,\hbar(f_n))) + (I-\mathcal{T})^2(I+\mathcal{T})\mathcal{T}^2R(f_n,\hbar(f_n))) \\ &+ \quad \mathcal{T}(R(f_n,\hbar(f_n))) \end{aligned}$$

$$(I-\mathcal{T})R(f_n,\hbar(f_n)) \preceq (I-\mathcal{T})^2(I+\mathcal{T})(I+\mathcal{T}^2)R(f_n,\hbar(f_n))). \text{ Thus}$$

$$\mathcal{T}^4(R(f_n,\hbar(f_n))) \prec \mathbf{0}_{\mathcal{E}},$$

that steers to contradiction. Thus, for every $n \geq 1$ and for some $s \geq 1$

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(f_n,\hbar(f_n))) \preceq sR(f_n,z^*).$$

This implies that (due to (5))

$$\mathcal{J}\left(\begin{array}{ll} R(\hbar(f_n),\hbar(z^*)), R(f_n,z^*), R(f_n,\hbar(f_n)), R(z^*,\hbar(z^*)), \\ R(f_{n-1},\hbar^2(z^*)), R(\hbar^2(f_n),\hbar(f_n)) \end{array}\right) \preceq \mathbf{0}_{\mathcal{E}}. \tag{7}$$

Now, we claim that $\|R(z^*,\hbar(z^*))\| = 0$, on contrary suppose that $\|R(z^*,\hbar(z^*))\| > 0$, as a result we get

$$R(\hbar(f_n),\hbar(z^*)) \preceq s[R(\hbar(f_n),\hbar(f_{n+1})) + R(\hbar(f_{n+1}),z^*)) + R(z^*,\hbar(z^*))]$$

$$lim_{n\to\infty}R(\hbar(f_n),\hbar(z^*)) \preceq sR(z^*,\hbar(z^*))$$

$$R(f_{n-1},\hbar^2(z^*)) \preceq s[R(f_{n-1},f_n) + R(f_n,z^*) + R(z^*,\hbar^2(z^*))]$$

$$lim_{n\to\infty}R(f_{n-1},\hbar^2(z^*)) \preceq sR(z^*,\hbar^2(z^*))$$

Taking $n \to \infty$, and in view of assumption (3) and (7) , we get

$$\mathcal{J}\left(sR(z^*,\hbar(z^*)), \mathbf{0}_{\mathcal{E}}, \mathbf{0}_{\mathcal{E}}, R(z^*,\hbar(z^*)), sR(z^*,\hbar^2(z^*)), \mathbf{0}_{\mathcal{E}}\right) \preceq \mathbf{0}_{\mathcal{E}}.$$

By $(\mathcal{J}_1)$ and condition (3), we have

$$\mathcal{J}\left(sR(z^*,\hbar(z^*)), \mathbf{0}_{\mathcal{E}}, \mathbf{0}_{\mathcal{E}}, R(z^*,\hbar(z^*)), sR(z^*,\hbar(z^*)), \mathbf{0}_{\mathcal{E}}\right) \preceq \mathbf{0}_{\mathcal{E}}.$$

This contradicts $(\mathcal{J}_3)$. Thus, $\|R(z^*,\hbar(z^*))\| = 0$. Hence, $R(z^*,\hbar(z^*)) = \mathbf{0}_{\mathcal{E}}$ and by (dRb1), we get $z^* = \hbar(z^*)$.

In next theorem, we present a result for non-increasing self-operators.

Theorem 14. *Let (W,R) be a complete rectangular cone b-metric space along with a cone $\mathcal{C} \subset \mathcal{E}$ and let $\hbar : W \to W$. Consider $\mathcal{T} \in B(\mathcal{E},\mathcal{E})$ with $\|\mathcal{T}\|_1 < \frac{1}{s}(s \geq 1)$, $\mathcal{I} : \mathcal{E} \to \mathcal{E}$ and $\mathcal{J} \in \mathcal{F}$ such that, for each pair of comparable elements $f,x \in W$ with $s \geq 1$, following holds*

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(f,\hbar(f))) \preceq sR(f,x) \text{ implies}$$

$$\mathcal{J}\left(R(\hbar(f),\hbar(x)), R(f,x), R(f,\hbar(f)), R(x,\hbar(x)), R(f,\hbar^2(x)), R(x,\hbar^2(f))\right) \preceq \mathbf{0}_{\mathcal{E}} \tag{8}$$

and

(1) there exists $f_0 \in W$ satisfying $\hbar(f_0)\Re f_0$;

(2) for every pair of $f, x \in W$, $f\Re x \Rightarrow \hbar(x)\Re\hbar(f)$;

(3) for a sequence $\{f_n\}$ satisfying $f_n\Re f_{n+1}$ and $f_n \to z^*$, we get $f_n\Re z^*\ \forall\ n \in \mathbb{N}$. Moreover, $R(z^*, \hbar(z^*)) \preceq R(z^*, \hbar^2(z^*))$.

Then, $\hbar$ admits a fixed point $z^* \in W$.

Proof. Let $f_0 \in W$ satisfies assumtion (1). Construct the sequence $\{f_n\}$ such that $f_n = \hbar(f_{n-1})\ \forall\ n \in N$. Since $f_1 = \hbar(f_0)\Re f_0$ and by using condition (2), $f_1 = \hbar(f_0)\Re\hbar(f_1) = f_2$, and assumption (2) further implies $f_n \preceq f_{n-1}$. Letting $f = f_1$ and $x = f_0$ in (8), we get

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(\hbar(f_0), f_0) = (I-\mathcal{T})^2(I+\mathcal{T})(R(f_1, f_0)) \preceq sR(f_1, f_0) \text{ implies}$$

$$\mathcal{J}\left(R(\hbar(f_1),\hbar(f_0)), R(f_1,f_0), R(f_1,\hbar(f_1)), R(f_0,\hbar(f_0)), R(f_1,\hbar^2(f_0)), R(f_0,\hbar^2(f_1))\right) \preceq \mathbf{0}_{\mathcal{E}}$$
$$\Rightarrow \mathcal{J}(R(f_1,f_2), R(f_0,f_1), R(f_1,f_2), R(f_0,f_1), R(f_1,f_2), R(f_0,f_3)) \preceq \mathbf{0}_{\mathcal{E}}.$$

By (dR3), we have

$$R(f_0, f_3) \preceq s[R(f_0, f_1) + R(f_1, f_2) + R(f_2, f_3)]$$

and then using $\mathcal{J}_1$, we obtain

$$\mathcal{J}(R(f_1,f_2), R(f_0,f_1), R(f_1,f_2), R(f_0,f_1), R(f_1,f_2), s[R(f_0,f_1)+R(f_1,f_2)+R(f_2,f_3)]) \preceq \mathbf{0}_{\mathcal{E}}.$$

By $(\mathcal{J}_2)$, there exists $\mathcal{T} \in B(\mathcal{E}, \mathcal{E})$ with $\|\mathcal{T}\|_1 < 1$ and

$$R(f_1,f_2) \preceq \frac{1}{s}\mathcal{T}(R(f_0,f_1)) \preceq \mathcal{T}(R(f_0,f_1)) \text{ and } R(f_2,f_3) \preceq \frac{1}{s}\mathcal{T}(R(f_1,f_2)) \preceq \mathcal{T}(R(f_1,f_2)).$$

By (2), $\hbar(f_0)\Re\hbar(f_1)$, letting $f = f_1$ and $x = f_2$ in (8) we get

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(f_1,\hbar(f_1)) = (I-\mathcal{T})^2(I+\mathcal{T})(R(f_1, f_2)) \preceq sR(f_1, f_2) \text{ implies}$$

$$\mathcal{J}\left(R(\hbar(f_1),\hbar(f_2)), R(f_1,f_2), R(f_1,\hbar(f_1)), R(f_2,\hbar(f_2)), R(f_1,\hbar^2(f_2)), R(f_2,\hbar^2(f_1))\right) \preceq \mathbf{0}_{\mathcal{E}}$$
$$\Rightarrow \mathcal{J}(R(f_2,f_3), R(f_1,f_2), R(f_1,f_2), R(f_2,f_3), R(f_1,f_4), R(f_2,f_3)) \preceq \mathbf{0}_{\mathcal{E}}.$$

By (dR3), $(\mathcal{J}_1)$ and $(\mathcal{J}_2)$, we get

$$R(f_3,f_4) \preceq \frac{1}{s}\mathcal{T}(R(f_2,f_3)) \preceq \frac{1}{s^2}\mathcal{T}^2(R(f_1,f_2)) \preceq \frac{1}{s^3}\mathcal{T}^3(R(f_0,f_1)) \preceq \mathcal{T}^3(R(f_0,f_1)).$$

Following this pattern, we get a sequence $\{f_n\}$ that satisfies

$$R(f_n,f_{n+1}) \preceq \frac{1}{s}\mathcal{T}(R(f_{n-1},f_n)) \preceq \frac{1}{s^2}\mathcal{T}^2(R(f_{n-2},f_{n-1})) \preceq \cdots \preceq \frac{1}{s^n}\mathcal{T}^n(R(f_0,f_1)).$$

Following the pattern for the proof of Theorem 13, there exists a convergence point z^* satisfying $z^* = \hbar(z^*)$.

The next result summarizes the results of Theorem 13 and Theorem 14.

Theorem 15. *Let (W, R) be a complete rectangular cone b-metric space with $\mathcal{C} \subset \mathcal{E}$ as a cone and $\hbar : W \to W$. If $\mathcal{T} \in B(\mathcal{E}, \mathcal{E})$ with $\|\mathcal{T}\|_1 < \frac{1}{s}(s \geq 1)$, $\mathcal{I} : \mathcal{E} \to \mathcal{E}$ and $\mathcal{J} \in \mathcal{F}$. Also let, for each pair of comparable elements $\varrho, x \in W$ and for some $s \geq 1$, the following holds*

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(f,\hbar(f))) \preceq sR(f,x) \text{ implies}$$

$$\mathcal{J}\left(R(\hbar(f),\hbar(x)), R(f,x), R(f,\hbar(f)), R(x,\hbar(x)), R(f,\hbar^2(x)), R(k,\hbar^2(f))\right) \preceq \mathbf{0}_{\mathcal{E}}, \tag{9}$$

and

(1) $\exists\ f_0 \in W$ with $f_0 \Re \hbar(f_0)$ or $\hbar(f_0) \Re f_0$;

(2) for a sequence $\{f_n\}$ satisfying $f_n \Re f_{n+1}$ and $f_n \to z^*$, we get $f_n \Re z^*\ \forall\ n \in \mathbb{N}$ and let $R(z^*, \hbar(z^*)) \preceq R(z^*, \hbar^2(z^*))$.

Then, $\hbar$ admits a fixed point $z^* \in W$.

Proof. Let $f_0 \in W$ and define the sequence $\{f_n\}$ by $f_n = \hbar(f_{n-1})\ \forall\ n \in \mathbb{N}$. By assumption (1), we get $f_0 \Re \hbar(f_0) = f_1$. Also $f_2 \Re f_1$(as f is order reversing). By (9), we obtain

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(f_0,\hbar(f_0))) = (I-\mathcal{T})^2(I+\mathcal{T})(R(f_0,f_1)) \preceq sR(f_0,f_1) \text{ implies}$$

$$\mathcal{J}\begin{pmatrix} R(\hbar(f_0),\hbar(f_1)), R(f_0,f_1), R(f_0,\hbar(f_0)), \\ R(f_1,\hbar(f_1)), R(f_0,\hbar(f_2)), R(f_1,\hbar(f_1)) \end{pmatrix} \preceq \mathbf{0}_{\mathcal{E}},$$

that is,

$$\mathcal{J}\begin{pmatrix} R(f_1,f_2), R(f_0,f_1), R(f_0,f_1), R(f_1,f_2), \\ R(f_0,f_3), R(f_1,f_2) \end{pmatrix} \preceq \mathbf{0}_{\mathcal{E}}. \tag{10}$$

By (dR3) and $(\mathcal{J}_1)$,

$$\mathcal{J}(R(f_1,f_2), R(f_0,f_1), R(f_0,f_1), R(f_1,f_2), s[R(f_0,f_1)+R(f_1,f_2)+R(f_2,f_3)], R(f_2,f_1) \preceq \mathbf{0}_{\mathcal{E}}.$$

By $(\mathcal{J}_2)$, $\exists\ \mathcal{T} \in B(\mathcal{E},\mathcal{E})$ so that $\|\mathcal{T}\|_1 < 1$ with

$$R(f_1,f_2) \preceq \frac{1}{s}\mathcal{T}(R(f_0,f_1)) \preceq \mathcal{T}(R(f_0,f_1)),$$

and

$$R(f_2,f_3) \preceq \frac{1}{s}\mathcal{T}(R(f_1,f_2)) \preceq \mathcal{T}(R(f_1,f_2)).$$

using assumption (2), we get $f_2 \Re f_1$(f being order preserving), hence using (9)

$$(I-\mathcal{T})^2(I+\mathcal{T})R(\hbar(f_1),f_1) = (I-\mathcal{T})^2(I+\mathcal{T})R(f_2,f_1) \preceq sR(f_2,f_1) \text{ implies}$$

$$\mathcal{J}\begin{pmatrix} R(\hbar(f_2),\hbar(f_1)), R(f_2,f_1), R(f_2,\hbar(f_2)), R(f_1,\hbar(f_1), \\ R(f_2,\hbar^2(f_1)), R(f_1,\hbar^2(f_2)) \end{pmatrix} \preceq \mathbf{0}_{\mathcal{E}}.$$

By (dR3), $(\mathcal{J}_1)$ and $(\mathcal{J}_2)$, we get

$$R(f_3,f_4) \preceq \frac{1}{s}\mathcal{T}(R(f_2,f_3)) \preceq \frac{1}{s^2}\mathcal{T}^2(R(f_1,f_2)) \preceq \frac{1}{s^3}\mathcal{T}^3(R(f_0,f_1)) \preceq \mathcal{T}^3(R(f_0,f_1)).$$

This pattern enable us to construct a sequence $\{f_n\}$ so that

$$R(f_n,f_{n+1}) \preceq \frac{1}{s}\mathcal{T}(R(f_{n-1},f_n)) \preceq \frac{1}{s^2}\mathcal{T}^2(R(f_{n-2},f_{n-1})) \preceq \cdots \preceq \frac{1}{s^n}\mathcal{T}^n(R(f_0,f_1)).$$

By following same pattern as for the proof of Theorem 13, we will obtain $z^* = \hbar(z^*)$.

Remark 16. *We get different ordered contractive conditions for different definitions of* $\mathcal{J}$. *Also, Theorem 15 generalizes the results in [8–11, 37, 38].*

Remark 17. *In a non-normal cone, by taking an" upper bound or lower bound" for every pair* $\varrho, \omega \in W$, *one may find a unique fixed point in Theorem 15.*

5. Examples and consequences

In this section, we give some examples for the explanation of hypotheses of main theorem.

Example 6. *Let* $\mathcal{E} = (\mathbb{R}^3, \|\cdot\|)$, *then it is a real Banach space. Define a cone* $\mathcal{C} \in \mathcal{E}$ *such that* $\mathcal{C} = \{(\vartheta, x, \nu) \in \mathbb{R}^3 : \vartheta, x, \nu \geq 0\}$ *and* $\|\vartheta\| = \max\{(|\vartheta_1|, |\vartheta_2|, |\vartheta_3|)\}$. *Consider* $X = \{(0,0,0), \left(\frac{1}{4},0,0\right), \left(\frac{1}{4},\frac{1}{4},0\right), \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\} \subset \mathcal{E}$ *with* $\hbar : X \to X$ *defined as*

$$\hbar(0,0,0) = \hbar\left(\frac{1}{4},0,0\right) = (0,0,0),\ \hbar\left(\frac{1}{4},\frac{1}{4},0\right) = \left(\frac{1}{4},0,0\right),\ \hbar\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) = \left(\frac{1}{4},\frac{1}{4},0\right).$$

Define $R : X \times X \to \mathcal{E}$,

$$R(\vartheta_1,\vartheta_2) = \begin{cases} (0,0,0) & \text{if } \vartheta_1 = \vartheta_2 \\ \left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right) & \text{if } \vartheta_1,\vartheta_2 \in \{(0,0,0), \left(\frac{1}{4},\frac{1}{4},0\right)\} \\ \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) & \text{if } \vartheta_1,\vartheta_2 \in \{\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right), \left(\frac{1}{4},\frac{1}{4},0\right)\} \\ \left(0,\frac{1}{4},\frac{1}{4}\right) & \text{otherwise.} \end{cases}$$

Define $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ *by*

$$\mathcal{T}(\vartheta) = \frac{\vartheta}{2}, \text{ clearly } \|\mathcal{T}\|_1 < 1.$$

$$\begin{aligned} \left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right) &= R\left((0,0,0), \left(\frac{1}{4},\frac{1}{4},0\right)\right) \\ &\succeq R\left((0,0,0), \left(\frac{1}{4},0,0\right)\right) + R\left(\left(\frac{1}{4},0,0\right), \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right) + R\left(\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right), \left(\frac{1}{4},\frac{1}{4},0\right)\right) \\ &= \left(0,\frac{1}{4},\frac{1}{4}\right) + \left(0,\frac{1}{4},\frac{1}{4}\right) + \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) \end{aligned}$$

$$= \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right).$$

Note that R is RCBM but not RBM, for $s = 3$. Consider $\vartheta = (0,0,0)$ and $\omega = \left(\frac{1}{4}, 0, 0\right)$. Then $\hbar(\vartheta) = (0,0,0) = \hbar(\omega) = \hbar^2(\omega)$

$$R(\vartheta, \hbar(\vartheta)) = R(\vartheta, \hbar^2(\omega)) = (0,0,0)\,, \;\; R(\vartheta, \omega) = \left(0, \frac{1}{4}, \frac{1}{4}\right),$$

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) = (0,0,0).$$

Take $\vartheta = \left(\frac{1}{4}, 0, 0\right)$ and $\omega = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$. Then $\hbar(\vartheta) = (0,0,0) = \hbar^2(\omega)$ and $\hbar(\omega) = \left(\frac{1}{4}, 0, 0\right)$.

$$R(\vartheta, \hbar(\vartheta)) = R(\vartheta, \hbar^2(\omega)) = R(\vartheta, \omega) = \left(0, \frac{1}{4}, \frac{1}{4}\right),$$

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) = \left(0, \frac{3}{32}, \frac{3}{32}\right).$$

For $\vartheta = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$ and $\omega = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Then $\hbar(\vartheta) = (\frac{1}{4}, 0, 0) = \hbar^2(\omega)$ and $\hbar(\omega) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$.

$$R(\hbar(\vartheta), \hbar(\omega)) = R(\vartheta, \hbar(\vartheta)) = R(\vartheta, \hbar^2(\omega)) = \left(0, \frac{1}{4}, \frac{1}{4}\right), \;\; R(\vartheta, \omega) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right),$$

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) = \left(0, \frac{3}{32}, \frac{3}{32}\right).$$

Take $\vartheta = (0,0,0)$ and $\omega = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$. Then $\hbar(\vartheta) = (0,0,0) = \hbar^2(\omega)$ and $\hbar(\omega) = \left(\frac{1}{4}, 0, 0\right)$.

$$R(\hbar(\vartheta), \hbar(\omega)) = R(\vartheta, \omega) = \left(0, \frac{1}{4}, \frac{1}{4}\right) \text{ and } R(\vartheta, \hbar^2(\omega)) = (0,0,0),$$

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) = (0,0,0)\,.$$

Take $\vartheta = (0,0,0)$ and $\omega = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Then $\hbar(\vartheta) = (0,0,0) = \hbar^2(\omega)$ and $\hbar(\omega) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$.

$$R(\hbar(\vartheta), \hbar(\omega)) = R(\vartheta, \omega) = \left(0, \frac{1}{4}, \frac{1}{4}\right) \text{ and } R(\vartheta, \hbar^2(\omega)) = (0,0,0),$$

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) = (0,0,0)\,.$$

Take $\vartheta = \left(\frac{1}{4}, 0, 0\right)$ and $\omega = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Then $\hbar(\vartheta) = (0,0,0) = \hbar^2(\omega)$ and $\hbar(\omega) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$.

$$R(\hbar(\vartheta), \hbar(\omega)) = R(\vartheta, \omega) = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right) \text{ and } R(\vartheta, \hbar^2(\omega)) = (0,0,0),$$

$$(I-\mathcal{T})^2(I+\mathcal{T})R(\vartheta,\hbar(\vartheta)) = \left(0, \frac{3}{32}, \frac{3}{32}\right).$$

Define for $\alpha \in [1,\infty)$, and $\mathcal{J} \in \mathcal{F}$

$$\mathcal{J}\left(\begin{array}{l} R(\hbar(\vartheta),\hbar(\omega)), R(\vartheta,\omega), R(\vartheta,\hbar(\vartheta)) \\ , R(\omega,\hbar(\omega)), R(\vartheta,\hbar^2(\omega)), R(\omega,\hbar^2(\vartheta)) \end{array}\right) = R(\vartheta,\hbar^2(\omega)) - [\alpha R(\hbar(\vartheta),\hbar(\omega)) + R(\vartheta,\omega)].$$

Thus, $(I-\mathcal{T})^2(I+\mathcal{T})R(\vartheta,\hbar(\vartheta)) \preceq R(\vartheta,\omega)$ implies

$$\mathcal{J}\left(\begin{array}{l} R(\hbar(\vartheta),\hbar(\omega)), R(\vartheta,\omega), R(\vartheta,\hbar(\vartheta)), R(\omega,\hbar(\omega)), \\ R(\vartheta,\hbar^2(\omega)), R(\omega,\hbar^2(\vartheta)) \end{array}\right) \preceq 0.$$

So all assumptions of Theorem 15 are justified and $\hbar(0,0,0) = (0,0,0)$.

Example 7. *Let $\mathcal{E} = (\mathbb{R}, \|\cdot\|)$, then it is a real Banach space and $\mathcal{C} = \{\vartheta \in \mathbb{R} : \vartheta \geq 0\}$ is a cone in $\mathcal{E}$. Let $X = \{1,2,3,4\}$ and define $\hbar : X \to X$ by $\hbar(1) = \hbar(2) = 1$ and $\hbar(4) = \hbar(3) = 2$. Define R by*

$$R(\vartheta_1,\vartheta_2) = \begin{cases} 0 & \text{if } \vartheta_1 = \vartheta_2 \\ \frac{e^t}{3} & \text{if } \vartheta_1,\vartheta_2 \in \{1,2\} \\ \frac{e^t}{2} & \text{if } (\vartheta_1,\vartheta_2) \in \{3,4\} \\ e^t & \text{otherwise }. \end{cases}$$

Note that

$$e^t = R(2,4) \geq R(2,1) + R(1,3) + R(3,4) = \frac{5e^t}{6}.$$

For $s = \frac{6}{5}$, R represents RCBM but not RCM. Define $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ by

$$\mathcal{T}(\vartheta) = \frac{\vartheta}{3}, \text{ clearly } \|\mathcal{T}\|_1 < 1.$$

Take $\vartheta = 1$ and $\omega = 2$, then

$$R(\omega,\hbar(\omega)) = R(\vartheta,\omega) = R(\omega,\hbar^2(\vartheta)) = \frac{e^t}{3}, \ \ R(\vartheta,\hbar(\vartheta)) = R(\hbar(\vartheta),\hbar(\omega)) = 0.$$

Take $\vartheta = 2$ and $\omega = 3$, then

$$R(\vartheta,\omega) = R(\vartheta,\hbar(\vartheta)) = R(\omega,\hbar^2(\vartheta)) = R(\omega,\hbar(\omega)) = \frac{e^t}{3}.$$

As

$$(I+\mathcal{T})(I-\mathcal{T})^2R(\vartheta,\hbar(\vartheta)) = (I-\mathcal{T})(I-\mathcal{T}^2)R(\vartheta,\hbar(\vartheta)).$$

We get

$$(I-\mathcal{T})(I-\mathcal{T}^2)R(\vartheta,\hbar(\vartheta)) = \frac{17e^t}{81}.$$

Take $\vartheta = 3$ *and* $\omega = 4$*, then*

$$R(\omega, \hbar(\omega)) = R(\omega, \hbar^2(\vartheta)) = e^t, \;\; R(\vartheta, \hbar(\vartheta)) = \frac{e^t}{3}, \;\; R(\vartheta, \omega) = \frac{e^t}{2}.$$

$$(I - \mathcal{T})(I - \mathcal{T}^2)R(\vartheta, \hbar(\vartheta)) = \frac{17e^t}{81}.$$

For $\mathcal{J} \in \mathcal{F}$*, define*

$$\mathcal{J}\left(\begin{array}{l} R(\hbar(\vartheta), \hbar(\omega)), R(\vartheta, \omega), R(\vartheta, \hbar(\vartheta)) \\ , R(\omega, \hbar(\omega)), R(\vartheta, \hbar^2(\omega)), R(\omega, \hbar^2(\vartheta)) \end{array} \right) = R(\omega, \hbar^2(\vartheta)) - \left[R(\vartheta, \hbar(\vartheta)) + R(\omega, \hbar(\omega))\right].$$

Thus, $(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) \preceq R(\vartheta, \omega)$ *implies*

$$\mathcal{J}\left(\begin{array}{l} R(\hbar(\vartheta), \hbar(\omega)), R(\vartheta, \omega), R(\vartheta, \hbar(\vartheta)), R(\omega, \hbar(\omega)), \\ R(\vartheta, \hbar^2(\omega)), R(\omega, \hbar^2(\vartheta)) \end{array} \right) \preceq 0.$$

Note that $\hbar(1) = 1$*.*

5.1. Consequences of the main theorem.

In this subsection, we state some corollaries that can be deduced directly from the main theorem.

Corollary 18. *Let* (X, R) *be a complete rectangular cone b-metric space along with self mapping* $\hbar : X \to X$ *and a cone* $\mathcal{C}$*. Let* $\mathcal{T} \in B(\mathcal{E}, \mathcal{E})$ *and* $I : \mathcal{E} \to \mathcal{E}$ *an identity operator,* $s \geq 1$*. If there exists* $\mathcal{J} \in \mathcal{F}$ *satisfying, for each pair of comparable elements* $\vartheta, \omega \in X$*,*

$$(I - \mathcal{T})^2(I + \mathcal{T})(R(\vartheta, \hbar(\vartheta))) \preceq sR(\vartheta, \omega) \text{ implies}$$

$$R(\hbar(\vartheta), \hbar(\omega)) \preceq \frac{1}{s}\mathcal{T}(R(\vartheta, \omega)), \tag{11}$$

and

(1) $\exists\, \vartheta \in X$ *so that* $\vartheta_0 \Re \hbar(\vartheta_0)$ *or* $\hbar(\vartheta_0) \Re \vartheta_0$*;*

(2) $\forall\, \vartheta, \omega \in X$*,* $\vartheta \Re \omega \Rightarrow \hbar(\vartheta) \Re \hbar(\omega)$ *or* $\hbar(\omega) \Re \hbar(\vartheta)$*;*

(3) for a sequence $\{\vartheta_n\}$ *such that* $\vartheta_n \Re \vartheta_{n+1}$ *and* $\vartheta_n \to z^*$*, we get* $\vartheta_n \Re z^*$ $\forall\, n \in \mathbb{N}$ *and let* $R(z^*, \hbar(z^*)) \preceq R(z^*, \hbar^2(z^*))$*.*

Then there exists $z^* \in X$ *so that* $z^* = \hbar(z^*)$*.*

Proof. Define $\mathcal{J}$ as in Example 5 (iv), and $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ by $\mathcal{T}(\vartheta) = \beta\vartheta \; \forall \; \vartheta \in \mathcal{E}$, $0 \leq \beta < 1$. Clearly $\|\mathcal{T}\|_1 < 1$, we get $\mathcal{T} \in B(\mathcal{E}, \mathcal{E})$, and the proof is obvious as an application of Theorem 15.

Corollary 19. *Let (X, R) be a complete rectangular cone b-metric space, $\hbar : X \to X$, $\mathcal{C}$ be a cone, $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ such that $\|\mathcal{T}\|_1 < \frac{1}{s}$, $I : \mathcal{E} \to \mathcal{E}$ ba an identity operator and $s \geq 1$. If there exists $\mathcal{J} \in \mathcal{F}$, for each pair of comparable elements $\vartheta, \omega \in X$, satisfying*

$$(I - \mathcal{T})^2 (I + \mathcal{T})(R(\vartheta, \hbar(\vartheta))) \preceq sR(\vartheta, \omega) \text{ implies}$$

$$R(\hbar(\vartheta), \hbar(\omega)) \preceq \frac{\mathcal{T}}{s}(R(\vartheta, \omega))$$

and

(1) there exists $\vartheta_0 \in X$ satisfying $\vartheta_0 \Re \hbar(\vartheta_0)$ or $\hbar(\vartheta_0) \Re \vartheta_0$;

(2) $\forall\ \vartheta, \omega \in X$, $\vartheta \Re \omega \Rightarrow \hbar(\vartheta) \Re \hbar(\omega)$ or $\hbar(\omega) \Re \hbar(\vartheta)$;

(3) for a sequence $\{\vartheta_n\}$ such that $\vartheta_n \Re \vartheta_{n+1}$ and $\vartheta_n \to z^*$, we get $\vartheta_n \Re z^*$ for all $n \in \mathbb{N}$ and let $R(z^*, \hbar(z^*)) \preceq R(z^*, \hbar^2(z^*))$.

Then there exists $z^* \in X$ satisfying $z^* = \hbar(z^*)$.

Proof. Define $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ by $\mathcal{T}(\vartheta) = \vartheta\ \forall\ \vartheta \in \mathcal{E}$. Then the proof is obvious from Corollary 18 .

Example 8. *Let $\mathcal{E} = (\mathbb{R}, \|\cdot\|)$, then it is real Banach space. The set $\mathcal{C} = \{\vartheta \in \mathbb{R} : \vartheta \geq 0\}$ is a cone. Let $W = \{0, 1, 2, 3\}$, define $\hbar : W \to W$ by $\hbar(0) = \hbar(2) = 0$, $\hbar(1) = 2$, $\hbar(3) = 1$ and $R : W \times W \to \mathcal{E}$ by*

$$R(\vartheta_1, \vartheta_2) = \begin{cases} 0 & \text{if } \vartheta_1 = \vartheta_2 \\ 3 & \text{if } \vartheta_1, \vartheta_2 \in \{1, 2\} \\ 10 & \text{if } \vartheta_1, \vartheta_2 \in \{0, 1\} \\ 22 & \text{if } \vartheta_1, \vartheta_2 \in \{2, 3\} \\ 0.5 & \text{otherwise .} \end{cases}$$

$$22 = R(2, 3) \geq R(2, 0) + R(0, 1) + R(1, 3) = 0.5 + 10 + 0.5 = 11.$$

We notice that for $s = 2$, R is a rectangular cone b-metric space, but not cone rectangular metric space. Define $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ by

$$\mathcal{T}(\vartheta) = \frac{\vartheta}{2}, \text{ clearly } \|\mathcal{T}\|_1 < 1.$$

For if $\vartheta = 0$, $\omega = 1$, then

$$R(\vartheta, \omega) = 10, \ \ R(\hbar(\vartheta), \hbar(\omega)) = 0.5, \ \ R(\vartheta, \hbar(\vartheta)) = 0.$$

$$\mathcal{T}R(\vartheta, \omega) = \frac{10}{2} = 5.$$

For if $\vartheta = 1$, $\omega = 2$, then

$$R(\vartheta, \omega) = R(\vartheta, \hbar(\vartheta)) = 3, \ \ R(\hbar(\vartheta), \hbar(\omega)) = 1.125.$$

$$\mathcal{T}R(\vartheta,\omega)=\frac{3}{2}=1.5,\ \ (I-\mathcal{T})^2(I+\mathcal{T})R(\vartheta,\hbar(\vartheta))=0.25.$$

For if $\vartheta=2,\ \omega=3$, *then*

$$R(\vartheta,\omega)=22,\ \ R(\hbar(\vartheta),\hbar(\omega))=10,\ \ R(\vartheta,\hbar(\vartheta))=0.5.$$

$$\mathcal{T}R(\vartheta,\omega)=\frac{0.5}{2}=0.25,\ \ (I-\mathcal{T})^2(I+\mathcal{T})R(\vartheta,\hbar(\vartheta))=0.1875.$$

Hence for all $\vartheta,\omega\in W$,

$$(I-\mathcal{T})^2(I+\mathcal{T})R(\vartheta,\hbar(\vartheta))\preceq sR(\vartheta,\omega)$$

implies

$$R(\hbar(\vartheta),\hbar(\omega))\preceq\mathcal{T}R(\vartheta,\omega).$$

Corollary 18 leads us to have a point 0 *satisfying* $\hbar(0)=0$.

Example 9. *Let* $\mathcal{E}=C^1_R[1,2]$, *and* $\|\vartheta\|=\|\vartheta\|_\infty+\|\acute{\vartheta}\|_\infty$, $\mathcal{C}=\{\vartheta(s)\in\mathcal{E}:\vartheta(s)>0,s\in[1,2]\}$. *For* $q\geq 1$, *let* $\vartheta=x$ *and* $\omega=x^{2k}$. *By definition* $\|\vartheta\|=2$ *and* $\|\omega\|=2k+1$ *also* $\vartheta\preceq\omega$, *with* $q\|\vartheta\|\ \leq\|\omega\|$. *So* $\mathcal{C}$ *represents a non normal cone. Define* $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ *by*

$$(\mathcal{T}\vartheta)(t)=\frac{1}{2}\int_0^t\vartheta(t)ds.$$

$$(\mathcal{T}(\vartheta+\omega))(t)=\frac{1}{2}\int_0^t(\vartheta+\omega)(t)ds=\frac{1}{2}\int_0^t\vartheta(t)ds+\frac{1}{2}\int_0^t\omega(t)ds.$$

$$(\mathcal{T}^n\vartheta)(t)\preceq\frac{\mathcal{T}^n}{2^n n!}\|\vartheta\|_\infty^{\frac{1}{2}}\preceq\frac{2^n}{2^n n!}\|\vartheta\|^{\frac{1}{2}}\ \textit{for}\ \ \textit{each}\ t\in[1,2]\ \textit{and}\ n\geq 1.$$

So

$$\|(\mathcal{T}^n\vartheta)(t)\|_\infty\preceq\frac{1}{n!}\|\vartheta\|^{\frac{1}{2}}.$$

$$\|(\mathcal{T}^n\vartheta)'(t)\|_\infty\preceq\frac{1}{\sqrt{(n-1)!}}\|\vartheta\|^{\frac{1}{4}}\ \textit{for}\ n\geq 2.$$

$$\|(\mathcal{T}^n\vartheta)(t)\|=\|(\mathcal{T}^n\vartheta)(t)\|_\infty+\|(\mathcal{T}^n\vartheta)'(t)\|_\infty\preceq\frac{1}{n!}\|\vartheta\|^{\frac{1}{2}}+\frac{1}{\sqrt{(n-1)!}}\|\vartheta\|^{\frac{1}{4}}\ \textit{for}\ n\geq 2,$$

$$\|(\mathcal{T}^n\vartheta)(t)\|=0\ \textit{when}\ n\geq n_1\ \textit{for}\ n_1\in N.$$

So $\mathcal{T}\in B(\mathcal{E},\mathcal{E})$. *Take* $X=\{\frac{e^t}{3},\frac{e^t}{2},e^t,2e^t\}$ *and* $\hbar:X\to X$, *such that* $\hbar(\frac{e^t}{3})=\hbar(\frac{e^t}{2})=\frac{e^t}{3}$, $\hbar(e^t)=2e^t$ *and* $\hbar(2e^t)=\frac{e^t}{2}$. *Define* $R:X\times X\to\mathcal{E}$ *by*

$$R(\vartheta_1,\vartheta_2)=\begin{cases}0 & \textit{if}\ \vartheta_1=\vartheta_2\\ 10e^t & \textit{if}\ (\vartheta_1,\vartheta_2)=(\frac{e^t}{3},e^t),\\ \frac{e^t}{3} & \textit{if}\ (\vartheta_1,\vartheta_2)=(\frac{e^t}{2},2e^t),\\ 6e^t & \textit{if}\ (\vartheta_1,\vartheta_2)\in\{(\frac{e^t}{2},e^t),(e^t,2e^t)\}\\ \frac{3e^t}{2} & \textit{otherwise}\ .\end{cases}$$

$$10e^t = R\left(\frac{e^t}{3}, e^t\right) \geq R\left(\frac{e^t}{3}, \frac{e^t}{2}\right) + R\left(\frac{e^t}{2}, 2e^t\right) + R\left(2e^t, e^t\right) = \frac{3e^t}{2} + \frac{e^t}{3} + 6e^t = \frac{29e^t}{6}.$$

One can easily check that (db3) holds for $s = 3$, so R represents rectangular cone b-metric space, but not rectangular cone metric space. For if $\vartheta = \frac{e^t}{3}$ and $\omega = \frac{e^t}{2}$, then

$$R(\vartheta, \hbar(\vartheta)) = R(\hbar(\vartheta), \hbar(\omega)) = 0, \;\; R(\vartheta, \omega) = \frac{3e^t}{2}$$

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) = 0.$$

For if $\vartheta = \frac{e^t}{2}$ and $\omega = e^t$, then

$$R(\vartheta, \omega) = 6e^t, \;\; R(\hbar(\vartheta), \hbar(\omega)) = R(\vartheta, \hbar(\vartheta)) = \frac{3e^t}{2}, \;\; \frac{\mathcal{T}}{s}R(\vartheta, \omega)(t) = e^t.$$

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) = \frac{15e^t}{4}.$$

For if $\vartheta = e^t$ and $\omega = 2e^t$, then

$$R(\vartheta, \omega) = 6e^t, \;\; R(\hbar(\vartheta), \hbar(\omega)) = \frac{3e^t}{2}, \;\; R(\vartheta, \hbar(\vartheta)) = e^t \;\; \frac{\mathcal{T}}{s}R(\vartheta, \omega)(t) = e^t.$$

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) = \frac{3e^t}{8}.$$

Now if $t \in [1, 2]$, and $s = 3$, then

$$(I - \mathcal{T})^2(I + \mathcal{T})R(\vartheta, \hbar(\vartheta)) \preceq sR(\vartheta, \omega)$$

implies

$$R(\hbar(\vartheta), \hbar(\omega)) \preceq \frac{1}{s}\mathcal{T}R(\vartheta, \omega).$$

By Corollary 19, we get $\hbar(\frac{e^t}{3}) = \frac{e^t}{3}$.

6. A homotopy result

In this section, we apply Corollary 19 to obtain the following homotopy result.

Theorem 20. *Let $(\mathcal{E}, \|.\|)$ be a real Banach space and $\mathcal{C} \subset \mathcal{E}$ be a cone. Let (X, R) be a complete rectangular cone b-metric space. Assume an open set $\mathcal{V} \subset X$. Let $\mathcal{T} \in B(\mathcal{E}, \mathcal{E})$ with $\|\mathcal{T}\|_1 < 1$ and $\mathcal{T}(\mathcal{C}) \subset \mathcal{C}$ and $g : \overline{\mathcal{V}} \times [0, 1] \to X$ be a monotonic mapping, and satisfies the axioms of Corollary 19 in the first variable and*

(1) $\vartheta \neq g(\vartheta, \wp)$ for each $\vartheta \in \partial\mathcal{V}$ ($\partial\mathcal{V}$ stands for the boundary of $\mathcal{V}$ in X);

(2) we have $M \geq 0$ with

$$\|R(g(\vartheta, \wp), g(\vartheta, \sigma))\| \leq M|\wp - \sigma|,$$

for some $\vartheta \in \overline{\mathcal{V}}$ and $\sigma, \wp \in [0, 1]$;

(3) for any $\vartheta \in \mathcal{V}$ there is $\omega \in X$ so that $\|R(\vartheta,\omega)\| \leq r$, then $\vartheta \Re \omega$, r is the radius of $\mathcal{V}$.

If $g(\cdot,0)$ admits a fixed point in $\mathcal{V}$, then $g(\cdot,1)$ also admits a fixed point in $\mathcal{V}$.

Proof. Let

$$C = \{\wp \in [0,1] \mid \vartheta = g(\vartheta,\wp); \text{ for } \vartheta \in \mathcal{V}\}.$$

Define the partial order $\preceq$ in $\mathcal{E}$ by $p \preceq h \Leftrightarrow \|p\| \leq \|h\| \ \forall\ p,h \in \mathcal{E}$. It is obvious that $0 \in C$, because $g(.,0)$ possesses a fixed point in $\mathcal{V}$. Hence $C \neq \phi$. Defining $R(\vartheta, g(\vartheta,\wp)) = R(\vartheta,\omega)$, $(I-\mathcal{T})^2(I+\mathcal{T})(R(\vartheta, g(\vartheta,\wp))) \preceq sR(\vartheta,\omega)\ \forall\ \vartheta\Re\omega$, then by using Corollary 19, we get

$$R(g(\vartheta,\wp), g(\omega,\wp)) \preceq \frac{1}{s}\mathcal{T}(R(\vartheta,\omega)).$$

Firstly, we need to show that C is closed in $[0,1]$. Take $\{\wp_n\}_{n=1}^{\infty} \subseteq C$ along with $\lim_{n\to\infty} \wp_n \to \wp \in [0,1]$. To show C is closed, we need to show that $\wp \in C$. As $\wp_n \in C\ \forall$ $n \in \mathbb{N}$, so we have $\vartheta_n \in \mathcal{V}$ so that $\vartheta_n = g(\vartheta_n, \wp_n)$. Because, $g(\vartheta,\cdot)$ is monotone, we get $\vartheta_m \Re \vartheta_n\ \forall\ n,m \in \mathbb{N}$. Also for $s \geq 1$,

$$(I-\mathcal{T})^2(I+\mathcal{T})(R(\vartheta_n, g(\vartheta_m,\wp_m))) = (I-\mathcal{T})^2(I+\mathcal{T})(R(\vartheta_n,\vartheta_m)) \preceq sR(\vartheta_n,\vartheta_m),$$

we have

$$R(g(\vartheta_n,\wp_m), g(\vartheta_m,\wp_m)) \preceq \frac{1}{s}\mathcal{T}(R(\vartheta_n\vartheta_m)),$$

and

$$\begin{aligned}
R(\vartheta_n,\vartheta_{m+1}) &= R(g(\vartheta_n,\wp_n), g(\vartheta_{m+1},\wp_{m+1})) \\
&\preceq s[R(g(\vartheta_n,\wp_n), g(\vartheta_n,\wp_m)) + R(g(\vartheta_n,\wp_m), g(\vartheta_n,\wp_{m+1}))] \\
&+ sR(g(\vartheta_n,\wp_{m+1}), g(\vartheta_{m+1},\wp_{m+1})) \\
\|R(\vartheta_n,\vartheta_m)\| &\leq s\left[M|\wp_n-\wp_m| + M|\wp_m-\wp_{m+1}| + \left\|\frac{1}{s}\mathcal{T}(R(\vartheta_n,\vartheta_m))\right\|\right] \\
\|R(\vartheta_n,\vartheta_m)\| &\leq \frac{sM}{1-\|\mathcal{T}\|}[|\wp_n-\wp_m| + |\wp_m-\wp_{m+1}|].
\end{aligned}$$

Since $\{\vartheta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $[0,1]$, so

$$\lim_{n,m\to\infty} R(\vartheta_n,\vartheta_m) = \mathbf{0}_{\mathcal{E}}.$$

This shows that $\{\vartheta_n\}$ represents a Cauchy sequence in X. Since X is a complete RCBM space, hence $\lim_{n\to\infty} R(\vartheta_n,\vartheta) \ll c$ for some $\vartheta \in \overline{U}$. As a result $\vartheta_n \Re \vartheta\ \forall\ n \in \mathbb{N}$. By $(dRb3)$

$$\begin{aligned}
R(\vartheta, g(\vartheta,\wp)) &\preceq s[R(\vartheta,\vartheta_n) + R(\vartheta_n, g(\vartheta_n,\wp)) + R(g(\vartheta_n,\wp), g(\vartheta,\wp))] \\
&\preceq s[R(\vartheta,\vartheta_n) + R(g(\vartheta_n,\wp_n), g(\vartheta_n,\wp)) + R(g(\vartheta_n,\wp), g(\vartheta,\wp))] \\
\|R(\vartheta, g(\vartheta,\wp))\| &\leq sR(\vartheta,\vartheta_n) + s\left[M|\wp_n-\wp| + \left\|\frac{1}{s}\mathcal{T}(R(\vartheta_n,\vartheta))\right\|\right].
\end{aligned}$$

Hence, $R(\vartheta, g(\vartheta, \wp)) = \mathbf{0}_{\mathcal{E}}$. So $\wp \in C$, hence C is closed in $[0,1]$. Now we show that C is open in $[0,1]$. Take $\wp_2 \in C$ with $g(\wp_2, \vartheta_2) = \vartheta_2$ for $\vartheta_2 \in \mathcal{V}$. Now $\mathcal{V}$ is open, so we can find some $r > 0$ with $C(\vartheta_2, r) \subseteq \mathcal{V}$. Assume

$$l = R(\vartheta_2, \partial\mathcal{V}) = \inf\{R(\vartheta_2, x) : x \in \partial\mathcal{V}\}.$$

We get $r = l > 0$. Given $\epsilon > 0$ taking $\epsilon < \frac{(1-\|\mathcal{T}\|)l}{2sM}$. Take $\wp \in (\wp_1 - \epsilon, \wp_1 + \epsilon)$ with $\wp_1 \in (\wp_2 - \epsilon, \wp_2 + \epsilon)$. Subsequently

$$\vartheta \in \overline{C(\vartheta_2, r)} = \{\vartheta \in X : \|R(\vartheta, \vartheta_2)\| \leq r\}, \text{ as } \vartheta\Re\vartheta_2.$$

Consider

$$\begin{aligned}
R(g(\vartheta, \wp), \vartheta_2) &= R(g(\vartheta, \wp), g(\vartheta_2, \wp_2)) \\
&\preceq s[R(g(\vartheta, \wp), g(\vartheta, \wp_1)) + R(g(\vartheta, \wp_1), g(\vartheta, \wp_2)) + R(g(\vartheta, \wp_2), g(\vartheta_2, \wp_2))] \\
\|R(g(\vartheta, \wp), \vartheta_2)\| &\leq sM|\wp_1 - \wp| + sM|\wp_2 - \wp_1| + \|\mathcal{T}(R(\vartheta_2, \vartheta))\| \\
&\leq sM\epsilon + sM\epsilon + \|\mathcal{T}\|\, l = 2sM\epsilon + \|\mathcal{T}\|\, l < l.
\end{aligned}$$

Thus for each $\wp \in (\wp_2 - \epsilon, \wp_2 + \epsilon)$, $g(\cdot, \wp) : \overline{C(\vartheta, r)} \to \overline{C(\vartheta, r)}$ has a fixed point in $\overline{\mathcal{V}}$ as an implementation of Corollary 19. So $\wp \in C$, for every $\wp \in (\wp_2 - \epsilon, \wp_2 + \epsilon)$ and hence C is open in $[0,1]$. Using connectedness, $C = [0,1]$. So $g(\cdot, 1)$ has a fixed point in $\mathcal{V}$.

6.1. Application of homotopy to human aging process

This segment use homotopy to explain the procedure of aging of human body. We assume aging process by taking appropriate values for the time parameters t of the homotopy $\eta(t, \alpha)$. Here t and α control the procedure of aging. Human body will be considered as one year old, if we have a homotopy $\eta(t, \alpha) : \hbar(\alpha) \to g(\alpha)$ for $t \in [0,1]$ along $\eta(0, \alpha) = \hbar(\alpha)$ and $\eta(1, \alpha) = g(\alpha)$. Human body will be considered as n years old, if we have $\eta(t, \alpha) : \hbar(\alpha) \to w(\alpha)$ for $t \in [1, n]$, $n > l$ with $\eta(l, \alpha) = \hbar(\alpha)$ and $\eta(n, \alpha) = w(\alpha)$. Now the true age of human body is the least upper bound $\eta(n, \alpha) = y(\alpha)$, where $t \in [l, n]$. Topologically the toddler is the same as a fully grown person, because the toddler constantly becomes an adult. We establish an algebraic method to relate homotopy with the procedure of aging of human body. The cylinder $X = S \times I$ is to be considered topologically equivalent to a compact connected human body, when S is taken as circle and $I = [0, \beta]$. The infant is topologically taken as $X = S \times I$. The family of continuous functions $\eta(t, \alpha)$ on the interval $I = [0, \beta]$ is called homotopy. Consider X as human body, then homotopy is an increasing sequence of the function $\eta(t, \alpha)$. We use homotopy to relate topologically a toddler to topologically an adult. Consider X as a human body, assume $\alpha \in X$ define growth of the body and $t \in I$ define age of the body. Because we are not sure about the life duration of human body, consider $t = \infty$ as the final age of the body, where $t \in [\theta, \infty]$ represents the age interval from $t = \theta$ to $t = \infty$. The aging procedure is the sequence of the functions $\eta(t, \alpha)$ so that, $\forall\ t \in [0, \infty]$ $\eta(0, \alpha) = \hbar(\alpha)$ and $\eta(\infty, \alpha) = y(\alpha)$.

Theorem 21. *Consider $X = S \times I$ as a cylinder. Take a homotopy $\eta(t, \alpha)$, if we have a fixed point for $\eta(0, \alpha)$, then $\eta(\infty, \alpha)$ also has fixed point.*

Proof. Because of compactness and connectedness, human body is topologically equivalent to a cylinder $X = S \times I$. Since constant changing in the shape of cylinder has many invariant points, which leads to proof.

7. The existence of a solution to Urysohn Integral Equation (UIE)

In this section, we will get a unique solution for an UIE as an application of Theorem 13:

$$\ell(\wp) = c(\wp) + \int_{IR} K_1(\wp, s, \varpi(s))ds. \tag{12}$$

Above mentioned equation is a summarization of Volterra Integral Equation (VIE) and Fredholm Integral Equation (FIE). This integral equation is dependent on the range of integration (IR). UIE will converted into VIE by fixing a in IR $= (a, x)$ and UIE will become FIE by fixing a, b in IR $= (a, b)$. Many authors have found a unique solution to UIE (see [41–43]). Here we get a unique solution to UIE by using a fixed point way, which further helps to find a convergence point to many mathematical structures.

Take IR as the set with finite measure along $\mathcal{J}^2_{\text{IR}} = \left\{\varpi \mid \int_{\text{IR}} |\varpi(s)|^2 ds < \infty\right\}$. Take the norm $\|.\| : \mathcal{J}^2_{\text{IR}} \to [0, \infty)$ such that

$$\|\varpi\|_2 = \sqrt{\int_{\text{IR}} |\varpi(s)|^2 ds}, \text{ for all } \varpi, r \in \mathcal{J}^2_{\text{IR}}.$$

Equivalently we define the norm in the following manner:

$$\|\varpi\|_{2,y} = \sqrt{\sup\{e^{-y \int_{\text{IR}} \alpha(s)ds} \int_{\text{IR}} |\varpi(s)|^2 ds\}}, \text{ for all } \varpi \in \mathcal{J}^2_{\text{IR}}, y > 1.$$

Then $\mathcal{E} = (\mathcal{J}^2_{\text{IR}}, \|.\|_{2,y})$ represents a Banach space. Consider a cone $\mathcal{D} = \{\varpi \in \mathcal{J}^2_{\text{IR}} : \varpi(s) > 0 \text{ for almost every s}\}$. The rectangular cone b-metric R_y is taken as $R_y(\varpi, r) = \varpi \|\varpi - r\|^2_{2,y}\ \forall\ \varpi,\ r \in \mathcal{D}$. Take $\preceq$ as a partial order on $\mathcal{D}$, so that

$$a \preceq \upsilon \Leftrightarrow a(s)\upsilon(s) \geq \upsilon(s), \forall a, \upsilon \in \mathcal{D}.$$

Subsequently $(\mathcal{E}, \preceq, R_y)$ represents a complete $RCBM$ space. Assume

(D1) The kernel $K_1 : \text{IR} \times \text{IR} \times \mathbb{R} \to \mathbb{R}$ holding Carathéodory axioms along with

$$|K_1(c, s, \varpi(s))| \leq w(\wp, s) + e(\wp, s)\,|\varpi(s)|\,;\ w, e \in \mathcal{J}^2(\text{IR} \times \text{IR}), e(\wp, s) > 0.$$

D2) Take $c : \text{IR} \to [1, \infty)$ as a continuous and bounded function over IR.

(D3) $\exists$ a scalar $C > 0$ so that

$$\sup_{\wp \in \mathrm{IR}} \int_{IR} |K_1(\wp, s)| \, ds \leq C.$$

(D4) For every $\varpi_0 \in \mathcal{J}^2_{\mathrm{IR}}$, $\exists\ \varpi_1 = R(\varpi_0)$ so that $\varpi_1 \preceq \varpi_0$ or $\varpi_0 \preceq \varpi_1$.

(D4') $\varpi_{n-1} \preceq \varpi_n$ and $\varpi_n \to p$ holds for any sequence $\{\varpi_n\}$, implies $\varpi_n \preceq p$, for each natural number n.

(D5) We get a non-negative integrable and measurable operator $q : \mathrm{IR} \times \mathrm{IR} \to \mathbb{R}$ over IR, so that

$$\alpha(\wp) := \int_{\mathrm{IR}} q^2(\wp, s) \, ds \leq \frac{1}{y}, \text{ where } y \geq 1$$

and

$$|K_1(\wp, s, \varpi(s)) - K_1(\wp, s, r(s))| \leq q(\wp, s)|\varpi(s) - r(s)|$$

for all $\wp, s \in \mathrm{IR}$ and $\varpi, r \in \mathcal{E}$ with $\varpi \preceq r$.

Theorem 22. *Assume c and K_1 satisfying all axioms* (D1)-(D5)*, then we get a unique solution for* UIE*.*

Proof.
Take a function $R : \mathcal{E} \to \mathcal{E}$, along with the mentioned symbols, so that

$$(R\varpi)(\wp) = c(\wp) + \int_{\mathrm{IR}} K_1(\wp, s, \varpi(s)) ds.$$

R is taken as $\preceq$-preserving:
Consider $\varpi, r \in \mathcal{E}$ such that $\varpi \preceq r$, then $\varpi(s)r(s) \geq r(s)$. Now, for almost every $\wp \in \mathrm{IR}$,

$$(R\varpi)(\wp) = c(\wp) + \int_{\mathrm{IR}} K_1(\wp, s, \varpi(s)) ds \geq 1,$$

implies that $(R\varpi)(\wp)(Rr)(\wp) \geq (Rr)(\wp)$. So, $(R\varpi) \perp (Rr)$.
Self-operator:
Using (D1) and (D3) we get $R : \mathcal{D} \to \mathcal{D}$ as continuous and compact function (see [41, Lemma 3]).

Using (D4), assures the existence of $\varpi_1 = R(\varpi_0)$ so that $\varpi_1 \preceq \varpi_0$ or $\varpi_0 \preceq \varpi_1$, for every $\varpi_0 \in \mathcal{D}$ and R is $\preceq$-preserving, so we get $\varpi_n = R^n(\varpi_0)$ with $\varpi_n \preceq \varpi_{n+1}$ or $\varpi_{n+1} \preceq \varpi_n\ \forall\ n \geq 0$.

Using (D5) and Holder inequality will lead us to the contractive condition of Theorem 13.

$$\varpi\, |(R\varpi)(\wp) - (Rr)(\wp)|^2 \;=\; \varpi \left| \int_{IR} K_1(\wp, s, \varpi(s)) \, ds - \int_{IR} K_1(\wp, s, r(s)) \, ds \right|^2$$

$$
\begin{aligned}
&\preceq \varpi \left(\int\limits_{IR} \left| K_1(\wp, s, \varpi(s)) - K_1(\wp, s, r(s)) \right| \, ds \right)^2 \\
&\preceq \varpi \left(\int\limits_{IR} q(\wp, s) |\varpi(s) - r(s)| \, ds \right)^2 \\
&\preceq \varpi \int\limits_{IR} q^2(\wp, s) \, ds \cdot \int\limits_{IR} |\varpi(s) - r(s)|^2 \, ds \\
&= \varpi\alpha(\wp) \int\limits_{IR} |\varpi(s) - r(s)|^2 \, ds.
\end{aligned}
$$

By integrating with respect to $\wp$, we get

$$
\begin{aligned}
\varpi \int\limits_{IR} \left| (R\varpi)(\wp) - (Rr)(\wp) \right|^2 \, d\wp &\preceq \varpi \int\limits_{IR} \left(\alpha(\wp) \int\limits_{IR} |\varpi(s) - r(s)|^2 \, ds \right) d\wp \\
&= \varpi \int\limits_{IR} \left(\alpha(\wp) e^{y \int_{\mathrm{IR}} \alpha(s) ds} \cdot e^{-y \int_{\mathrm{IR}} \alpha(s) ds} \int\limits_{IR} |\varpi(s) - r(s)|^2 \, ds \right) d\wp \\
&\preceq \varpi \left\| \varpi - r \right\|_{2,y}^2 \int\limits_{IR} \alpha(\wp) e^{y \int_{\mathrm{IR}} \alpha(s) ds} \, d\wp \\
&\preceq \varpi \frac{1}{y} \left\| \varpi - r \right\|_{2,y}^2 e^{y \int_{\mathrm{IR}} \alpha(s) ds}.
\end{aligned}
$$

Hence, we get

$$
\varpi e^{-y \int_{\mathrm{IR}} \alpha(s) ds} \int\limits_{IR} \left| (R\varpi)(\wp) - (Rr)(\wp) \right|^2 \, d\wp \preceq \varpi \frac{1}{y} \left\| \varpi - r \right\|_{2,y}^2 .
$$

Which further implies that

$$
\varpi \left\| R\varpi - Rr \right\|_{2,y}^2 \preceq \varpi \frac{1}{y} \left\| \varpi - r \right\|_{2,y}^2 .
$$

So,

$$
c_y(R\varpi, Rr) \preceq \frac{1}{y} c_y(\varpi, r) \text{ implies}
$$

$$
y c_y(R\varpi, Rr) \preceq c_y(\varpi, r)
$$

Define $\mathcal{J} : \mathcal{E}^6 \to \mathcal{E}$ by

$$
\mathcal{J}(p_1, p_2, p_3, p_4, p_5, p_6) = kp_1 - p_2; \;\; k > 1,
$$

we have

$$\mathcal{J}\left(c(R\varpi, Rr), c(\varpi, r), c(\varpi, R\varpi), c(r, Rr), c(\varpi, Rr), c(r, R\varpi)\right) \preceq \mathbf{0}_{\mathcal{E}}.$$

By using 13, we get a unique fixed point for R, which implies UIE (12) has a unique solution.

8. Conclusion

Fixed point results show that rectangular cone b-metric space can be applied to obtain the general existence results for implicit contractions subject to implicit ordered relation. These results generalize many theorems in [1, 4, 34]. The obtained results can be applied to obtain more general homotopy results and existence results for integral equations. This idea can be further applied to obtain fixed point results in cone A-metric space. We refer a cmparison between this paper and [44] for further studies.

We suggest the readers and interested researchers to compare the results presented in this paper with the results appearing in [44] for further study.

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9. Data Availability

No data were used to support this study.

Competing interests

The authors declare that they have no competing interests.

10. Author's Contributions

All authors contributed equally to this work.

References

[1] V. Popa. Fixed point theorems for implicit contractive mappings. *Studii şi Cercetări Ştiinţifice. Seria Matematică. Universitatea Bacău*, 7:127–134, 1997.

[2] R. George, H. A. Nabwey, R. Rajagopalan, S. Radenović, and K. P. Reshma. Rectangular cone b-metric spaces over banach algebra and contraction principle. *Fixed Point Theory and Applications*, page 14, 2017.

[3] Z. Ercan. On the end of the cone metric spaces. *Topology and its Applications*, 166:10–14, 2014.

[4] A. C. M. Ran and M. C. B. Reurings. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proceedings of the American Mathematical Society*, 132:1435–1443, 2004.

[5] S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3:133–181, 1922.

[6] J. J. Nieto and R. Rodríguez-López. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. *Acta Mathematica Sinica*, 23:2205–2212, 2007.

[7] D. O'Regan and A. Petruşel. Fixed point theorems for generalized contractions in ordered metric spaces. *Journal of Mathematical Analysis and Applications*, 341:1241–1252, 2008.

[8] I. Beg and A. R. Butt. Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 71:3699–3704, 2009.

[9] I. Beg and A. R. Butt. Fixed points for weakly compatible mappings satisfying an implicit relation in partially ordered metric spaces. *Carpathian Journal of Mathematics*, 25:1–12, 2009.

[10] V. Berinde. Stability of picard iteration for contractive mappings satisfying an implicit relation. *Carpathian Journal of Mathematics*, 27:13–23, 2011.

[11] V. Berinde and F. Vetro. Common fixed points of mappings satisfying implicit contractive conditions. *Fixed Point Theory and Applications*, page 105, 2012.

[12] S. Sedghi, I. Altun, and N. Shobe. A fixed point theorem for multi maps satisfying an implicit relation on metric spaces. *Applicable Analysis and Discrete Mathematics*, 2:189–196, 2008.

[13] I. Altun and H. Simsek. Some fixed point theorems on ordered metric spaces and application. *Fixed Point Theory and Applications*, page 621469, 2010.

[14] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan. Generalized contractions in partially ordered metric spaces. *Applicable Analysis*, 87:1–8, 2008.

[15] J. Jachymski. The contraction principle for mappings on a metric space with a graph. *Proceedings of the American Mathematical Society*, 136:1359–1373, 2008.

[16] S. B. Nadler. Multivalued contraction mappings. *Pacific Journal of Mathematics*, 30:475–488, 1969.

[17] T. Rasham, P. Agarwal, L. S. Abbasi, and S. Jain. A study of some new multivalued fixed point results in a modular like metric space with graph. *The Journal of Analysis*, 30:833–844, 2022.

[18] H. A. Hammad, P. Agarwal, and Juan L. G. Guirao. Applications to boundary value problems and homotopy theory via tripled fixed point techniques in partially metric spaces. *Mathematics*, 9:2012, 2021.

[19] B. Samet, C. Vetro, and P. Vetro. Fixed point theorems for (α, ψ)-contractive type mappings. *Nonlinear Analysis: Theory, Methods & Applications*, 75:2154–2165, 2012.

[20] E. Karapinar, A. Fulga, and R. P. Agarwal. A survey: F-contractions with related

fixed point results. *Journal of Fixed Point Theory and Applications*, 22:69, 2020.

[21] L. G. Huang and X. Zhang. Cone metric spaces and fixed point theorems of contractive mappings. *Journal of Mathematical Analysis and Applications*, 332(2):1468–1476, 2007.

[22] S. Rezapour and R. Hamlbarani. Some notes on the paper "cone metric spaces and fixed point theorems of contractive mappings". *Journal of Mathematical Analysis and Applications*, 345:719–724, 2008.

[23] S. Czerwik. Nonlinear set-valued contraction mappings in b-metric spaces. *Atti del Seminario Matematico e Fisico dell'Università di Modena*, 46:263–276, 1998.

[24] N. Hussian and M. H. Shah. Kkm mappings in cone b-metric spaces. *Computers & Mathematics with Applications*, 62:1677–1684, 2011.

[25] H. Huang and S. Xu. Fixed point theorems of contractive mappings in cone b-metric spaces and applications. *Fixed Point Theory and Applications*, page 112, 2013.

[26] A. Arif, M. Nazam, A. Hussain, and M. Abbas. The ordered implicit relations and related fixed point problems in the cone b-metric spaces. *AIMS Mathematics*, 7(4):5199–5219, 2022.

[27] S. Aleksić, Z. Kadelburg, Z. D. Mitrović, and S. Radenović. A new survey: cone metric spaces. *Journal of the International Mathematical Virtual Institute*, 9:93–121, 2019.

[28] A. Bera, P. Mondal, H. Garai, and L. K. Dey. On maia type fixed point results via implicit relation. *AIMS Mathematics*, 8(9):22067–22080, 2023.

[29] M. Boriceanu, M. Bota, and A. Petruşel. Multivalued fractals in b-metric spaces. *Central European Journal of Mathematics*, 8(2):367–377, 2010.

[30] M. Bota, A. Molnár, and V. Csaba. On ekeland's variational principle in b-metric spaces. *Fixed Point Theory*, 12:21–28, 2011.

[31] K. Javed, F. Uddin, H. Aydi, A. Mukheimer, and M. Arshad. Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application. *Journal of Mathematics*, page 6663707, 2021.

[32] K. Javed, H. Aydi, F. Uddin, and M. Arshad. On orthogonal partial b-metric spaces with an application. *Journal of Mathematics*, page 6692063, 2021.

[33] E. Karapinar, S. Czerwik, and H. Aydi. (α, ψ)-meir-keeler contraction mappings in generalized b-metric spaces. *Journal of Function Spaces*, page 3264620, 2018.

[34] A. Azam, M. Arshad, and I. Beg. Banach contraction principle on cone rectangular metric spaces. *Applicable Analysis and Discrete Mathematics*, 3:236–241, 2009.

[35] Sonam, V. Rathore, A. Pal, R. Bhardwaj, and S. Narayan. Fixed-point results for mappings satisfying implicit relation in orthogonal fuzzy metric spaces. *Advances in Fuzzy Systems*, page 5037401, 2023.

[36] R. George, S. Radenović, and K. P. Reshma. Rectangular b-metric space and contraction principles. *Journal of Nonlinear Science and Applications*, 8:1005–1013, 2015.

[37] V. Popa. Some fixed point theorems for compatible mappings satisfying an implicit relation. *Demonstratio Mathematica*, 32:157–163, 1999.

[38] V. Popa. A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation. *Demonstratio Mathematica*, 33:159–164, 2000.

[39] I. Altun, F. Sola, and H. Simsek. Generalized contractions on partial metric spaces. *Topology and its Applications*, 157:2778–2785, 2010.

[40] I. Altun and D. Turkoglu. Some fixed point theorems for weakly compatible mappings satisfying an implicit relation. *Taiwanese Journal of Mathematics*, 13:1291–1304, 2009.

[41] M. Joshi. Existence theorems for urysohn's integral equation. *Proceedings of the American Mathematical Society*, 49(2):387–392, 1975.

[42] K. Maleknejad, H. Derili, and S. Sohrabi. Numerical solution of urysohn integral equations using the iterated collocation method. *International Journal of Computer Mathematics*, 85(1):143–154, 2008.

[43] R. Singh, G. Nelakanti, and J. Kumar. Approximate solution of urysohn integral equations using the adomian decomposition method. *The Scientific World Journal*, page 150483, 2014.

[44] M. Nazam, H. Aydi, and A. Hussain. Existence theorems for (ϕ, ψ)-orthogonal interpolative contractions and an application to fractional differential equations. *Optimization*, 71, 2022.