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A New Subclass of Bi-Univalent Functions Involving Bell and Meixner-Pollaczek Polynomials

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Abstract. In this work, we present a novel subclass of bi-univalent functions defined by Meixner-Pollaczek and Bell polynomials. Deriving coefficient estimates is the primary focus, especially for the second and third Taylor-Maclaurin coefficients, a_2 and a_3. Fekete-Szegö functional inequalities related to these subclasses are also examined. By extending and generalizing current subclasses, the proposed class offers fresh perspectives on the geometric and analytic characteristics of bi-univalent functions. Our findings demonstrate the theoretical originality and possible uses of orthogonal-polynomial-based function classes.

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Key Words and Phrases: Bell polynomials, Fekete-Szegö problem functions, bi-univalent functions, analytic functions, Meixner-Pollaczek polynomials

1. Preliminaries

When considering a particular weight function across a specified interval, orthogonal polynomials are a particular kind of polynomial that meets a specific orthogonality criterion. Numerous branches of mathematics, including approximation theory, numerical

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analysis, and mathematical physics, have devoted a significant amount of time and energy to the study of these polynomials. The fact that they constitute a basis for the space of square-integrable functions with respect to the weight function is one of the most important characteristics of these functions. Consequently, this makes it possible to express and approximate functions in an effective manner by utilizing polynomial expansions. Several well-known families of orthogonal polynomials are available, such as Legendre polynomials, Chebyshev polynomials, Meixner-Pollaczek polynomials, and Jacobi polynomials. Each of these families has its unique weight function and orthogonality features, specially designed to cater to particular applications (see [1, 2] for more information). Building upon these classical families, several subclasses of bi-univalent functions have been constructed using orthogonal polynomials such as Chebyshev, Gegenbauer, and Horadam polynomials. In the present work, we extend these developments by introducing a new subclass defined through Bell and Meixner–Pollaczek polynomials. This construction not only generalizes the previous subclasses, but also establishes new analytical connections among these polynomial families within the framework of geometric function theory.

As a result of their orthogonality property in connection to a certain weight function on the real line, the mathematicians Wolfgang Meixner and Erwin Pollaczek got a lot of attention. The study of stochastic processes, such as random walks and queuing systems, frequently benefits from the application of Meixner-Pollaczek polynomials. In the context of differential equations or difference equations with a discrete spectrum, they are frequently utilized as solutions. As a result of their links to special functions, such as hypergeometric functions and q-series, these polynomials have been the subject of a significant amount of research. They also possess intriguing combinatorial properties. When it comes to examination of the spectral properties of differential operators and analysis of probabilistic models, Meixner-Pollaczek polynomials are crucial instruments due to their versatility and analytical properties, as stated in studies ([3], [4]).

$\mathcal{A}$ represents the class of all analytic functions f defined on the disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. These functions are normalized by the constraints $f(0) = 0$ and $f'(0) = 1$.

As a result, every f that is a member of the mathematical category A possesses a Taylor-Maclaurin series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1}$$

In addition, denote by $\mathcal{S}$ the collection of all functions f that belong to $\mathcal{A}$ and are univalent in $\mathbb{U}$.

In the discipline of geometric function theory, the robust tools that differential subordination of analytic functions provides have the potential to make substantial contributions to the field's overall advancement. Miller and Mocanu [5] were the ones who initially presented the differential subordination problem, and more references can be found in [6]. The book written by Miller and Mocanu cite5aa provides a detailed documentation of the developments that have taken place in this particular field, including the publishing dates.

Each and every function f that belongs to $\mathcal{S}$ is known to have an inverse f^{-1} that is defined by the following equation:

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{2}$$

Assuming that both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$, a function is considered bi-univalent in $\mathbb{U}$.

Given that equation (1) defines the class of bi-univalent functions in $\mathbb{U}$, let Σ be assigned the role of representing this class. The class Σ contains a number of different implementations of functions

$$\frac{z}{1-z}, \qquad \log\frac{1}{1-z}, \qquad \log\sqrt{\frac{1+z}{1-z}}.$$

It is important to note that the well-known Koebe function is not included in the set Σ. In addition, there are other examples of functions in $\mathbb{U}$ that are well-known, specifically the following:

$$\frac{2z - z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ.

An exact upper limit for functional $\eta a_2^2 - a_3$, where η is a real number $(0 \leq \eta \leq 1)$, applied to a univalent function f, was established in 1933 by Fekete and Szegö [7]. Find the best bounds for this functional across all compact families of functions f belonging to $\mathcal{A}$, regardless of the complex value of η.

2. Both Bell polynomials and Meixner-Pollaczek polynomials are represented here

In 2018 Castellares et al. presented Bell polynomials [8], which is an appropriate polynomial for count data that exhibit over-dispersion. The Bell polynomials are an advance in comparison to the Bell numbers, as stated in the references [9, 10]. The expression for the probability density function of a discrete random variable X, which is based on the Bell distribution, is as follows:

$$\lambda(X = n) = \frac{\daleth^n e^{e^{(-\daleth^2)+1}} \Upsilon_n}{n!}; \qquad n = 1, 2, 3, \cdots . \tag{3}$$

where $\Upsilon_n = \frac{1}{e}\sum\limits_{k=0}^{\infty}\frac{k^n}{k!}$ are the Bell numbers, $n \geq 1$, and $\daleth > 0$.

Example of Bell numbers are $\Upsilon_2 = 2, \Upsilon_3 = 5, \Upsilon_4 = 15$ and $\Upsilon_5 = 52$.

Next, we will show a novel power series with coefficients that accurately reflect the probability connected to the Bell polynomials,

$$\Upsilon(\daleth, z) = z + \sum_{n=2}^{\infty} \frac{\daleth^{n-1}\Upsilon_n}{(n-1)!\, e^{\daleth^2-1}} z^n, \qquad (z \in \mathbb{U}), \tag{4}$$

where $\daleth > 0$.

Following that, we look at the linear operator $\Phi_{\daleth} : \mathcal{A} \to \mathcal{A}$: The Hadamard product, often known as convolution, is defined.

$$\begin{aligned} \Phi_{\daleth} f(z) &= \Upsilon(\daleth, z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\daleth^{n-1} e^{1-\daleth^2}\Upsilon_n}{(n-1)!} a_n z^n, (z \in \mathbb{U}), \\ &= z + \frac{2\daleth}{e^{\daleth^2-1}} a_2 z^2 + \frac{5\daleth^2}{2e^{\daleth^2-1}} a_3 z^3 + \frac{15\daleth^3}{3!e^{\daleth^2-1}} a_4 z^4 + \cdots . \end{aligned} \tag{5}$$

The Meixner–Pollaczek polynomials $\lambda_n^{(\eth)}(x;\ell)$ (see [11]) of a real variable x as coefficients of

$$\varrho^{\eth}(\widetilde{q}, \ell; z) = \frac{1}{(1 - ze^{i\ell})^{\eth - i\widetilde{q}} (1 - ze^{i\ell})^{\eth + i\widetilde{q}}} = \sum_{n=0}^{\infty} \lambda_n^{(\eth)}(\widetilde{q};\ell) z^n, \tag{6}$$

where

$$\lambda_n^{(\eth)}(\widetilde{q};\ell) = \frac{(2\eth)_n}{n!} e^{in\ell} \left(\frac{e^{2i\ell}}{e^{2i\ell}-1} \right)^n {}_2F_1 \left(\left. \begin{matrix} -n, \eth + i\widetilde{q} \\ 2\eth \end{matrix} \right| 1 - \frac{1}{e^{2i\ell}} \right), \tag{7}$$

are orthogonal with respect to the continuous weight;

$$\omega(x;\ell) = \left|\Gamma(\eth + i\widetilde{q})\right|^2 e^{(2\ell - \pi)\widetilde{q}}, \tag{8}$$

For $n \in \mathbb{N}$, $\eth > 0$, and $0 < \ell < \pi$ in the interval $(-\infty, \infty)$, observe that the complex Gamma function in Equation (7) has the form [12],

$$\left|\Gamma(\eth + i\widetilde{q})\right|^2 = \Gamma(\eth + i\widetilde{q})\Gamma(\eth - i\widetilde{q}).$$

Special cases:

1) $\lim\limits_{\ell \to \frac{\pi}{2}} \lambda_n^{(\frac{\alpha+1}{2})}(\frac{-\widetilde{q}}{2\ell};\ell)$ is called Laguerre polynomial $L_n^{\alpha}(x)$.

2) $\lim\limits_{\eth \to \infty} n!\eth^{\frac{-n}{2}} \lambda_n^{(\eth)}(\frac{-\widetilde{q}\sqrt{\eth} - \eth\cos\ell}{\sin\ell};\ell)$ is called Hermite polynomial $H_n(x)$.

By means of a three-term recurrence relation, the Meixner-Pollaczek polynomials can be represented.

$$\lambda_n^{(\eth)}(\widetilde{q};\ell) = \left(\widetilde{q} + \alpha_n^{(\eth,\ell)}\right) \lambda_{n-1}^{(\eth)}(\widetilde{q};\ell) - C_n^{(\eth,\ell)} \lambda_{n-2}^{(\eth)}(\widetilde{q};\ell), \tag{9}$$

where

$$\alpha_n^{(\eth,\ell)} := \frac{\eth + n - 1}{\tan \ell}; \quad \text{and} \quad C_n^{(\eth,\ell)} := \frac{(n-1)(2\eth + n - 2)}{4\sin^2 \ell}, \tag{10}$$

with $\lambda_{-1}^{(\eth)}(\widetilde{q}) = 0$, $\lambda_0^{(\eth)}(\widetilde{q}) = 1$ and $\alpha_n^{(\eth,\frac{\pi}{2})} = \lim_{\ell \to \frac{\pi}{2}} \alpha_n^{(\eth,\ell)} = 0$.

The initial polynomials can be constructed $\lambda_n^{(\eth)}(x;\delta)$ using Equation (13) as described below (for further reference, see [13]).

$$\begin{aligned} &\lambda_0^{(\eth)}(\widetilde{q};\delta) = 1 \\ &\lambda_1^{(\eth)}(\widetilde{q};\delta) = \widetilde{q} + \delta\eth \\ &\lambda_2^{(\eth)}(\widetilde{q};\delta) = \widetilde{q}^2 + \left(\delta\eth + \eth + 1\right)\widetilde{q} - 2\delta^2\eth + \delta\eth^2 + \delta\eth - 2\eth \end{aligned} \tag{11}$$

Subclasses of bi-univalent functions that are connected with orthogonal polynomials have recently drawn the attention of a group of scholars who have begun their investigation. Estimates for the coefficients that initially are associated with these functions have been determined. Nevertheless, the difficulty in identifying accurate boundaries for coefficients $|a_n|$, $(n = 3, 4, 5, \cdots)$ has not yet been resolved, as has been mentioned in a number of sources [14–46].

Many researchers have used a variety of probability distributions, including the Pascal, Poisson, and Borel distributions, to examine certain subclasses of analytic functions (for an example, see [30, 47, 48]) and other applications can be found in [49–55]).

The primary purpose of this research is to analyze the characteristics of bi-univalent functions in a new class from a mathematical perspective. The following definitions serve as the starting point for the investigation.

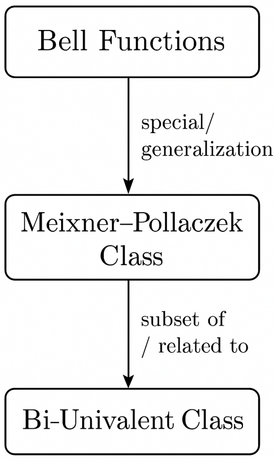

Figure 1: Hierarchical relationship among Bell functions, Meixner–Pollaczek class, and the bi-univalent class.

3. Definition and Examples

In this section, a novel subclass of bi-univalent functions within the unit disk will be defined and investigated. This will be accomplished by using the subordination principle. The Bell polynomials and subordination through Meixner-Pollaczek polynomials will be used to create this new class.

Definition 1. *The function $f \in \Sigma$, indicated by (1), belongs to the class $\mathfrak{G}_{\Sigma}\left(⅂, \xi, m, \psi, \varrho^{\partial}(\widetilde{q}, \ell; z)\right)$, if the conditions in subsequent subordinations are fulfilled. That is*

$$(1 + me^{i\psi}) \left\{ (1 - \xi) \frac{\Phi \neg f(z)}{z} + \xi (\Phi \neg f(z))' \right\} - me^{i\psi} = \varrho^{\partial}(\widetilde{q}, \ell; w) \tag{12}$$

and

$$(1 + me^{i\psi}) \left\{ (1 - \xi) \frac{\Phi \neg g(w)}{w} + \xi (\Phi \neg g(w))' \right\} - me^{i\psi} = \varrho^{\partial}(\widetilde{q}, \ell; v), \tag{13}$$

when x falls within the interval $[-1, 1], the function g(\mathrm{w}), defined by (2), is provided.$, $m \geq 0, -\pi < \psi \leq \pi$, and $0 < \ell < \pi$. The Meixner-Pollaczek polynomials $\varrho^{\partial}(\widetilde{q}, \ell; z)$ are provided by (6).

Example 1. *Consider ξ to be a positive integer. The function $f \in \Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_{\Sigma}\left(⅂, 0, m, \psi, \varrho^{\partial}(\widetilde{q}, \ell; z)\right)$ if the requirements listed below are met:*

$$(1 + me^{i\psi}) \left\{ \frac{\Phi \neg f(z)}{z} \right\} - me^{i\psi} \prec \varrho^{\partial}(\widetilde{q}, \ell; z) \tag{14}$$

and

$$(1+me^{i\psi})\left\{\frac{\Phi\neg g(w)}{w}\right\}-me^{i\psi}\prec\varrho^{\partial}(\widetilde{q},\ell;w), \tag{15}$$

when x falls within the interval $[-1,1], the function g(w), defined by (2), is provided.,$ $m\geq 0,-\pi<\psi\leq\pi$, and $0<\ell<\pi$. The Meixner-Pollaczek polynomials $\varrho^{\partial}(\widetilde{q},\ell;z)$ are provided by (6).

Example 2. *Consider ξ to be a positive integer. The function $f\in\Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_{\Sigma}\left(\daleth,1,m,\psi,\varrho^{\partial}(\widetilde{q},\ell;z)\right)$ if the requirements listed below are met:*

$$(1+me^{i\psi})\left\{(\Phi\neg f(z))'\right\}-me^{i\psi}\prec\varrho^{\partial}(\widetilde{q},\ell;z) \tag{16}$$

and

$$(1+me^{i\psi})\left\{(\Phi\neg g(w))'\right\}-me^{i\psi}\prec\varrho^{\partial}(\widetilde{q},\ell;w), \tag{17}$$

when x falls within the interval $[-1,1], the function g(w), defined by (2), is provided.,$ $m\geq 0,-\pi<\psi\leq\pi$, and $0<\ell<\pi$. The Meixner-Pollaczek polynomials $\varrho^{\partial}(\widetilde{q},\ell;z)$ are provided by (6).

Example 3. *Consider m to be a positive integer. The function $f\in\Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_{\Sigma}\left(\daleth,\xi,0,\psi,\varrho^{\partial}(\widetilde{q},\ell;z)\right)$ if the requirements listed below are met and go back to [56]:*

$$(1-\xi)\frac{\Phi\neg f(z)}{z}+\xi(\Phi\neg f(z))'\prec\varrho^{\partial}(\widetilde{q},\ell;z) \tag{18}$$

and

$$(1-\xi)\frac{\Phi\neg g(w)}{w}+\xi(\Phi\neg g(w))'\prec\varrho^{\partial}(\widetilde{q},\ell;v), \tag{19}$$

when x falls within the interval $[-1,1], the function g(w), defined by (2), is provided.,$ $m\geq 0,-\pi<\psi\leq\pi$, and $0<\ell<\pi$. The Meixner-Pollaczek polynomials $\varrho^{\partial}(\widetilde{q},\ell;z)$ are provided by (6).

Example 4. *Consider m,ξ to be a positive integer. The function $f\in\Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_{\Sigma}\left(\daleth,0,0,\psi,\varrho^{\partial}(\widetilde{q},\ell;z)\right)$ if the requirements listed below are met and go back to [56]:*

$$\frac{\Phi\neg f(z)}{z}\prec\varrho^{\partial}(\widetilde{q},\ell;z) \tag{20}$$

and

$$\frac{\Phi\neg g(w)}{w}\prec\varrho^{\partial}(\widetilde{q},\ell;w), \tag{21}$$

when x falls within the interval $[-1,1], the function g(\mathrm{w}), defined by (2), is provided.$, $m \geq 0, -\pi < \psi \leq \pi$, and $0 < \ell < \pi$. The Meixner-Pollaczek polynomials $\varrho^{\eth}(\widetilde{q}, \ell; z)$ are provided by (6).

Example 5. *Consider m, ξ to be a positive integer. The function $f \in \Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_{\Sigma}\left(\daleth, 1, 0, \psi, \varrho^{\eth}(\widetilde{q}, \ell; z)\right)$ if the requirements listed below are met and go back to [56]:*

$$(\Phi_{\neg} f(z))' \prec \varrho^{\eth}(\widetilde{q}, \ell; z) \tag{22}$$

and

$$(\Phi_{\neg} g(w))' \prec \varrho^{\eth}(\widetilde{q}, \ell; w), \tag{23}$$

when x falls within the interval $[-1,1], the function g(\mathrm{w}), defined by (2), is provided.$, $m \geq 0, -\pi < \psi \leq \pi$, and $0 < \ell < \pi$. The Meixner-Pollaczek polynomials $\varrho^{\eth}(\widetilde{q}, \ell; z)$ are provided by (6).

4. Bounds of the class $\mathfrak{G}_{\Sigma}(\daleth, \xi, m, \psi)$ for equations

To begin, let us present the estimates of the coefficients for the class $\mathfrak{G}_{\Sigma}\left(\daleth, \xi, m, \psi, \varrho^{\eth}(\widetilde{q}, \ell; z)\right)$ using the definition given in Definition 12.

Theorem 1. *Function $f \in \Sigma$, indicated by (1), belongs to the class $\mathfrak{G}_{\Sigma}\left(\daleth, \xi, m, \psi, \varrho^{\eth}(\widetilde{q}, \ell; z)\right)$, if the conditions in the subsequent subordinations are fulfilled. That is*

$$|a_2| \leq \frac{\frac{e^{\daleth^2-1}}{\daleth}\left|\widetilde{q}+\delta\eth\right|\sqrt{2\left(\widetilde{q}+\delta\eth\right)}}{\sqrt{\left|\begin{aligned}&(5(1+me^{i\psi})(1+2\xi)e^{\daleth^2-1} - 8(1+me^{i\psi})^2(1+\xi)^2)\widetilde{q}^2\\ &\quad + (10(1+me^{i\psi})^2\delta\eth(1+2\xi)e^{\daleth^2-1} - 8(1+me^{i\psi})^2(\delta\eth+\eth+1)(1+\xi)^2)\widetilde{q}\\ &\quad + 8(1+me^{i\psi})^2(1+\xi)^2\left(2\delta^2\eth - \delta\eth^2 - \delta\eth + 2\eth\right)\end{aligned}\right|}}$$

and

$$|a_3| \leq \frac{\left(e^{\daleth^2-1}\right)^2(\widetilde{q}+\delta\eth)^2}{4(1+me^{i\psi})^2(1+\xi)^2\daleth^2} + \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\eth|}{5(1+me^{i\psi})(1+2\xi)\daleth^2}.$$

Proof. Consider $f \in \mathfrak{G}_{\Sigma}\left(\daleth, \xi, m, \psi, \varrho^{\eth}(\widetilde{q}, \ell; z)\right)$. Definition 12 states that there are analytic functions w and v where $w(0) = v(0) = 0$ and $|w(z)| < 1$. If $|v(w)| < 1$ for any $z, w \in \mathbb{U}$, it can be expressed as follows:

$$(1+me^{i\psi})\left\{(1-\xi)\frac{\Phi_{\neg} f(z)}{z} + \xi(\Phi_{\neg} f(z))'\right\} - me^{i\psi} = \varrho^{\eth}(\widetilde{q}, \ell; w(z)) \tag{24}$$

and

$$(1+me^{i\psi})\left\{(1-\xi)\frac{\Phi\daleth g(w)}{w}+\xi(\Phi\daleth g(w))'\right\}-me^{i\psi}=\varrho^{\backepsilon}(\widetilde{q},\ell;v(w)), \tag{25}$$

Using equalities (24 and (25), we can conclude that

$$\begin{aligned}&(1+me^{i\psi})\left\{(1-\xi)\frac{\Phi\daleth f(z)}{z}+\xi(\Phi\daleth f(z))'\right\}-me^{i\psi}\\&=1+\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)c_1z+\left[\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)c_2+\lambda_2^{(\backepsilon)}(\widetilde{q};\delta)c_1^2\right]z^2+\cdots\end{aligned} \tag{26}$$

and

$$\begin{aligned}&(1+me^{i\psi})\left\{(1-\xi)\frac{\Phi\daleth g(w)}{w}+\xi(\Phi\daleth g(w))'\right\}-me^{i\psi}\\&=1+\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)d_1w+\left[\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)d_2+\lambda_2^{(\backepsilon)}(\widetilde{q};\delta)d_1^2\right])w^2+\cdots.\end{aligned} \tag{27}$$

It's commonly understood that if

$$|w(z)|=\left|c_1z+c_2z^2+c_3z^3+\cdots\right|<1,\quad (z\in\mathbb{U})$$

and

$$|v(w)|=\left|d_1w+d_2w^2+d_3w^3+\cdots\right|<1,\quad (w\in\mathbb{U}),$$

then

$$|c_j|\le 1 \text{ and } |d_j|\le 1 \text{ for all } j\in\mathbb{N}. \tag{28}$$

Comparing the coefficients in (26) with (27) yields

$$\frac{2(1+me^{i\psi})(1+\xi)\daleth}{e^{\daleth^2-1}}a_2=\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)c_1, \tag{29}$$

$$\frac{5(1+me^{i\psi})(1+2\xi)\daleth^2}{2e^{\daleth^2-1}}a_3=\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)c_2+\lambda_2^{(\backepsilon)}(\widetilde{q};\delta)c_1^2, \tag{30}$$

and

$$-\frac{2(1+me^{i\psi})(1+\xi)\daleth}{e^{\daleth^2-1}}a_2=\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)d_1, \tag{31}$$

$$\frac{5(1+me^{i\psi})(1+2\xi)\daleth^2}{2e^{\daleth^2-1}}\left(2a_2^2-a_3\right)=\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)d_2+\lambda_2^{(\backepsilon)}(\widetilde{q};\delta)d_1^2, \tag{32}$$

According to (29) and (31),

$$c_1=-d_1 \tag{33}$$

and

$$\begin{aligned}2\left(\frac{2(1+me^{i\psi})(1+\xi)\daleth}{e^{\daleth^2-1}}\right)^2a_2^2&=\left[\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)\right]^2\left(c_1^2+d_1^2\right)\\c_1^2+d_1^2&=\frac{8(1+me^{i\psi})^2(1+\xi)^2\daleth^2}{\left(e^{\daleth^2-1}\right)^2\left[\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)\right]^2}a_2^2\end{aligned} \tag{34}$$

If we add (30) and (32), we get

$$\frac{5(1+me^{i\psi})(1+2\xi)\daleth^2}{e^{\daleth^2-1}} a_2^2 = \lambda_1^{(\Game)}(\widetilde{q};\delta)\,(c_2+d_2) + \lambda_2^{(\Game)}(\widetilde{q};\delta)\left(c_1^2+d_1^2\right). \tag{35}$$

Substituting the expression for $c_1^2+d_1^2$ from (34) into the right-hand side of (35) and rearranging the resulting identity, we obtain an equality relating a_2^2 to c_2+d_2. Moreover, by (28) — which follows from Carathéodory's lemma for analytic Schwarz functions — we have $|c_j| \le 1$ and $|d_j| \le 1$ for all j. Hence $|c_1^2+d_1^2| \le 2$ and $|c_2+d_2| \le 2$; applying these bounds and carrying out elementary algebraic simplifications yields Equation (36).

$$\left(5(1+2\xi)(1+me^{i\psi}) - \frac{8(1+me^{i\psi})^2(1+\xi)^2\lambda_2^{(\Game)}(\widetilde{q};\delta)}{\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\Game)}(\widetilde{q};\delta)\right]^2}\right)\frac{\daleth^2}{e^{\daleth^2-1}}\,a_2^2 = \lambda_1^{(\Game)}(\widetilde{q};\delta)\,(c_2+d_2)$$

$$a_2^2 = \frac{\left(e^{\daleth^2-1}\right)^2\left[\lambda_1^{(\Game)}(\widetilde{q};\delta)\right]^3}{\daleth^2\left(5(1+me^{i\psi})(1+2\xi)\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\Game)}(\widetilde{q};\delta)\right]^2 - 8(1+me^{i\psi})^2(1+\xi)^2\lambda_2^{(\Game)}(\widetilde{q};\delta)\right)}(c_2+d_2) \tag{36}$$

Furthermore, computations utilising (11), (28), and (36) reveal that

$$|a_2| \le \frac{\frac{e^{\daleth^2-1}}{\daleth}\left|\widetilde{q}+\delta\Game\right|\sqrt{2\left(\widetilde{q}+\delta\Game\right)}}{\sqrt{\left|\begin{array}{l}(5(1+me^{i\psi})(1+2\xi)e^{\daleth^2-1} - 8(1+me^{i\psi})^2(1+\xi)^2)\widetilde{q}^2 \\ \quad + (10(1+me^{i\psi})^2\delta\Game(1+2\xi)e^{\daleth^2-1} - 8(1+me^{i\psi})^2(\delta\Game+\Game+1)(1+\xi)^2)\widetilde{q} \\ \quad + 8(1+me^{i\psi})^2(1+\xi)^2\left(2\delta^2\Game - \delta\Game^2 - \delta\Game + 2\Game\right)\end{array}\right|}}$$

Furthermore, when we subtract (32) from (30), we get

$$\frac{5(1+me^{i\psi})(1+2\xi)\daleth^2}{e^{\daleth^2-1}}\left(a_3-a_2^2\right) = \lambda_1^{(\Game)}(\widetilde{q};\delta)\,(c_2-d_2) + \lambda_2^{(\Game)}(\widetilde{q};\delta)\left(c_1^2-d_1^2\right). \tag{37}$$

Then, in view of (28) and (34), Eq. (37) becomes

$$a_3 = \frac{\left(e^{\daleth^2-1}\right)^2\left[\lambda_1^{(\Game)}(\widetilde{q};\delta)\right]^2}{8(1+me^{i\psi})^2(1+\xi)^2\daleth^2}\left(c_1^2+d_1^2\right) + \frac{e^{\daleth^2-1}\lambda_1^{(\Game)}(\widetilde{q};\delta)}{5(1+me^{i\psi})(1+2\xi)\daleth^2}(c_2-d_2)$$

Thus, using (11) and (28), we deduce that

$$|a_3| \le \frac{\left(e^{\daleth^2-1}\right)^2\left(\widetilde{q}+\delta\Game\right)^2}{4(1+me^{i\psi})^2(1+\xi)^2\daleth^2} + \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\Game|}{5(1+me^{i\psi})(1+2\xi)\daleth^2}.$$

This completes the proof of Theorem.

Theorem 2. *Function $f \in \Sigma$, indicated by (1), belongs to the class $\mathfrak{G}_\Sigma\left(\daleth, \xi, m, \psi, \varrho^{\eth}(\widetilde{q}, \ell; z)\right)$, if the conditions in the subsequent subordinations are fulfilled. That is*

$$\left|a_3 - \Phi a_2^2\right| \leq \begin{cases} \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\eth|}{5(1+me^{i\psi})(1+2\xi)\daleth^2}, & |1-\Phi| \leq \mathcal{J}(\xi, \widetilde{q}, \eth, \delta, \daleth), \\ \\ 2\left|\widetilde{q}+\delta\eth\right|\left|\mathcal{K}(\Phi)\right|, & |1-\Phi| \geq \mathcal{J}(\xi, \widetilde{q}, \eth, \delta, \daleth), \end{cases}$$

where

$$\mathcal{J}(\xi, \widetilde{q}, \eth, \delta, \daleth) = \left|1 - \frac{8(1+me^{i\psi})^2(1+\xi)^2\left(\widetilde{q}^2 + \left(\delta\eth + \eth + 1\right)\widetilde{q} - 2\delta^2\eth + \delta\eth^2 + \delta\eth - 2\eth\right)}{5(1+me^{i\psi})(1+2\xi)e^{\daleth^2-1}\left(\widetilde{q}+\delta\eth\right)^2}\right|,$$

and

$$\mathcal{K}(\Phi) = \frac{e^{\daleth^2-1}\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2(1-\Phi)}{\daleth^2\left(5(1+me^{i\psi})(1+2\xi)\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2 - 8(1+me^{i\psi})^2(1+\xi)^2\lambda_2^{(\eth)}(\widetilde{q};\delta)\right)}.$$

Proof. From (36) and (37)

$$\begin{aligned} a_3 - \Phi a_2^2 &= \frac{e^{\daleth^2-1}\lambda_1^{(\eth)}(\widetilde{q};\delta)}{5(1+me^{i\psi})(1+2\xi)\daleth^2}(c_2-d_2) \\ &\quad + \frac{(1-\Phi)\left(e^{\daleth^2-1}\right)^2\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^3(c_2+d_2)}{\daleth^2\left(5(1+me^{i\psi})(1+2\xi)\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2 - 8(1+me^{i\psi})^2(1+\xi)^2\lambda_2^{(\eth)}(\widetilde{q};\delta)\right)} \\ &= \lambda_1^{(\eth)}(\widetilde{q};\delta)\left(\left[\mathcal{K}(\Phi) + \frac{e^{\daleth^2-1}}{5(1+me^{i\psi})(1+2\xi)\daleth^2}\right]c_2 + \left[\mathcal{K}(\Phi) - \frac{e^{\daleth^2-1}}{5(1+me^{i\psi})(1+2\xi)\daleth^2}\right]d_2\right), \end{aligned}$$

where

$$\mathcal{K}(\Phi) = \frac{\left(e^{\daleth^2-1}\right)^2\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2(1-\Phi)}{\daleth^2\left(5(1+me^{i\psi})(1+2\xi)\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2 - 8(1+me^{i\psi})^2(1+\xi)^2\lambda_2^{(\eth)}(\widetilde{q};\delta)\right)},$$

Then, in view of (11), we conclude that

$$\left|a_3 - \Phi a_2^2\right| \leq \begin{cases} \frac{2e^{\daleth^2-1}\left|\lambda_1^{(\eth)}(\widetilde{q};\delta)\right|}{5(1+me^{i\psi})(1+2\xi)\daleth^2}, & |\mathcal{K}(\Phi)| \leq \frac{e^{\daleth^2-1}}{5(1+me^{i\psi})(1+2\xi)\daleth^2}, \\ \\ 2\left|\lambda_1^{(\eth)}(\widetilde{q};\delta)\right|\left|\mathcal{K}(\Phi)\right|, & |\mathcal{K}(\Phi)| \geq \frac{e^{\daleth^2-1}}{5(1+me^{i\psi})(1+2\xi)\daleth^2}. \end{cases}$$

Which completes the proof of Theorem 2.

5. Corollaries and Consequences

The results that are obtained from the application of Theorems 1 and 2 are in close agreement with the examples 1, 2, 3, 4, and 5.

Corollary 1. *Consider ξ to be a positive integer. The function $f \in \Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_\Sigma\left(\daleth, 0, m, \psi, \varrho^{\eth}(\widetilde{q}, \ell; z)\right)$ if the requirements listed below:*

$$|a_2| \le \frac{\frac{e^{\daleth^2-1}}{\daleth}\left|\widetilde{q}+\delta\eth\right|\sqrt{2\left(\widetilde{q}+\delta\eth\right)}}{\sqrt{\left|\begin{array}{l}(5(1+me^{i\psi})e^{\daleth^2-1}-8(1+me^{i\psi})^2)\widetilde{q}^2\\ \quad+\left(10(1+me^{i\psi})\delta\eth e^{\daleth^2-1}-8(1+me^{i\psi})^2(\delta\eth+\eth+1)\right)\widetilde{q}\\ \quad+8(1+me^{i\psi})^2\left(2\delta^2\eth-\delta\eth^2-\delta\eth+2\eth\right)\end{array}\right|}}$$

$$|a_3| \le \frac{\left(e^{\daleth^2-1}\right)^2\left(\widetilde{q}+\delta\eth\right)^2}{4(1+me^{i\psi})^2\daleth^2}+\frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\eth|}{5(1+me^{i\psi})\daleth^2}.$$

and

$$\left|a_3-\Phi a_2^2\right| \le \begin{cases} \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\eth|}{5(1+me^{i\psi})\daleth^2}, & |1-\Phi| \le \mathcal{J}(0,\widetilde{q},\eth,\delta,\daleth),\\ \\ 2\left|\widetilde{q}+\delta\eth\right|\left|\mathcal{K}(\Phi)\right|, & |1-\Phi| \ge \mathcal{J}(0,\widetilde{q},\eth,\delta,\daleth),\end{cases}$$

where

$$\mathcal{J}(0,\widetilde{q},\eth,\delta,\daleth)=\left|1-\frac{8(1+me^{i\psi})^2\left(\widetilde{q}^2+\left(\delta\eth+\eth+1\right)\widetilde{q}-2\delta^2\eth+\delta\eth^2+\delta\eth-2\eth\right)}{5(1+me^{i\psi})e^{\daleth^2-1}\left(\widetilde{q}+\delta\eth\right)^2}\right|,$$

and

$$\mathcal{K}(\Phi)=\frac{e^{\daleth^2-1}\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2(1-\Phi)}{\daleth^2\left(5(1+me^{i\psi})\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2-8(1+me^{i\psi})^2\lambda_2^{(\eth)}(\widetilde{q};\delta)\right)}.$$

Corollary 2. *Consider ξ to be a positive integer. The function $f \in \Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_\Sigma\left(\daleth, 1, m, \psi, \varrho^{\eth}(\widetilde{q}, \ell; z)\right)$ if the requirements listed below:*

$$|a_2| \le \frac{\frac{e^{\daleth^2-1}}{\daleth}\left|\widetilde{q}+\delta\eth\right|\sqrt{2\left(\widetilde{q}+\delta\eth\right)}}{\sqrt{\left|\begin{array}{l}(15(1+me^{i\psi})e^{\daleth^2-1}-32(1+me^{i\psi})^2)\widetilde{q}^2\\ \quad+\left(30(1+me^{i\psi})\delta\eth e^{\daleth^2-1}-32(1+me^{i\psi})^2(\delta\eth+\eth+1)\right)\widetilde{q}\\ \quad+32(1+me^{i\psi})^2\left(2\delta^2\eth-\delta\eth^2-\delta\eth+2\eth\right)\end{array}\right|}}$$

$$|a_3| \leq \frac{\left(e^{\daleth^2-1}\right)^2(\widetilde{q}+\delta\backepsilon)^2}{16(1+me^{i\psi})^2\daleth^2} + \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\backepsilon|}{15(1+me^{i\psi})\daleth^2}.$$

and

$$\left|a_3-\Phi a_2^2\right| \leq \begin{cases} \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\backepsilon|}{15(1+me^{i\psi})\daleth^2}, & |1-\Phi| \leq \mathcal{J}(1,\widetilde{q},\backepsilon,\delta,\daleth), \\ \\ 2\left|\widetilde{q}+\delta\backepsilon\right|\left|\mathcal{K}(\Phi)\right|, & |1-\Phi| \geq \mathcal{J}(1,\widetilde{q},\backepsilon,\delta,\daleth), \end{cases}$$

where

$$\mathcal{J}(1,\widetilde{q},\backepsilon,\delta,\daleth) = \left|1-\frac{32(1+me^{i\psi})^2\big(\widetilde{q}^2+\big(\delta\backepsilon+\backepsilon+1\big)\widetilde{q}-2\delta^2\backepsilon+\delta\backepsilon^2+\delta\backepsilon-2\backepsilon\big)}{15(1+me^{i\psi})e^{\daleth^2-1}\left(\widetilde{q}+\delta\backepsilon\right)^2}\right|,$$

and

$$\mathcal{K}(\Phi) = \frac{e^{\daleth^2-1}\left[\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)\right]^2(1-\Phi)}{\daleth^2\left(15(1+me^{i\psi})\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\backepsilon)}(\widetilde{q};\delta)\right]^2-32(1+me^{i\psi})^2\lambda_2^{(\backepsilon)}(\widetilde{q};\delta)\right)}.$$

Corollary 3. *Consider m to be a positive integer. The function $f \in \Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_\Sigma\left(\daleth,\xi,0,\psi,\varrho^{\backepsilon}(\widetilde{q},\ell;z)\right)$ if the requirements listed below:*

$$|a_2| \leq \frac{\frac{e^{\daleth^2-1}}{\daleth}\left|\widetilde{q}+\delta\backepsilon\right|\sqrt{2\left(\widetilde{q}+\delta\backepsilon\right)}}{\sqrt{\left|\begin{array}{l}(5(1+2\xi)e^{\daleth^2-1}-8(1+\xi)^2)\widetilde{q}^2 \\ \quad +(10\delta\backepsilon(1+2\xi)e^{\daleth^2-1}-8(\delta\backepsilon+\backepsilon+1)(1+\xi)^2)\widetilde{q} \\ \quad +8(1+\xi)^2\big(2\delta^2\backepsilon-\delta\backepsilon^2-\delta\backepsilon+2\backepsilon\big)\end{array}\right|}}$$

and

$$|a_3| \leq \frac{\left(e^{\daleth^2-1}\right)^2(\widetilde{q}+\delta\backepsilon)^2}{4(1+\xi)^2\daleth^2} + \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\backepsilon|}{5(1+2\xi)\daleth^2}.$$

and

$$\left|a_3-\Phi a_2^2\right| \leq \begin{cases} \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\backepsilon|}{5(1+2\xi)\daleth^2}, & |1-\Phi| \leq \mathcal{J}(\xi,\widetilde{q},\backepsilon,\delta,\daleth), \\ \\ 2\left|\widetilde{q}+\delta\backepsilon\right|\left|\mathcal{K}(\Phi)\right|, & |1-\Phi| \geq \mathcal{J}(\xi,\widetilde{q},\backepsilon,\delta,\daleth), \end{cases}$$

where

$$\mathcal{J}(\xi,\widetilde{q},\backepsilon,\delta,\daleth) = \left|1-\frac{8(1+\xi)^2\big(\widetilde{q}^2+\big(\delta\backepsilon+\backepsilon+1\big)\widetilde{q}-2\delta^2\backepsilon+\delta\backepsilon^2+\delta\backepsilon-2\backepsilon\big)}{5(1+2\xi)e^{\daleth^2-1}\left(\widetilde{q}+\delta\backepsilon\right)^2}\right|,$$

and

$$\mathcal{K}(\Phi) = \frac{e^{\daleth^2-1}\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2(1-\Phi)}{\daleth^2\left(5(1+2\xi)\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2-8(1+\xi)^2\lambda_2^{(\eth)}(\widetilde{q};\delta)\right)}.$$

Corollary 4. *Consider m,ξ to be a positive integer. The function $f\in\Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_\Sigma\left(\daleth,0,0,\psi,\varrho^{\eth}(\widetilde{q},\ell;z)\right)$ if the requirements listed below:*

$$|a_2| \leq \frac{\frac{e^{\daleth^2-1}}{\daleth}\left|\widetilde{q}+\delta\eth\right|\sqrt{2\left(\widetilde{q}+\delta\eth\right)}}{\sqrt{\begin{array}{c}\left|\left(5e^{\daleth^2-1}-8\right)\widetilde{q}^2+\left(10\delta\eth e^{\daleth^2-1}-8(\delta\eth+\eth+1)\right)\widetilde{q}\right.\\ \left.+8\left(2\delta^2\eth-\delta\eth^2-\delta\eth+2\eth\right)\right|\end{array}}}.$$

$$|a_3| \leq \frac{\left(e^{\daleth^2-1}\right)^2\left(\widetilde{q}+\delta\eth\right)^2}{4\daleth^2}+\frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\eth|}{5\daleth^2}.$$

and

$$\left|a_3-\Phi a_2^2\right| \leq \begin{cases}\frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\eth|}{5\daleth^2}, & |1-\Phi|\leq\mathcal{J}(0,\widetilde{q},\eth,\delta,\daleth),\\ \\ 2\left|\widetilde{q}+\delta\eth\right|\left|\mathcal{K}(\Phi)\right|, & |1-\Phi|\geq\mathcal{J}(0,\widetilde{q},\eth,\delta,\daleth),\end{cases}$$

where

$$\mathcal{J}(0,\widetilde{q},\eth,\delta,\daleth) = \left|1-\frac{8\left(\widetilde{q}^2+\left(\delta\eth+\eth+1\right)\widetilde{q}-2\delta^2\eth+\delta\eth^2+\delta\eth-2\eth\right)}{5e^{\daleth^2-1}\left(\widetilde{q}+\delta\eth\right)^2}\right|,$$

and

$$\mathcal{K}(\Phi) = \frac{e^{\daleth^2-1}\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2(1-\Phi)}{\daleth^2\left(5\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2-8\lambda_2^{(\eth)}(\widetilde{q};\delta)\right)}.$$

Corollary 5. *Consider m,ξ to be a positive integer. The function $f\in\Sigma$, which is represented by the equation (1), is considered to be a member of the class $\mathfrak{G}_\Sigma\left(\daleth,1,0,\psi,\varrho^{\eth}(\widetilde{q},\ell;z)\right)$ if the requirements listed below:*

$$|a_2| \leq \frac{\frac{e^{\daleth^2-1}}{\daleth}\left|\widetilde{q}+\delta\eth\right|\sqrt{2\left(\widetilde{q}+\delta\eth\right)}}{\sqrt{\begin{array}{c}\left|\left(15e^{\daleth^2-1}-32\right)\widetilde{q}^2+\left(30\delta\eth e^{\daleth^2-1}-32(\delta\eth+\eth+1)\right)\widetilde{q}\right.\\ \left.+32\left(2\delta^2\eth-\delta\eth^2-\delta\eth+2\eth\right)\right|\end{array}}}.$$

$$|a_3| \leq \frac{\left(e^{\daleth^2-1}\right)^2\left(\widetilde{q}+\delta\eth\right)^2}{16\daleth^2}+\frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\eth|}{15\daleth^2}.$$

and

$$\left|a_3-\Phi a_2^2\right| \leq \begin{cases} \frac{2e^{\daleth^2-1}|\widetilde{q}+\delta\eth|}{15\daleth^2}, & |1-\Phi| \leq \mathcal{J}(1,\widetilde{q},\eth,\delta,\daleth), \\ \\ 2\left|\widetilde{q}+\delta\eth\right|\left|\mathcal{K}(\Phi)\right|, & |1-\Phi| \geq \mathcal{J}(1,\widetilde{q},\eth,\delta,\daleth), \end{cases}$$

where

$$\mathcal{J}(1,\widetilde{q},\eth,\delta,\daleth)=\left|1-\frac{32\big(\widetilde{q}^2+\big(\delta\eth+\eth+1\big)\widetilde{q}-2\delta^2\eth+\delta\eth^2+\delta\eth-2\eth\big)}{15e^{\daleth^2-1}\left(\widetilde{q}+\delta\eth\right)^2}\right|,$$

and

$$\mathcal{K}(\Phi)=\frac{e^{\daleth^2-1}\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2(1-\Phi)}{\daleth^2\left(15\left(e^{\daleth^2-1}\right)\left[\lambda_1^{(\eth)}(\widetilde{q};\delta)\right]^2-32\lambda_2^{(\eth)}(\widetilde{q};\delta)\right)}.$$

Concluding Remark: We have presented and analyzed the problems that arise with the coefficients of a new subclass of bi-univalent functions! This particular subclass is referred to as $\mathfrak{G}_\Sigma\left(\daleth,\xi,m,\psi,\varrho^{\eth}(\widetilde{q},\ell;z)\right)$ due to the presence of the Bell polynomials and the Meixner-Pollaczek polynomials. When it comes to functions that belong to this newly introduced subclass, we possess estimations for the Fekete-Szegö functional issues as well as the Taylor-Maclaurin coefficients, which are denoted as a_2 and a_3, respectively.

The purpose of this study is to investigate the connection that exists between Meixner-Pollaczek polynomials that belong to specific families and the Bell polynomials. The estimations on the bounds of $|a_n|$ for $n \geq 4; n \in \mathbb{N}$ for the classes that have been detailed throughout this article are an example of how this finding might motivate further research in other domains.

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