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The Imaginary Error Function and New Classes of Bi-Univalent Functions Subordinate to Jacobi Polynomials

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Abstract. A unique family of bi-univalent functions, commonly known as functions that are defined on the symmetric domain, is presented and investigated in this paper. We also presented and examined the subfamily of the functions. The imaginary error function establishes a connection between the relevant subfamily and the Jacobi polynomial. In addition to this, we obtained the initial coefficients of the Maclaurin series for functions that are members of this subfamily. Additionally, we proceed to do an analysis of the Fekete-Szegö inequality associated with these functions.

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Key Words and Phrases: Imaginary error function, jacobi polynomials, bi-univalent functions, fekete-szegö inequality, maclaurin series

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1. Introduction and Preliminaries

The solution of ordinary differential equations that are able to satisfy model constraints is frequently accomplished through the utilisation of orthogonal polynomials [1]. It is important to note that these polynomials are not only fundamental in the fields of physics and engineering, but they also hold great relevance in current mathematics, particularly in the field of approximation theory. A number of disciplines, such as quantum physics, probability theory, interpolation, differential equation theory, and mathematical statistics, are among the areas in which orthogonal polynomials find their application. When it comes to signal processing, image processing, and data analysis, they are utilised for modelling and analysing complicated systems and datasets (see [2–4] for more information).

Both $\widetilde{q}_\xi$ and $\widetilde{q}_\varepsilon$, which belong to orders ξ and ε, respectively, are orthogonal if

$$\langle \widetilde{q}_\xi,\ \widetilde{q}_\varepsilon \rangle = \int_{\sigma_1}^{\sigma_2} \widetilde{q}_\xi(y)\widetilde{q}_\varepsilon(y)r(y)dy = 0, \qquad \text{for} \qquad \xi \neq \varepsilon. \tag{1}$$

In the interval (σ_1, σ_2), as $r(y)$ is a non-negative function, and the integral of all finite-order polynomials $\widetilde{q}_\xi(y)$ is correctly defined (see [5]).

Over the years, numerous families of orthogonal polynomials have gained widespread recognition, including Laguerre, Legendre, Hermite, Chebyshev, and numerous others. This group of polynomials possesses a multitude of features and applications that are beneficial. The weight function and interval of each family are the distinguishing characteristics of that family.

Jacobi polynomials have a generating function that is specified so that

$$O_\alpha(t,z) = 2^{\Im+\mu}Q^{-1}\left(1-t+Q\right)^{-\Im}\left(1+t+Q\right)^{-\mu},$$

with $Q := Q(t,z) = (1-2zt+t^2)^{0.5} astheequation.$, $\Im$, and $\mu > -1$, with t falling between the range of $[-1,1]$, $\alpha, \alpha+\Im, \alpha+\mu$ are all non-negative integers, and z is a member of the open unit disc $\mathbb{J} = \{z \in \mathbb{C} : |z| < 1\}$, as stated in [6].

For a constant t, the function $O_\alpha(t,z)$ is analytic in $\mathbb{J}$, which enables it to be represented by a Taylor series expansion in the following manner:

$$O_\alpha(t,z) = \sum_{\alpha=0}^{\infty} P_\alpha^{(\Im,\mu)}(t)z^\alpha, \tag{2}$$

The Jacobi polynomial of degree α is denoted by the expression $P_\alpha^{(\Im,\mu)}(t)$.

In the second-order linear homogeneous differential equation, the Jacobi polynomial $P_\alpha^{(\Im,\mu)}(t)$ is a satisfactory solution,

$$(1-t^2)y'' + (\mu - \Im - (\Im+\mu+2)t)y' + \alpha(\alpha+\Im+\mu+1)y = 0.$$

An additional way to characterise Jacobi polynomials is by referring to the recursive relationship that follows:

$$P_\alpha^{(\Im,\mu)}(t) = (h_{\alpha-1}z - s_{\alpha-1})P_{\alpha-1}^{(\Im,\mu)}(t) - C_{\alpha-1}P_{\alpha-2}^{(\Im,\mu)}(t),\ \alpha \geqslant 2, \tag{3}$$

where
$h_\alpha = \frac{(2\alpha+\Im+\mu+1)(2\alpha+\Im+\mu+2)}{2(\alpha+1)(\alpha+\Im+\mu+1)}$, $s_\alpha = \frac{(2\alpha+\Im+\mu+1)\left(\mu^2-\Im^2\right)}{2(\alpha+1)(\alpha+\Im+\mu+1)(2\alpha+\Im+\mu)}$, and
$C_\alpha = \frac{(2\alpha+\Im+\mu+2)(\alpha+\Im)(\alpha+\mu)}{(\alpha+1)(\alpha+\Im+\mu+1)(2\alpha+\Im+\mu)}$, with the initial values

$$P_0^{(\Im,\mu)}(t) = 1,\ P_1^{(\Im,\mu)}(t) = (\Im+1) + \frac{1}{2}(\Im+\mu+2)(t-1) \tag{4}$$

and

$$P_2^{(\Im,\mu)}(t) = \frac{(\Im+1)(\Im+2)}{2} + \frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1) + \frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2.$$

In order to initiate the process, we will introduce particular instances of the polynomials $P_\alpha^{(\Im,\mu)}$. "The polynomials reduce to" the Legendre polynomials when the values of $\Im$ and μ are equal to zero. The Chebyshev polynomials of the first kind are obtained by setting $\Im = \mu = -0.5$, whereas the Chebyshev polynomials of the second kind are obtained by setting $\Im = \mu = 0.5$. Furthermore, when $\Im$ is equal to μ, the polynomials reduce to Gegenbauer polynomials, with $\Im$ being substituted by $(\Im - 0.5)$.

In the open unit disc $\mathbb{J}$, let $\mathcal{U}$ be the family of functions B that are both analytic and univalent as well as having the form

$$B(z) = z + A_2 z^2 + A_3 z^3 + \cdots, \tag{5}$$

In the field of geometric function theory, the concept of differential subordination, which was initially presented by Miller and Mocanu [7], is an essential framework. Their ground-breaking work established the foundation for further research and applications of differential subordination of analytic functions, which were made possible from their work. As an additional point of interest, their book [8] offers a detailed summary of the advancements and references that have been made in the subject up until the time that it was published.

If there exists a function $\mathcal{L}$ via $\mathcal{L}(0) = 0$ and $|\mathcal{L}(z)| < 1$, then B subordination to V (represented by $B \prec V$) for all $z \in \mathbb{J}$

$$B(z) = V(\mathcal{L}(z)).$$

In addition, according to [9], if the variable V is univalent in the field of $\mathbb{J}$, then

$$B(z) \prec V(z) \text{ if and only if } B(0) = V(0)$$

and

$$B(\mathbb{J}) \subset V(\mathbb{J}).$$

At any function $B \in \mathcal{U}$, there exists an inverse B^{-1}, which is defined by (see to [10] for further reference).

$$B^{-1}(B(z)) = z \qquad (z \in \mathbb{J})$$

and

$$\mathcal{L} = B(B^{-1}(\mathcal{L})) \qquad (|\mathcal{L}| < r_0(B);\ r_0(B) \geq \frac{1}{4}),$$

where

$$V(\mathcal{L}) := B^{-1}(\mathcal{L}) = \mathcal{L} - A_2\mathcal{L}^2 + (2A_2^2 - A_3)\mathcal{L}^3 - (A_4 + 5A_2^3 - 5A_3A_2)\mathcal{L}^4 + \cdots . \tag{6}$$

According to [11–15], a function $B \in \mathcal{U}$ is considered bi-univalent in $\mathbb{J}$ (the family of bi-univalent functions in $\mathbb{J}$ represents by Σ)) if both $B(z)$ and $B^{-1}(z)$ are univalent in $\mathbb{J}$.

In the field of geometric function theory, the study of bi-univalent functions is of utmost importance, particularly when it comes to the investigation of functions that are univalent in both a domain and its inverse. The link that they have with orthogonal polynomials, such as Jacob polynomials, is beneficial to the process of analysing structural properties and coefficient bounds. There are numerous applications that encompass a wide range of fields, such as low-light imaging for the purpose of enhancing contrast, picture edge detection for the purpose of precision, and stealth combat aircraft for the purpose of optimising radar signatures (see [16–18]).

For a variety of scientific disciplines, such as probability, statistics, partial differential equations, and other engineering applications, the error function is an extremely important component. Because of this, it has received a significant amount of attention in the field of mathematics. There have been a multitude of studies that have investigated inequalities and the properties of the error function that are associated with them; for example, see [19–22]. Furthermore, the error function and its approximations are utilised extensively in the process of forecasting events that have extremely high or extremely low probability or probabilities. Thus, polynomials bridge the gap between abstract complex analysis and computational modeling, allowing deeper exploration of geometric mappings and their analytic behavior [23–27]. Some applications in q-Calculus can be found in [28–38].

[39] is the source that defines the error function, which is represented by the symbol erf.

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha z^{2\alpha+1}}{(2\alpha+1)\alpha!}, \quad z \in \mathbb{C}. \tag{7}$$

Figure 1 offers a graphical representation of the error function erf over real numbers, whereas Figure 2 illustrates the function in the complex plane.

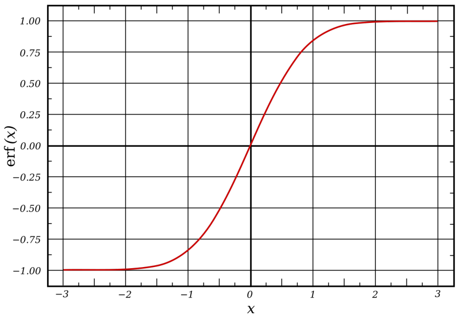

Figure 1: For more information on the error function over real numbers, please refer to [40].

It is possible to derive the series by expanding the integrand e^{-t^2} into its Maclaurin series and then integrating each term individually, as demonstrated in the previous illustration. According to the explanation provided by (see [41, 42]), the Maclaurin series of the imaginary error function $erfi$ is quite comparable

$$erfi(z) = \frac{2}{\sqrt{\pi}} \sum_{\alpha=0}^{\infty} \frac{z^{2\alpha+1}}{(2\alpha+1)\alpha!}, \quad z \in \mathbb{C}. \tag{8}$$

Figure 3 serves as a graphical representation of the imaginary error function $erfi(z)$ in the complex plane.

In their study, Ramachandran and colleagues [43] utilised the function (7) to evaluate the normalised analytical error function in relation to the form

$$\mathrm{Erf}(z) = \frac{\sqrt{\pi z}}{2} erf(\sqrt{z}) = z + \sum_{\alpha=2}^{\infty} \frac{(-1)^{\alpha-1} z^{\alpha}}{(2\alpha-1)(\alpha-1)!}. \tag{9}$$

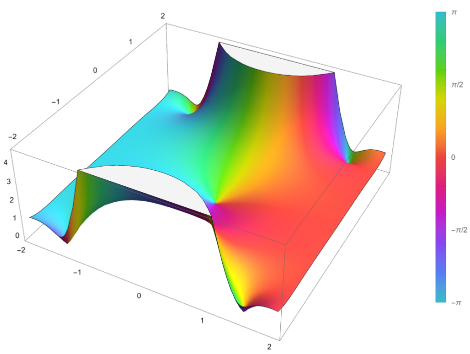

Figure 2: For more information on the error function in the complex plane, please refer to [40].

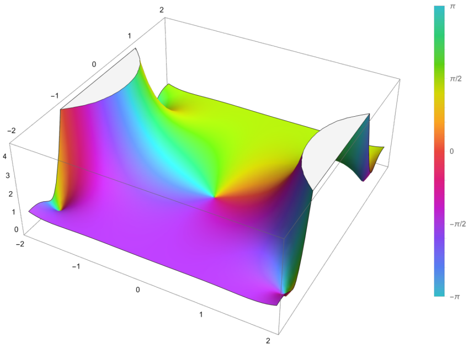

Figure 3: For more information on the imaginary error function in the complex plane; see [40]

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Convolution product is an operation that combines two power series [44], where the coefficients of the new series are the products of the coefficients from the original series. The combination of these two power series is known as the convolution product. A new series $h(z) = \sum_{\alpha=2}^{\infty}(a_\alpha b_\alpha)z^\alpha$ is produced by the product of two series $f(z) = \sum_{\alpha=2}^{\infty} a_\alpha z^\alpha$ and $g(z) = \sum_{\alpha=2}^{\infty} b_\alpha z^\alpha$. This new series is defined by making use of the convolution product, which results in the following family:

$$\mathrm{Erf} * \mathcal{U} = \left\{ w : w(z) = (\mathrm{Erf} * B)(z) = z + \sum_{\alpha=2}^{\infty} \frac{(-1)^{\alpha-1} c_\alpha}{(2\alpha-1)(\alpha-1)!} z^\alpha, \quad B \in \mathcal{U} \right\}. \tag{10}$$

The normalised analytic imaginary error function Erfi is defined by the following equation, deduced from (8):

$$\mathrm{Erfi}(z) = \frac{\sqrt{\pi z}}{2} erfi(\sqrt{z}) = z + \sum_{\alpha=2}^{\infty} \frac{z^\alpha}{(2\alpha-1)(\alpha-1)!},$$

and, in accordance with the convolution product, we define

$$\partial B(z) = (\mathrm{Erfi} * B)(z) = z + \sum_{\alpha=2}^{\infty} \frac{A_\alpha}{(2\alpha-1)(\alpha-1)!} z^\alpha, \ \ B \in \mathcal{U}.$$

Following the definition of the Jacobi polynomials and the normalised analytic imaginary error function, we shall proceed to present the subsequent subfamily of bi-univalent functions.

Definition 1. *If a function $B \in \Sigma$ given by (5) meets the two requirements below, it is considered to belong to the family $\mathfrak{J}_\Sigma(m, \psi, t, \Upsilon, \Omega)$.*

$$(1 + me^{i\psi}) \left\{ (1-\Omega) \frac{\partial B(z)}{z} + \Omega \left(\partial B(z)\right)' + \Upsilon z \left(\partial B(z)\right)'' \right\} - me^{i\psi} \prec O_\alpha(t, z) \tag{11}$$

and

$$(1+me^{i\psi})\left\{(1-\Omega)\frac{\partial V(\mathcal{L})}{\mathcal{L}}+\Omega\left(\partial V(\mathcal{L})\right)'+\Upsilon\mathcal{L}\left(\partial V(\mathcal{L})\right)''\right\}-me^{i\psi}\prec O_\alpha(t,\mathcal{L}), \tag{12}$$

where $z,\mathcal{L}\in\mathbb{J}$, $m\geq 0, -\pi<\psi\leq\pi, \Omega, \Upsilon\geq 0$, $\Im,\mu>-1$, $t\in(\frac{1}{2},1]$, *and the function* $V=B^{-1}$ *is given by (6).*

For the purpose of establishing that the family $\mathfrak{J}_\Sigma$ contains functions that are not trivial, we would like to direct the reader to [45], where a rigorous proof is presented. In particular, the reference [45] provides a demonstration of the building of an explicit example of a function that is a member of the $\mathfrak{J}_\Sigma$ family. This serves to guarantee that the family does not contain any empty members.

Subfamily 1. *When* $\Upsilon=0$, *we obtain* $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. *Here,* $(m,\psi,t,0,\Omega)$ *is the set of functions* $B\in\Sigma$ *that satisfy the following criteria and are provided by (5),*

$$(1+me^{i\psi})\left\{(1-\Omega)\frac{\partial B(z)}{z}+\Omega\left(\partial B(z)\right)'\right\}-me^{i\psi}\prec O_\alpha(t,z)$$

and

$$(1+me^{i\psi})\left\{(1-\Omega)\frac{\partial V(\mathcal{L})}{\mathcal{L}}+\Omega\left(\partial V(\mathcal{L})\right)'\right\}-me^{i\psi}\prec O_\alpha(t,\mathcal{L}),$$

where $z,\mathcal{L}\in\mathbb{J}$, $m\geq 0, -\pi<\psi\leq\pi, \Omega\geq 0$, $\Im,\mu>-1$, *and* $t\in(\frac{1}{2},1]$.

Subfamily 2. *When* $\Upsilon=0$ *and* $\Omega=1$, *we obtain* $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. *Here,* $(m,\psi,t,0,1)$ *is the set of functions* $B\in\Sigma$ *that satisfy the following criteria and are provided by (5),*

$$(1+me^{i\psi})\left\{\left(\partial B(z)\right)'\right\}-me^{i\psi}\prec O_\alpha(t,z)$$

and

$$(1+me^{i\psi})\left\{\left(\partial V(\mathcal{L})\right)'\right\}-me^{i\psi}\prec O_\alpha(t,\mathcal{L}),$$

where $z,\mathcal{L}\in\mathbb{J}$, $m\geq 0, -\pi<\psi\leq\pi, \Im,\mu>-1$, *and* $t\in(\frac{1}{2},1]$.

Subfamily 3. *When* $\Upsilon=0$ *and* $\Omega=0$, *we obtain* $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. *Here,* $(m,\psi,t,0,0)$ *is the set of functions* $B\in\Sigma$ *that satisfy the following criteria and are provided by (5),*

$$(1+me^{i\psi})\left\{\frac{\partial B(z)}{z}\right\}-me^{i\psi}\prec O_\alpha(t,z)$$

and

$$(1+me^{i\psi})\left\{\frac{\partial V(\mathcal{L})}{\mathcal{L}}\right\}-me^{i\psi}\prec O_\alpha(t,\mathcal{L}),$$

where $z,\mathcal{L}\in\mathbb{J}$, $m\geq 0, -\pi<\psi\leq\pi, \Im,\mu>-1$, *and* $t\in(\frac{1}{2},1]$.

Subfamily 4. *When $m = 0$, we obtain $\mathfrak{J}_\Sigma(m, \psi, t, \Upsilon, \Omega)$. Here, $(0, \psi, t, \Upsilon, \Omega)$ is the set of functions $B \in \Sigma$ that satisfy the following criteria and are provided by (5),*

$$(1-\Omega)\frac{\partial B(z)}{z} + \Omega\left(\partial B(z)\right)' + \Upsilon z\left(\partial B(z)\right)'' \prec O_\alpha(t, z)$$

and

$$(1-\Omega)\frac{\partial V(\mathcal{L})}{\mathcal{L}} + \Omega\left(\partial V(\mathcal{L})\right)' + \Upsilon \mathcal{L}\left(\partial V(\mathcal{L})\right)'' \prec O_\alpha(t, \mathcal{L}),$$

where $z, \mathcal{L} \in \mathbb{J}$, $\Upsilon \geq 0, -\pi < \psi \leq \pi, \Omega \geq 0$, $\Im, \mu > -1$, and $t \in (\frac{1}{2}, 1]$.

Subfamily 5. *When $m = \Upsilon = 0$, we obtain $\mathfrak{J}_\Sigma(m, \psi, t, \Upsilon, \Omega)$. Here, $(0, \psi, t, 0, \Omega)$ is the set of functions $B \in \Sigma$ that satisfy the following criteria and are provided by (5),*

$$(1-\Omega)\frac{\partial B(z)}{z} + \Omega\left(\partial B(z)\right)' \prec O_\alpha(t, z)$$

and

$$(1-\Omega)\frac{\partial V(\mathcal{L})}{\mathcal{L}} + \Omega\left(\partial V(\mathcal{L})\right)' \prec O_\alpha(t, \mathcal{L}),$$

where $z, \mathcal{L} \in \mathbb{J}$, $-\pi < \psi \leq \pi, \Omega \geq 0$, $\Im, \mu > -1$, and $t \in (\frac{1}{2}, 1]$.

Subfamily 6. *When $m = \Upsilon = 0$ and $\Omega = 1$, we obtain $\mathfrak{J}_\Sigma(m, \psi, t, \Upsilon, \Omega)$. Here, $(0, \psi, t, 0, 1)$ is the set of functions $B \in \Sigma$ that satisfy the following criteria and are provided by (5),*

$$\left(\partial B(z)\right)' \prec O_\alpha(t, z)$$

and

$$\left(\partial V(\mathcal{L})\right)' \prec O_\alpha(t, \mathcal{L}),$$

where $z, \mathcal{L} \in \mathbb{J}$, $-\pi < \psi \leq \pi, \Im, \mu > -1$, and $t \in (\frac{1}{2}, 1]$.

Subfamily 7. *When $m = \Upsilon = 0$ and $\Omega = 0$, we obtain $\mathfrak{J}_\Sigma(m, \psi, t, \Upsilon, \Omega)$. Here, $(0, \psi, t, 0, 0)$ is the set of functions $B \in \Sigma$ that satisfy the following criteria and are provided by (5),*

$$\frac{\partial B(z)}{z} \prec O_\alpha(t, z)$$

and

$$\frac{\partial V(\mathcal{L})}{\mathcal{L}} \prec O_\alpha(t, \mathcal{L}),$$

where $z, \mathcal{L} \in \mathbb{J}$, $-\pi < \psi \leq \pi, \Im, \mu > -1$, and $t \in (\frac{1}{2}, 1]$.

Recently, many researchers have examined bi-univalent functions associated with orthogonal polynomials, obtaining non-sharp estimates for the Maclaurin coefficients $|c_2|$ and $|c_3|$ (see [46–55]).

Additionally, in recent years, numerous studies have utilised a variety of special functions, including Borel, Poisson, Rabotnov, Pascal, Wright, and Bessel, in order to investigate essential aspects of geometric function theory. These aspects include the estimation of coefficients, the establishment of inclusion relations, and the determination of criteria for membership in particular families (refer to [56–66]).

Following is an outline of the content that is included in this paper. In Section 2, we present the bounds for the coefficients $|A_2|$ and $|A_3|$ in the Maclaurin expansions. Additionally, we provide an estimation of the Fekete–Szegő inequality for functions that belong to the family $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. Specifically, Section 3 draws attention to the significant connections that exist between specific instances of the key outcomes. As a last point of conclusion, Section 4 brings the study to a close with a few observations.

2. Main Results

The initial segment of this section 2 commences by establishing constraints for the coefficients $|A_2|$ and $|A_3|$ in the Maclaurin expansions of functions inside the family $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$.

Theorem 1. *If a function $B\in\Sigma$ given by (5) meets the two requirements below, it is considered to belong to the family $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$,*

$$|A_2| \le \frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2\left(\Im+1\right)+(\Im+\mu+2)(t-1)}}{\sqrt{|\Upsilon(t,\Im,\Upsilon,\mu)|}}$$

and

$$|A_3| \le \frac{9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}{(2\Upsilon+\Omega+1)^2\,(1+me^{i\psi})^2}+\frac{10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})},$$

where

$$\begin{aligned}\Upsilon(t,\Im,\Upsilon,\mu) &= \frac{1}{5}\left(6\Upsilon+2\Omega+1\right)\left(1+me^{i\psi}\right)\left[\begin{array}{c}(\Im+1)\\ +\frac{1}{2}(\Im+\mu+2)(t-1)\end{array}\right]^2\\ &\quad -\frac{2}{9}\left(2\Upsilon+\Omega+1\right)^2\left(1+me^{i\psi}\right)^2\left[\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}\left(\Im+2\right)\left(\Im+\mu+3\right)(t-1)\\ +\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right].\end{aligned}$$

Proof. Let $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. According to Definition 1, we can express

$$(1+me^{i\psi})\left\{(1-\Omega)\frac{\eth B(z)}{z}+\Omega\left(\eth B(z)\right)'+\Upsilon z\left(\eth B(z)\right)''\right\}-me^{i\psi}=O_\alpha(t,p(z)) \quad (13)$$

and

$$(1+me^{i\psi})\left\{(1-\Omega)\frac{\eth V(\mathcal{L})}{\mathcal{L}}+\Omega\left(\eth V(\mathcal{L})\right)'+\Upsilon\mathcal{L}\left(\eth V(\mathcal{L})\right)''\right\}-me^{i\psi}=O_\alpha(t,q(\mathcal{L})), \quad (14)$$

In this case, both p and q are analytical and represent the form

$$p(z)=j_1z+j_2z^2+j_3z^3+\cdots,\quad (z\in\mathbb{J})$$

and

$$q(\mathcal{L}) = d_1\mathcal{L} + d_2\mathcal{L}^2 + d_3\mathcal{L}^3 + \cdots, \quad (\mathcal{L} \in \mathbb{J}),$$

to the extent that $p(0) = q(0) = 0$ The expression $|p(z)| < 1$ and $|q(\mathcal{L})| < 1$ holds true for all $z, \mathcal{L} \in \mathbb{J}$. By utilising the equalities (13) and (14), we are able to acquire without exception.

$$\begin{aligned}&(1+me^{i\psi})\left\{(1-\Omega)\frac{\partial B(z)}{z} + \Omega\left(\partial B(z)\right)' + \Upsilon z\left(\partial B(z)\right)''\right\} - me^{i\psi}\\ &= 1 + P_1^{(\Im,\mu)}(t)j_1 z + \left[\ P_1^{(\Im,\mu)}(t)j_2 + P_2^{(\Im,\mu)}(t)j_1^2\ \right]z^2 + \cdots\end{aligned} \tag{15}$$

and

$$\begin{aligned}&(1+me^{i\psi})\left\{(1-\Omega)\frac{\partial V(\mathcal{L})}{\mathcal{L}} + \Omega\left(\partial V(\mathcal{L})\right)' + \Upsilon \mathcal{L}\left(\partial V(\mathcal{L})\right)''\right\} - me^{i\psi}\\ &= 1 + P_1^{(\Im,\mu)}(t)d_1 \mathcal{L} + \left[\ P_1^{(\Im,\mu)}(t)d_2 + P_2^{(\Im,\mu)}(t)d_1^2\ \right]\mathcal{L}^2 + \cdots.\end{aligned} \tag{16}$$

It is common knowledge that if

$$|p(z)| = \left|j_1 z + j_2 z^2 + j_3 z^3 + \cdots\right| < 1, \quad (z \in \mathbb{J})$$

and

$$|q(\mathcal{L})| = \left|d_1\mathcal{L} + d_2\mathcal{L}^2 + d_3\mathcal{L}^3 + \cdots\right| < 1, \quad \mathcal{L} \in \mathbb{J},$$

then

$$|j_i| \leq 1 \text{ and } |d_i| \leq 1 \text{ for all } i \in \mathbb{N}. \tag{17}$$

When we take the coefficients of both sides in (15) and (16) and put them into equation, we get the following:

$$\frac{1}{3}\left(2\Upsilon + \Omega + 1\right)(1+me^{i\psi})A_2 = P_1^{(\Im,\mu)}(t)j_1, \tag{18}$$

$$\frac{1}{10}\left(6\Upsilon + 2\Omega + 1\right)(1+me^{i\psi})A_3 = P_1^{(\Im,\mu)}(t)j_2 + P_2^{(\Im,\mu)}(t)j_1^2, \tag{19}$$

$$-\frac{1}{3}\left(2\Upsilon + \Omega + 1\right)(1+me^{i\psi})c_2 = P_1^{(\Im,\mu)}(t)d_1, \tag{20}$$

and

$$\frac{1}{10}\left(6\Upsilon + 2\Omega + 1\right)(1+me^{i\psi})\left[2A_2^2 - A_3\right] = P_1^{(\Im,\mu)}(t)d_2 + P_2^{(\Im,\mu)}(t)d_1^2. \tag{21}$$

Given the findings of (18) and (20), it deduces that

$$j_1 = -d_1 \tag{22}$$

and

$$\frac{2}{9}\left(2\Upsilon+\Omega+1\right)^{2}(1+me^{i\psi})^{2}A_2^2=\left[P_1^{(\Im,\mu)}(t)\right]^2\left(j_1^2+d_1^2\right). \tag{23}$$

The result that we get when we add (19) and (21) is

$$\frac{1}{5}\left(6\Upsilon+2\Omega+1\right)(1+me^{i\psi})A_2^2=P_1^{(\Im,\mu)}(t)\left(j_2+d_2\right)+P_2^{(\Im,\mu)}(t)\left(j_1^2+d_1^2\right). \tag{24}$$

By substituting the function $\left(j_1^2+d_1^2\right)$ from (23) into the right-hand side of (24), we are able to obtain the following result:

$$\left[\frac{1}{5}\left(6\Upsilon+2\Omega+1\right)(1+me^{i\psi})-\frac{2}{9}\left(2\Upsilon+\Omega+1\right)^{2}(1+me^{i\psi})^{2}\frac{P_2^{(\Im,\mu)}(t)}{\left[P_1^{(\Im,\mu)}(t)\right]^2}\right]A_2^2=P_1^{(\Im,\mu)}(t)\left(j_2+d_2\right). \tag{25}$$

When we use the formulas (4) and (17) in (25), we discover that

$$|A_2|\leq\frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2\left(\Im+1\right)+(\Im+\mu+2)(t-1)}}{\sqrt{|\Upsilon(t,\Im,\Upsilon,\mu)|}},$$

where

$$\begin{aligned}\Upsilon(t,\Im,\Upsilon,\mu) \;=\;& \frac{1}{5}\left(6\Upsilon+2\Omega+1\right)(1+me^{i\psi})\left[\begin{array}{c}(\Im+1)\\+\frac{1}{2}(\Im+\mu+2)(t-1)\end{array}\right]^2\\ &-\frac{2}{9}\left(2\Upsilon+\Omega+1\right)^{2}(1+me^{i\psi})^{2}\left[\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}\left(\Im+2\right)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right].\end{aligned}$$

Also, if we take (21) and subtract it from (19), we get the following:

$$\frac{1}{5}\left(6\Upsilon+2\Omega+1\right)(1+me^{i\psi})\left(A_3-A_2^2\right)=P_1^{(\Im,\mu)}(t)\left(j_2-d_2\right)+P_2^{(\Im,\mu)}(t)\left(j_1^2-d_1^2\right). \tag{26}$$

Then, from (22) and (23), equation (26) becomes

$$A_3=\frac{9\left[P_1^{(\Im,\mu)}(t)\right]^2}{2\left(2\Upsilon+\Omega+1\right)^{2}(1+me^{i\psi})^{2}}\left(j_1^2+d_1^2\right)+\frac{5P_1^{(\Im,\mu)}(t)}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})}\left(j_2-d_2\right).$$

Using the formula (4), we are able to express that

$$|A_3|\leq\frac{9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}{\left(2\Upsilon+\Omega+1\right)^{2}(1+me^{i\psi})^{2}}+\frac{10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})}.$$

According to the values of A_2 and A_3, we do an estimation of the functional $\left|A_3-\varphi A_2^2\right|$ for functions that belong to the family of bi-univalent functions $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$.

Theorem 2. *If a function $B \in \Sigma$ given by (5) meets the two requirements below, it is considered to belong to the family $\mathfrak{J}_\Sigma(m, \psi, t, \Upsilon, \Omega)$,*

$$\left|c_3 - \varphi c_2^2\right| \leq \begin{cases} \frac{10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right|}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})} & |1-\varphi| \leq \Pi_1, \\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\Upsilon,\mu)|} & |1-\varphi| \geq \Pi_1, \end{cases}$$

where

$$\Pi_1 = 1 - \frac{\frac{10}{9}\left(2\Upsilon+\Omega+1\right)^2\left(1+me^{i\psi}\right)^2\begin{pmatrix} \frac{(\Im+1)(\Im+2)}{2} + \frac{1}{2}\left(\Im+2\right)\left(\Im+\mu+3\right)(t-1) \\ +\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2 \end{pmatrix}}{\left(6\Upsilon+2\Omega+1\right)\left(1+me^{i\psi}\right)\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

Proof. From (25) and (26), we have

$$\begin{aligned} c_3 - \varphi c_2^2 &= \frac{5P_1^{(\Im,\mu)}(t)}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})}\left(j_2 - d_2\right) \\ &+ (1-\varphi)\frac{\left[P_1^{(\Im,\mu)}(t)\right]^3\left(j_2+d_2\right)}{\frac{1}{5}\left(6\Upsilon+2\Omega+1\right)\left(1+me^{i\psi}\right)\left[P_1^{(\Im,\mu)}(t)\right]^2 - \frac{2}{9}\left(2\Upsilon+\Omega+1\right)^2\left(1+me^{i\psi}\right)^2 P_2^{(\Im,\mu)}(t)} \\ &= P_1^{(\Im,\mu)}(t)\left[F(\varphi) + \frac{5}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})}\right]j_2 \\ &+ P_1^{(\Im,\mu)}(t)\left[F(\varphi) - \frac{5}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})}\right]d_2, \end{aligned}$$

where

$$F(\varphi) = \frac{\left[P_1^{(\Im,\mu)}(t)\right]^2(1-\varphi)}{\frac{1}{5}\left(6\Upsilon+2\Omega+1\right)\left(1+me^{i\psi}\right)\left[P_1^{(\Im,\mu)}(t)\right]^2 - \frac{2}{9}\left(2\Upsilon+\Omega+1\right)^2\left(1+me^{i\psi}\right)^2 P_2^{(\Im,\mu)}(t)}.$$

Then, from (4), we deduce that

$$\left|c_3 - \varphi c_2^2\right| \leq \begin{cases} \frac{10\left|P_1^{(\Im,\mu)}(t)\right|}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})} & |F(\varphi)| \leq \frac{5}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})}, \\ 2\left|P_1^{(\Im,\mu)}(t)\right|\left|F(\varphi)\right| & |F(\varphi)| \geq \frac{5}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})}. \end{cases}$$

$$\equiv \begin{cases} \frac{10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right|}{(6\Upsilon+2\Omega+1)(1+me^{i\psi})} & |1-\varphi| \leq \Pi_1, \\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\Upsilon,\mu)|} & |1-\varphi| \geq \Pi_1, \end{cases}$$

where

$$\Pi_1 = 1 - \frac{\frac{10}{9}\left(2\Upsilon+\Omega+1\right)^2\left(1+me^{i\psi}\right)^2\left(\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}\left(\Im+2\right)\left(\Im+\mu+3\right)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right)}{\left(6\Upsilon+2\Omega+1\right)\left(1+me^{i\psi}\right)\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

3. Particular Cases

By specialising the parameters m, Υ, and Ω in the aforementioned Theorems in section 2, the following corollaries can be produced.

Corollary 1. *When $\Upsilon = 0$, we obtain $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. Here, $(m,\psi,t,0,\Omega)$ is the set of functions $B\in\Sigma$ that satisfy the following criteria and are provided by (5),*

$$|A_2| \leq \frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2\left(\Im+1\right)+\left(\Im+\mu+2\right)(t-1)}}{\sqrt{|\Upsilon(t,\Im,0,\mu)|}},$$

$$|A_3| \leq \frac{9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}{\left(\Omega+1\right)^2\left(1+me^{i\psi}\right)^2} + \frac{10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]}{(2\Omega+1)(1+me^{i\psi})},$$

and

$$\left|A_3-\varphi A_2^2\right| \leq \begin{cases} \frac{10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right|}{(2\Omega+1)(1+me^{i\psi})} & |1-\varphi|\leq\Pi_2, \\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\mu)|} & |1-\varphi|\geq\Pi_2, \end{cases}$$

where

$$\begin{aligned}\Upsilon(t,\Im,\mu) &= \frac{1}{5}\left(2\Omega+1\right)\left(1+me^{i\psi}\right)\left[\begin{array}{c}(\Im+1)\\+\frac{1}{2}(\Im+\mu+2)(t-1)\end{array}\right]^2 \\ &\quad -\frac{2}{9}\left(\Omega+1\right)^2\left(1+me^{i\psi}\right)^2\left[\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}\left(\Im+2\right)\left(\Im+\mu+3\right)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right]\end{aligned}$$

and

$$\Pi_2 = 1 - \frac{\frac{10}{9}\left(\Omega+1\right)^2\left(1+me^{i\psi}\right)^2\left(\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}\left(\Im+2\right)\left(\Im+\mu+3\right)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right)}{\left(2\Omega+1\right)\left(1+me^{i\psi}\right)\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

Corollary 2. *When $\Upsilon = 0$ and $\Omega = 1$, we obtain $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. Here, $(m,\psi,t,0,1)$ is the set of functions $B\in\Sigma$ that satisfy the following criteria and are provided by (5),*

$$|A_2| \leq \frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2\left(\Im+1\right)+\left(\Im+\mu+2\right)(t-1)}}{\sqrt{|\Upsilon(t,\Im,\mu)|}},$$

$$|A_3| \leq \frac{9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}{4(1+me^{i\psi})^2}+\frac{10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]}{3(1+me^{i\psi})},$$

and

$$\left|A_3-\varphi A_2^2\right| \leq \begin{cases} \frac{10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right|}{3(1+me^{i\psi})} & |1-\varphi| \leq \Pi_3, \\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\mu)|} & |1-\varphi| \geq \Pi_3, \end{cases}$$

where

$$\begin{aligned} \Upsilon(t,\Im,\Upsilon,\mu) &= \frac{3}{5}(1+me^{i\psi})\left[\begin{array}{c}(\Im+1)\\+\frac{1}{2}(\Im+\mu+2)(t-1)\end{array}\right]^2 \\ &\quad -\frac{8}{9}(1+me^{i\psi})^2\left[\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right] \end{aligned}$$

and

$$\Pi_3 = 1-\frac{\frac{40}{9}(1+me^{i\psi})^2\left(\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right)}{3(1+me^{i\psi})\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

Corollary 3. *When $\Upsilon=0$ and $\Omega=0$, we obtain $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. Here, $(m,\psi,t,0,0)$ is the set of functions $B\in\Sigma$ that satisfy the following criteria and are provided by (5),*

$$|A_2| \leq \frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2(\Im+1)+(\Im+\mu+2)(t-1)}}{\sqrt{|\Upsilon(t,\Im,\mu)|}},$$

$$|A_3| \leq \frac{9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}{(1+me^{i\psi})^2}+\frac{10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]}{(1+me^{i\psi})},$$

and

$$\left|A_3-\varphi A_2^2\right| \leq \begin{cases} \frac{10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right|}{(1+me^{i\psi})} & |1-\varphi| \leq \Pi_4, \\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\mu)|} & |1-\varphi| \geq \Pi_4, \end{cases}$$

where

$$\begin{aligned} \Upsilon(t,\Im,\mu) &= \frac{1}{5}(1+me^{i\psi})\left[\begin{array}{c}(\Im+1)\\+\frac{1}{2}(\Im+\mu+2)(t-1)\end{array}\right]^2 \\ &\quad -\frac{2}{9}(1+me^{i\psi})^2\left[\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right] \end{aligned}$$

and

$$\Pi_4 = 1-\frac{\frac{10}{9}(1+me^{i\psi})^2\left(\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right)}{(1+me^{i\psi})\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

Corollary 4. *When $m = 0$, we obtain $\mathfrak{J}_{\Sigma}(m, \psi, t, \Upsilon, \Omega)$. Here, $(0, \psi, t, \Upsilon, \Omega)$ is the set of functions $B \in \Sigma$ that satisfy the following criteria and are provided by (5),*

$$|A_2| \le \frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2\,(\Im+1)+(\Im+\mu+2)(t-1)}}{\sqrt{|\Upsilon(t,\Im,\Upsilon,\mu)|}}$$

$$|A_3| \le \frac{9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}{(2\Upsilon+\Omega+1)^2} + \frac{10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]}{6\Upsilon+2\Omega+1},$$

and

$$\left|A_3-\varphi A_2^2\right| \le \begin{cases} \frac{10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right|}{6\Upsilon+2\Omega+1} & |1-\varphi| \le \Pi_5, \\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\Upsilon,\mu)|} & |1-\varphi| \ge \Pi_5, \end{cases}$$

where

$$\begin{aligned} \Upsilon(t,\Im,\Upsilon,\mu) &= \frac{1}{5}\,(6\Upsilon+2\Omega+1)\begin{bmatrix} (\Im+1) \\ +\frac{1}{2}(\Im+\mu+2)(t-1) \end{bmatrix}^2 \\ &\quad -\frac{2}{9}\,(2\Upsilon+\Omega+1)^2\begin{bmatrix} \frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}\,(\Im+2)\,(\Im+\mu+3)(t-1) \\ +\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2 \end{bmatrix}. \end{aligned}$$

and

$$\Pi_5 = 1-\frac{\frac{10}{9}\,(2\Upsilon+\Omega+1)^2\begin{pmatrix} \frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}\,(\Im+2)\,(\Im+\mu+3)(t-1) \\ +\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2 \end{pmatrix}}{(6\Upsilon+2\Omega+1)\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

Corollary 5. *When $m = \Upsilon = 0$, we obtain $\mathfrak{J}_{\Sigma}(m, \psi, t, \Upsilon, \Omega)$. Here, $(0, \psi, t, 0, \Omega)$ is the set of functions $B \in \Sigma$ that satisfy the following criteria and are provided by (5),*

$$|A_2| \le \frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2\,(\Im+1)+(\Im+\mu+2)(t-1)}}{\sqrt{|\Upsilon(t,\Im,0,\mu)|}},$$

$$|A_3| \le \frac{9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}{(\Omega+1)^2} + \frac{10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]}{2\Omega+1},$$

and

$$\left|A_3-\varphi A_2^2\right| \le \begin{cases} \frac{10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right|}{2\Omega+1} & |1-\varphi| \le \Pi_6, \\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\mu)|} & |1-\varphi| \ge \Pi_6, \end{cases}$$

where

$$\Upsilon(t,\Im,\mu) = \frac{1}{5}\,(2\Omega+1)\begin{bmatrix} (\Im+1) \\ +\frac{1}{2}(\Im+\mu+2)(t-1) \end{bmatrix}^2$$

$$-\frac{2}{9}(\Omega+1)^2\left[\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right]$$

and

$$\Pi_6=1-\frac{\frac{10}{9}(\Omega+1)^2(1+me^{i\psi})^2\left(\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right)}{(2\Omega+1)(1+me^{i\psi})\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

Corollary 6. *When $m=\Upsilon=0$ and $\Omega=1$, we obtain $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. Here, $(0,\psi,t,0,1)$ is the set of functions $B\in\Sigma$ that satisfy the following criteria and are provided by (5),*

$$|A_2|\leq\frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2(\Im+1)+(\Im+\mu+2)(t-1)}}{\sqrt{|\Upsilon(t,\Im,\mu)|}},$$

$$|A_3|\leq\frac{9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}{4}+\frac{10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]}{3},$$

and

$$\left|A_3-\varphi A_2^2\right|\leq\begin{cases}\frac{10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right|}{3} & |1-\varphi|\leq\Pi_7,\\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\mu)|} & |1-\varphi|\geq\Pi_7,\end{cases}$$

where

$$\begin{aligned}\Upsilon(t,\Im,\Upsilon,\mu) &= \frac{3}{5}\left[\begin{array}{c}(\Im+1)\\+\frac{1}{2}(\Im+\mu+2)(t-1)\end{array}\right]^2\\&\quad-\frac{8}{9}\left[\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right]\end{aligned}$$

and

$$\Pi_7=1-\frac{\frac{40}{9}\left(\begin{array}{c}\frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1)\\+\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2\end{array}\right)}{3\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

Corollary 7. *When $m=\Upsilon=0$ and $\Omega=0$, we obtain $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$. Here, $(0,\psi,t,0,0)$ is the set of functions $B\in\Sigma$ that satisfy the following criteria and are provided by (5),*

$$|A_2|\leq\frac{\left((\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right)\sqrt{2(\Im+1)+(\Im+\mu+2)(t-1)}}{\sqrt{|\Upsilon(t,\Im,\mu)|}},$$

$$|A_3|\leq 9\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2+10\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right],$$

and

$$\left|A_3-\varphi A_2^2\right| \leq \begin{cases} 10\left|(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right| & |1-\varphi| \leq \Pi_8, \\ \frac{2\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^3|1-\varphi|}{|\Upsilon(t,\Im,\mu)|} & |1-\varphi| \geq \Pi_8, \end{cases}$$

where

$$\begin{aligned} \Upsilon(t,\Im,\mu) &= \frac{1}{5}\left[\begin{array}{c} (\Im+1) \\ +\frac{1}{2}(\Im+\mu+2)(t-1) \end{array}\right]^2 \\ &\quad -\frac{2}{9}\left[\begin{array}{c} \frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1) \\ +\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2 \end{array}\right] \end{aligned}$$

and

$$\Pi_8 = 1-\frac{\frac{10}{9}\left(\begin{array}{c} \frac{(\Im+1)(\Im+2)}{2}+\frac{1}{2}(\Im+2)(\Im+\mu+3)(t-1) \\ +\frac{1}{8}(\Im+\mu+3)(\Im+\mu+4)(t-1)^2 \end{array}\right)}{\left[(\Im+1)+\frac{1}{2}(\Im+\mu+2)(t-1)\right]^2}.$$

4. Conclusions

We have detailed in this work an extensive family of analytic and bi-univalent functions that are related to the imaginary error function and subordinated to Jacobi polynomials. These functions are also related to the Jacobi polynomials. $\mathfrak{J}_\Sigma(m,\psi,t,\Upsilon,\Omega)$ is the symbol that is used to denote these functions. We offered estimates for the Maclaurin coefficients $|A_2|$ and $|A_3|$, and additionally, we handled the Fekete–Szegő . Furthermore, by specialising the parameters m, Υ, and Ω, the findings for the subfamilies $\mathfrak{J}_\Sigma(m,\psi,t,\Omega)$, which are presented in the next sentence, are as follows: The equations $\mathfrak{J}_\Sigma(m,\psi,t)$ and $\mathfrak{J}_\Sigma(\psi,t,\Upsilon,\Omega)$. It is possible to deduce the functions called $\mathfrak{J}_\Sigma(\psi,t,\Omega)$ and $\mathfrak{J}_\Sigma(\psi,t)$.

It is possible that researchers will be inspired to construct coefficient estimates for functions that belong to a new subfamily of bi-univalent functions as a result of the use of the normalised error function. These estimations encompass the figures $|A_2|$, $|A_3|$, as well as the Fekete–Szegő problems. In addition, we anticipate that other researchers will be motivated to broaden the scope of this family to incorporate harmonic functions and symmetric q-calculus as a consequence of our study. In addition to the domain that is currently being utilised, the approach can be adjusted to make use of the symmetric q-sine and q-cosine domains. This is an extra alternative to the domain that is currently being utilised.

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