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A Numerical Comparison Between the Standard and Modified Versions of OHAM Procedure for Solving Volterra Integro-Delay Differential Equations

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Abstract. In this paper, we formulate and enhance a robust semi-analytical method, known as the Optimal Homotopy Asymptotic Method (OHAM), for solving Volterra delay integro-differential equations (VDIDEs). A comparative analysis between the standard OHAM and its modified version is presented, emphasizing the improvements introduced through the modification. The modification is based on a refined construction of the auxiliary function $H(p)$, which plays a crucial role in enhancing both the accuracy and the convergence of the method. The effectiveness and efficiency of the proposed technique are demonstrated through the solution of various numerical problems. Notably, the modified OHAM achieves higher accuracy within a single order of iteration, in contrast to the four orders required by the standard OHAM. This advancement significantly reduces computational effort, simplifies calculations, and decreases overall time consumption, thereby establishing the modified OHAM as a more efficient and practical tool for addressing such equations.

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Key Words and Phrases: OHAM, MOHAM, nonlinear delay Volterra integro-differential equations

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1. Introduction

Differential and integral equations play a central role in science and engineering, with applications spanning control theory, economics, electrical engineering, medicine, and many other disciplines. Since exact solutions to nonlinear systems are often difficult to obtain, numerous researchers have contributed alternative approaches for constructing approximate solutions. For example, Yang et al. [1] applied the modified reproducing kernel method to linear Volterra integral equations, while Anakira et al. [2] and Raftari [3] successfully implemented the homotopy perturbation method (HPM) for Volterra integro-differential and delay equations. The Chebyshev wavelet method was developed and applied by Biazar and Ebrahimi [4] and later extended by Sahu and Ray [5] for Lane–Emden type equations, Anakira et al. [6] proposed a modified version for handling nonlinear delay integro-differential problems. whereas Dawar et al. [7] further advanced the residual power series method with an improved formulation In the context of homotopy-based strategies, Hashim et al. [8], Jameel et al. [9, 10], and Nawaz et al. [11] demonstrated the flexibility of the OHAM for fractional, fuzzy, and integro-differential equations. Jameel et al. [12] applied the differential transformation method (DTM) for high-order fuzzy problems, while Matinfar et al. [13] employed the homotopy analysis method (HAM) to systems of Volterra integral equations. Hamoud et al. [14] investigated the modified variational iteration method, and Hong et al. [15] applied the relaxed Monte Carlo method for Volterra integral systems. Other notable contributions include Agarwal et al. [16], who explored collocation methods for stochastic equations, while Anakira et al. [17, 18] introduced Multistage MOHAM for solving fuzzy and delay differential equations. Finally, Jameel et al. [19] developed a six-stage Runge–Kutta method of order five for $n'th$ order fuzzy initial value problems. Together, these studies underscore the rich diversity of techniques proposed by different authors, each of whom has advanced the state of the art in approximating solutions for complex differential and integro-differential equations [20–24].

This research employs the OHAM [25, 26] to solve the Volterra delay integro-differential equations (VDIDE). The motivation stems from the need to achieve highly accurate approximate solutions for such systems by employing a robust semi-analytical methodology. A key advantage of OHAM lies in its flexible convergence, which provides an effective framework for constructing and controlling approximation series. employed it to solve Klein–Gordon equations [27], Sheikholeslami et al. used it to investigate laminar viscous flow [28]; Hashmi et al. applied it to Fredholm integral equations [29], and Nawaz et al. derived optimal solutions for fractional Zakharov–Kuznetsov equations [30], three-dimensional Volterra integral equations [31], and fractional integro-differential equations [32].

In this study, we build on these foundations by modifying the standard OHAM to overcome some of its inherent difficulties and limitations. The proposed modification enhances convergence and accuracy while reducing computational effort. We then perform a comparative study between the standard OHAM and its modified version, supported by two numerical examples, to clearly demonstrate the efficiency and reliability of the improved approach in solving systems of VIEs.

Modified OHAM for Volterra Integro-Delay Differential Equations

We consider the general nonlinear Volterra integro-delay differential equation of the form

$$L[u(x)] + g(x) + N[u(x)] + \int_0^x K(x,s)\,u(s-\tau)\,ds = 0, \qquad x \in [0,T], \quad \tau > 0, \tag{1}$$

subject to the boundary/initial conditions

$$u(0) = u_0, \qquad B\left(\frac{du}{dx}\right)_{x=0} = 0, \tag{2}$$

where L is a linear operator, N is a nonlinear operator, $K(x,s)$ is a known kernel, τ is a constant time-delay, and $g(x)$ is a given source function.

According to the basic idea of OHAM, we construct the homotopy

$$(1-p)\,L[v(x,p)] = H(p)\Big[L(v(x,p)) + g(x) + N(v(x,p)) + \int_0^x K(x,s)\,v(s-\tau,p)\,ds\Big], \tag{3}$$

with the boundary operator;

$$u(0) = u_0, \qquad B\left(\frac{du}{dx}\right)_{x=0} = 0, \tag{4}$$

where $p \in [0,1]$ is the embedding parameter, $v(x,p)$ is the unknown function, and $H(p)$ is a non-zero auxiliary function with $H(0) = 0,\ H(1) = 1$.

To improve convergence, we introduce a **modified auxiliary function** in the form

$$H(q) = \sum_{n=1}^{N} \left(\sum_{k=0}^{M} c_{nk} x^k \right) q^n, \tag{5}$$

where c_{nk} are the convergence control parameters to be determined.

Expanding $v(x,p)$ into a power series of p, we have

$$v(x,p) = u_0(x) + \sum_{k=1}^{\infty} u_k(x, C_1, C_2, \ldots, C_m)\,p^k. \tag{6}$$

When $p \to 1$, the approximate solution becomes

$$u(x) \approx u_0(x) + \sum_{k=1}^{m} u_k(x, C_1, C_2, \ldots, C_m). \tag{7}$$

The residual for the truncated m-term approximation is given by

$$R(x; C_1, C_2, \ldots, C_m) = L[\tilde{u}(x)] + g(x) + N[\tilde{u}(x)] + \int_0^x K(x,s)\,\tilde{u}(s-\tau)\,ds, \tag{8}$$

where

$$\tilde{u}(x) = u_0(x) + \sum_{k=1}^{m} u_k(x, C_1, C_2, \ldots, C_m). \tag{9}$$

The optimal values of the unknown convergence control constants $C_1, C_2, \ldots, C_m$ are obtained by minimizing the functional value.

$$J(C_1, C_2, \ldots, C_m) = \int_a^b R^2(x; C_1, C_2, \ldots, C_m)\, dx, \tag{10}$$

which leads to the conditions

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \cdots = \frac{\partial J}{\partial C_m} = 0. \tag{11}$$

Thus, with the modified auxiliary function, the OHAM procedure provides an approximate analytical solution to Volterra integro-delay differential equations with improved convergence properties.

2. Applications

In this section, we present two examples with known exact solutions to evaluate the performance and accuracy of the MOHAM algorithm in comparison with the standard OHAM.

Example 1 As a first step, we consider the following nonlinear VDIDE as a test case. It is solved using OHAM and then with MOHAM, enabling a direct comparison that demonstrates the efficiency, accuracy, and reliability of the modified approach. [5]

$$u'(x) = u^2\left(\frac{x}{2}\right) - e^x + \int_0^x u^2(\frac{t}{2})dt + 1, \qquad u(x) = e^x, \quad x \leq 0, \tag{12}$$

with the exact solution

$$u(x) = e^x. \tag{13}$$

2.1. OHAM solution

Following the OHAM formulation described in [25], we begin by introducing the corresponding operators that will be used in constructing the solution.

$$\begin{aligned} L[v(x,p)] &= \frac{dv(x,p)}{dx} = 1, \\ N[v(x,p)] &= u'(x) - u^2\left(\frac{x}{2}\right) - \psi(x) + \int_0^x u^2(\frac{t}{2})dt + 1, \quad v(0,p) = 1, \end{aligned} \tag{14}$$

where $\psi(x)$ is the expansion Taylor series of e^x with respect to p, which can be written as

$$\psi(x;p)=\sum_{k=0}^{\infty}\left(\frac{x}{k!}\right)^k p^k.$$

Then, we construct a family of homtopy equations $h(u(x;p):R->[0,1]$ as follows

$$(1-p)\,L[v(x,p)]=H(p)\big[L(v(x,p))+g(x)+N(v(x,p))\big], \tag{15}$$

substituting $L[u(x)]$, we have

$$u'(x;p)-u_0(x)+\sum_{i=1}^{m}C_m q^m\left((u'(x)-u^2\left(\frac{x}{2}\right)-\psi(x)+\int_0^x u^2(\frac{t}{2})dt+1\right)=0. \tag{16}$$

Thus, as p varies from 0 to 1, the solution approach from $u_0(x)$ to $u(x)$, where $u_0(x)$ is the zeroth -order problem that can be obtained from the solution of initial guess

$$u_0'(x)=1,\quad u_0(0)=1. \tag{17}$$

Then, by expanding $u(x;p;c_i), i=1,...,4$ in Taylor series about p, we have

$$u(x;p;c_i)=u_0(x)+\sum_{k=1}^{m}u_k(x,C_1,C_2,\ldots,C_m)\,p^k. \tag{18}$$

Substituting Eq.(18) into Eq. (16) and by equating the coefficient of like power p, yields

The first-order deformation problem

$$u_1'(x,C_1)=c_1(-x)-\frac{1}{4}\left(c_1x^2\right)+\frac{c_1x^3}{12}+\frac{c_1x^4}{24}+\frac{c_1x^5}{120}+\frac{c_1x^6}{720}+\frac{c_1x^7}{5040},\quad u_0(0)=0, \tag{19}$$

The second-order deformation problem is

$$\begin{aligned}u_2'(x,C_1,C_2) \;=\; & c_1(-x)-\frac{1}{4}\left(c_1x^2\right)+\frac{c_1x^3}{12}+\frac{c_1x^4}{24}+\frac{c_1x^5}{120}+\frac{c_1x^6}{720}+\frac{c_1x^6}{720}\\ &+\frac{c_1x^7}{5040}c_1^2(-x)+\frac{5}{16}c_1^2x^3+\frac{11}{128}c_1^2x^4+\frac{31c_1^2x^5}{3840}+\frac{c_1^2x^6}{1280}+\frac{c_1^2x^7}{7680}\\ &-\frac{5c_1^2x^8}{1032192}-\frac{c_1^2x^9}{3440640}-\frac{c_1^2x^{10}}{103219200}-c_2x-\frac{1}{4}\left(c_2x^2\right)+\frac{c_2x^3}{12}\\ &+\frac{c_2x^4}{24}+\frac{c_2x^5}{120}+\frac{c_2x^6}{720}+\frac{c_2x^7}{5040},\quad u_2(0)=0.\end{aligned} \tag{20}$$

The third-order deformation problem

$$\begin{aligned}u_3'(x,C_1,C_2,C_3) \;=\; & -\frac{5c_1^2x^8}{172032}+\frac{c_1^2x^7}{1280}+\frac{3}{640}c_1^2x^6+\frac{31}{640}c_1^2x^5+\frac{33}{64}c_1^2x^4+\frac{15}{8}c_1^2x^3-6c_1^2x\\ &-\frac{c_1^2x^{10}}{17203200}-\frac{c_1^2x^9}{573440}+\frac{141}{256}c_1^3x^4+\frac{25}{8}c_1^3x^3+\frac{3}{2}c_1^3x^2-6c_1^3x\end{aligned}$$

$$\frac{8509c_1^3x^9}{1981808640}+\frac{283c_1^3x^8}{3932160}+\frac{509c_1^3x^7}{573440}-\frac{743c_1^3x^6}{122880}-\frac{171c_1^3x^5}{5120}$$
$$-\frac{13801c_1^3x^{12}}{871995801600}-\frac{12659c_1^3x^{11}}{108999475200}-\frac{3509c_1^3x^{10}}{8808038400}+\ldots,\quad u_3(0)=0, \tag{21}$$

and the fourth order deformation problem

$$\begin{aligned}u_4'(x,C_1,C_2,C_3,C_4) &= \frac{c_1x^7}{5040}+\frac{c_1x^6}{720}+\frac{c_1x^5}{120}+\frac{c_1x^4}{24}+\frac{c_1x^3}{12}-\frac{c_1x^2}{4}-3c_1^2x-c_1x\\
&-\frac{5c_1^2x^8}{344064}+\frac{c_1^2x^7}{2560}+\frac{3c_1^2x^6}{1280}+\frac{31c_1^2x^5}{1280}+\frac{33}{128}c_1^2x^4+\frac{15}{16}c_1^2x^3\\
&-\frac{c_1^2x^{10}}{34406400}-\frac{c_1^2x^9}{1146880}+\frac{25}{16}c_1^3x^3+\frac{3}{4}c_1^3x^2-3c_1^3x\\
&\frac{283c_1^3x^8}{7864320}+\frac{509c_1^3x^7}{1146880}-\frac{743c_1^3x^6}{245760}-\frac{171c_1^3x^5}{10240}+\frac{141}{512}c_1^3x^4\\
&-\frac{12659c_1^3x^{11}}{217998950400}-\frac{3509c_1^3x^{10}}{17616076800}+\frac{8509c_1^3x^9}{3963617280}+\ldots,\quad u_4(0)=0\end{aligned} \tag{22}$$

By solving Eqs. (17), (19), (20), (21) and (22) and substituting them into Eq.(18), the fourt-order approximate solution by OHAM for $p=1$, is:

$$\begin{aligned}\widetilde{u}(x,C_1,C_2,C_3,C_4) &= u_0(x)+u_1(x,C_1)+u_2(x,C_1,C_2)\\
&+u_3(x,C_1,C_2,C_3)++u_4(x,C_1,C_2,C_3,c_4).\end{aligned} \tag{23}$$

By using the proposed method of section 2 on $[0,1]$, we use the residual error,

$$\begin{aligned}R &= \widetilde{u}'\left(x,C_1,\cdots,C_4\right)-\widetilde{u}^2\left(\frac{x}{2},C_1,\cdots,C_4\right)\\
&+e^x-\int_0^x u^2\left(\frac{t}{2},C_1,\cdots,C_4\right)dt-1.\end{aligned} \tag{24}$$

The Less Square error can be formed as

$$J(C_1,\cdots,C_4) = \int_0^1 R^2dx, \tag{25}$$

and

$$\frac{\partial J(C_1,\cdots,C_4)}{\partial C_1}=\frac{\partial J(C_1,C_2,C_3)}{\partial C_2}=\frac{\partial J(C_1,C_2,C_3)}{\partial C_3}=\frac{\partial J(C_1,C_2,C_3,C_4)}{\partial C_4}=0. \tag{26}$$

Thus, the following optimal values of C_i's are obtained:

$C_1=-1.03257,\quad C_2=3.75206\times10^{-6},\quad C_3=-8.89991\times10^{-7}\quad C_4=-4.48187\times10^{-6}.$

In this case, our approximate solution is

$$\begin{aligned}\widetilde{u}(x, C_1, \cdots, C_4) \quad = \quad & 1 + x + 0.5000016432167662x^2 + 0.16667895652842996x^3 \\ & 58556x^4 + 0.00830125122279865x^5 + 0.0014533846838279182x^6 \\ & +0.000205626800537755x^7 + 0.000006767376975033287x^8 \\ & -0.000001498570429380608x^9 - 6.299736808070288 \times 10^{-7}x^{10} \\ & -6.739802323017767 \times 10^{-8}x^11 - 3.758169088190813 \times 10^{-9}x^{12} \\ & -7.089767587924317 \times 10^{-11}x^{13} + 9.59982558191048 \times 10^{-12}x^{14} \\ & +1.309622277385672 \times 10^{-12}x^{15} + 8.75719622963593 \times 10^{-14}x^{16} \\ & +3.636632322436979 \times 10^{-15}x^17 + 9.172211052795001 \times 10^{-17}x^{18} \\ & +1.443596286575417 \times 10^{-18}x^{19} + 2.522219242033677 \times 10^{-20}x^{20} \\ & +2.258535766371391 \times 10^{-22}x^{21} \qquad (27)\end{aligned}$$

As we observed, the accuracy of the OHAM strongly depends on the number of approximation terms used in the series solution. While increasing the number of terms generally improves the precision of the obtained results and reduces the error, it also leads to heavier computational work and longer calculation time, which can limit the efficiency of the method when applied to complex nonlinear problems. These practical difficulties—namely the high computational cost, extended time requirements, and the necessity of employing higher-order approximation terms to achieve remarkably low errors motivated the modification of OHAM. The proposed modification is built upon the construction of the auxiliary function $H(p)$ $(Eq.(5))$ in the standard OHAM. This formulation offers enhanced flexibility in controlling the convergence of the series solution and enables the method to effectively applied to a wider class of nonlinear problems with only one order of approximation. This feature represents the main advantage of the proposed modification, as it significantly save time, reduces effort, and minimizes computational calculations as we will see on next part.

2.2. MOHAM Solution

The MOHAM solution is obtained by constructing a family of homotopy equations similar to the standard OHAM framework, but with an important modification in the design of the auxiliary function. This new construction of the auxiliary function introduces additional flexibility through adjustable convergence-control parameters, which allows the method to achieve accurate results using only a single term of the OHAM order of approximation. Based on Eq. (16), and by employing Eq. (5) as follows

$$u'(x;p) - u_0(x) + (\sum_{n=1}^{N} \left(\sum_{k=0}^{M} c_{nk} x^k \right) q^n) \left((u'(x) - u^2 \left(\frac{x}{2} \right) - \psi(x) + \int_0^x u^2(\frac{t}{2}) dt + 1 \right) = 0. \qquad (28)$$

Thus, as p varies from 0 to 1, the solution approach from $u_0(x)$ to $u(x)$, where $u_0(x)$ is the zeroth -order problem that can be obtained from the solution of initial guess

$$u_0'(x) = 1, \quad u_0(0) = 1. \qquad (29)$$

Then, by expanding $u(x;p;c_i), i = 1, ..., 5$, and $m = 1$ in Taylor series about p, we have

$$u(x;p;c_i) = u_0(x) + \sum_{k=1}^{m} u_k(x, C_1, C_2, \ldots, C_i)\, p^k. \tag{30}$$

Substituting Eq.(30) into Eq. (28) and by equating the coefficient of like power p, yields to the first order deformation problem

$$\begin{aligned} u_1'(x) \;=\; & \frac{c_5x^{12}}{5040} + \frac{c_4x^{11}}{5040} + \frac{c_5x^{11}}{720} + \frac{c_3x^{10}}{5040} + \frac{c_4x^{10}}{720} + \frac{c_5x^{10}}{120} + \frac{c_2x^9}{5040} + \frac{c_3x^9}{720} + \frac{c_4x^9}{120} \\ & + \frac{c_5x^9}{24} + \frac{c_1x^8}{5040} + \frac{c_2x^8}{720} + \frac{c_3x^8}{120} + \frac{c_4x^8}{24} + \frac{c_5x^8}{12} + \frac{c_0x^7}{5040} + \frac{c_1x^7}{720} + \frac{c_2x^7}{120} + \frac{c_3x^7}{24} \\ & + \frac{c_4x^7}{12} - \frac{c_5x^7}{4} + \frac{c_0x^6}{720} + \frac{c_1x^6}{120} + \frac{c_2x^6}{24} + \frac{c_3x^6}{12} - c_5x^6 - \frac{c_4x^6}{4} + \frac{c_0x^5}{120} + \frac{c_1x^5}{24} \\ & + \frac{c_2x^5}{12} - c_4x^5 - \frac{c_3x^5}{4} + \frac{c_0x^4}{24} + \frac{c_1x^4}{12} - c_3x^4 - \frac{c_2x^4}{4} + \frac{c_0x^3}{12} - c_2x^3 - \frac{c_1x^3}{4} \\ & - c_1x^2 - \frac{c_0x^2}{4} - c_0x \end{aligned} \tag{31}$$

By solving Eqs. (31) and (30), and substituting them into Eq.(30), we obtain MOHAM approximate solution of order one as follows:

$$\begin{aligned} u(x, c_1, \ldots, c_5) \;=\; & \frac{c_5x^{13}}{65520} + \frac{c_4x^{12}}{60480} + \frac{c_5x^{12}}{8640} + \frac{c_3x^{11}}{55440} + \frac{c_4x^{11}}{7920} + \frac{c_5x^{11}}{1320} + \frac{c_2x^{10}}{50400} + \frac{c_3x^{10}}{7200} \\ & + \frac{c_4x^{10}}{1200} + \frac{c_5x^{10}}{240} + \frac{c_1x^9}{45360} + \frac{c_2x^9}{6480} + \frac{c_3x^9}{1080} + \frac{c_4x^9}{216} + \frac{c_5x^9}{108} + \frac{c_0x^8}{40320} \\ & + \frac{c_1x^8}{5760} + \frac{c_2x^8}{960} + \frac{c_3x^8}{192} + \frac{c_4x^8}{96} - \frac{c_5x^8}{32} + \frac{c_0x^7}{5040} + \frac{c_1x^7}{840} + \frac{c_2x^7}{168} + \frac{c_3x^7}{84} \\ & - \frac{c_5x^7}{7} - \frac{c_4x^7}{28} + \frac{c_0x^6}{720} + \frac{c_1x^6}{144} + \frac{c_2x^6}{72} - \frac{c_4x^6}{6} - \frac{c_3x^6}{24} + \frac{c_0x^5}{120} + \frac{c_1x^5}{60} \\ & - \frac{c_3x^5}{5} - \frac{c_2x^5}{20} + \frac{c_0x^4}{48} - \frac{c_2x^4}{4} - \frac{c_1x^4}{16} - \frac{c_1x^3}{3} - \frac{c_0x^3}{12} - \frac{c_0x^2}{2} + x + 1. \end{aligned} \tag{32}$$

the residual error,

$$\begin{aligned} R \;=\; & \widetilde{u}'(x, C_1, \cdots, C_5) - \widetilde{u}^2\left(\frac{x}{2}, C_1, \cdots, C_5\right) \\ & + e^x - \int_0^x u^2\left(\frac{t}{2}, C_1, \cdots, C_5\right) dt - 1. \end{aligned} \tag{33}$$

The Less Square error can be formed as

$$J(C_1, \cdots, C_5) \;=\; \int_0^1 R^2 dx, \tag{34}$$

Table 1: Numerical result of example 1

x	4'th order OHAM Absolute Error	First-order MOHAM Absolute Error
0.0	0.0000000000	0.0000000000
0.2	5.41×10^{-9}	2.51×10^{-7}
0.4	1.52×10^{-7}	4.58×10^{-7}
0.6	7.59×10^{-7}	5.86×10^{-7}
0.8	5.213×10^{-7}	4.30×10^{-7}
1.0	9.88×10^{-6}	1.71×10^{-6}

and

$$\frac{\partial J(C_1, \cdots, C_5)}{\partial C_1} = \cdots = \frac{\partial J(C_1, C_2, C_3, C_5)}{\partial C_5} = 0. \tag{35}$$

Thus, the following optimal values of C_i's are obtained:

$C_1 = -0.999749, \quad C_2 = -0.254198, \quad C_3 = -0.122836 \quad C_4 = 0.0566211 \quad C_5 = -0.0578005.$

In this case, our approximate solution is

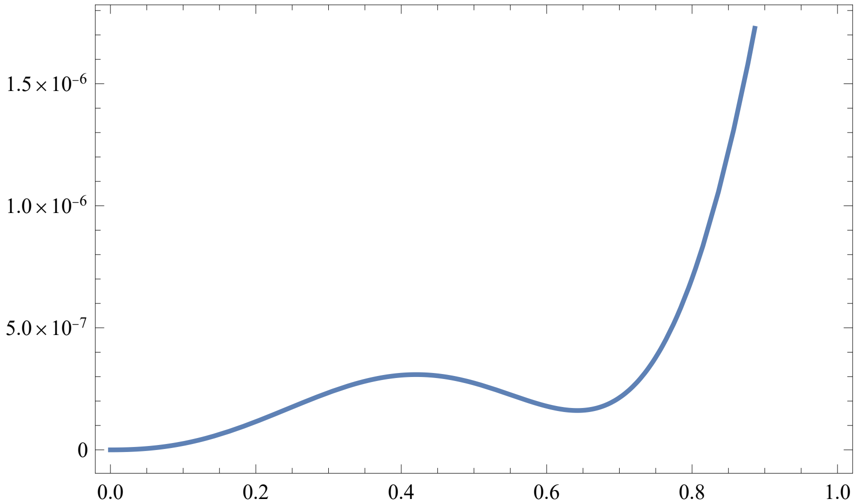
$$\begin{aligned} u(x) \;=\; & -8.821804171278825 \times 10^{-7} x^{13} - 5.753673082235477 \times^1 0{-}6x^{12} \\ & -0.0000388548x^{11} - 0.000213946x^{10} - 0.00041755x^9 + 0.00151758x^8 \\ & +0.00330149x^7 - 0.00973638x^6 + 0.0201492x^5 + 0.035809x^4 \\ & +0.168045x^3 + 0.499875x^2 + x + 1 \end{aligned} \tag{36}$$

The numerical results of Example 1 highlight the superior efficiency of MOHAM compared with OHAM. As shown in Table 1, the fourth-order OHAM solution produces extremely small errors (as low as 5.41×10^{-9} at $x = 0.2$), while the first-order MOHAM approximation already achieves comparable accuracy, with absolute errors ranging from 10^{-7} to 10^{-6}. This indicates that MOHAM can provide reliable results with only a single term, thereby reducing computational complexity and effort. Furthermore, the error distribution presented in Figure 1 demonstrates that MOHAM ensures smooth and stable convergence across the interval, whereas OHAM requires higher-order expansions to attain a similar level of accuracy. Overall, these findings clearly confirm the advantage of MOHAM in achieving high accuracy at lower approximation orders.

2.3. Example 2

The second example considered in this study is the following non-linear DVIDE

$$u(x) = u\left(\frac{x}{2}\right) - \frac{3}{2}\sin x - \frac{x}{2} - \cos\left(\frac{x}{2}\right) + \int_0^x u^2\left(\frac{s}{2}\right) ds, u(0) = 1. \tag{37}$$

a)


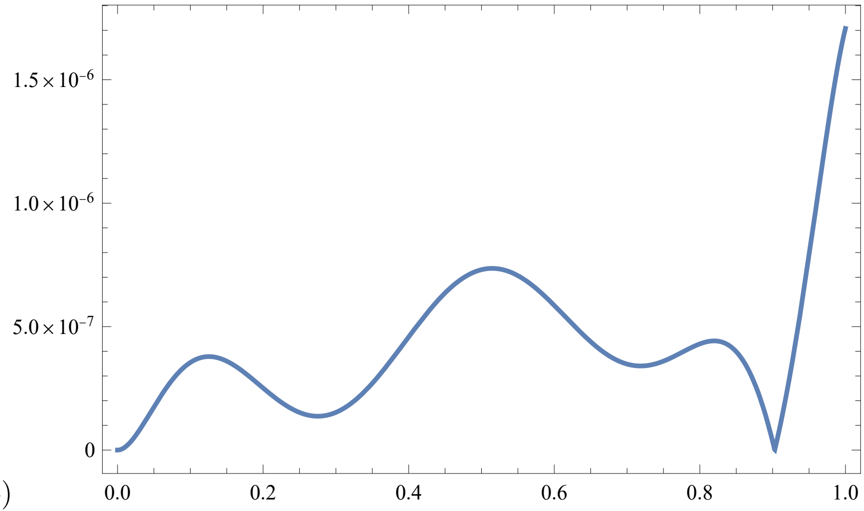
b)


Figure 1: (a) OHAM absolute error resulted from fourth -order of approximation. (b) MOHAM absolute error resulted from asingle -order of approximation foe example 1

with exact solution

$$u(x) = cos(x). \tag{38}$$

According to the method which was described in above section, and by following the same procedure that was applied on the first example, we have the fourth order OHAM approximate solutions

$$\begin{aligned}\widetilde{u}(x, C_1, C_2, C_3) \quad = \quad & 6.874157120897826 \times 10^{-37}x^{26} - 8.463767623593088 \times 10^{-32}x^{25} \\ & +2.184023696562375 \times 10^{-27}x^{24} + 1.0133589762100683 \times 10^{-26}x^{23} \\ & +3.460786587767694 \times 10^{-22}x^{22} - 2.7806197391988356 \times 10^{-23}x^{21} \\ & -2.9629995732517603 \times 10^{-19}x^{20} + 5.564560645958861 \times 10^{-20}x^{19} \\ & +1.3918670220163003 \times 10^{-16}x^{18} - 9.700555588641355 \times 10^{-17}x^{17}\end{aligned}$$

Table 2: Numerical result of example 2

x	4'th order OHAM Absolute Error	First-order MOHAM Absolute Error
0.0	0.00	0.00
0.2	6.06×10^{-8}	3.58×10^{-7}
0.4	1.81×10^{-7}	7.29×10^{-7}
0.6	2.36×10^{-7}	6.74×10^{-7}
0.8	1.47×10^{-7}	1.01×10^{-6}
1.0	2.57×10^{-7}	1.25×10^{-6}

$$
\begin{aligned}
&-4.222517784937938 \times 10^{-14}x^{16} + 4.202560325267928 \times 10^{-14}x^{15}\\
&+9.028868136417469 \times 10^{-12}x^{14} - 1.2725695013294234 \times 10^{-11}x^{13}\\
&-1.3653035357656884 \times 10^{-9}x^{12} + 2.555212476061491 \times 10^{-9}x^{11}\\
&-2.6421049209252045 \times 10^{-7}x^{10} - 2.710683819309949 \times 10^{-7}x^{9}\\
&+0.0000239393x^{8} + 6.931541036055155 \times 10^{-8}x^{7} - 0.00138905x^{6}\\
&+3.5759065280388524 \times 10^{-7}x^{5} + 0.041668x^{4}\\
&+1.0621242323427538 \times 10^{-6}x^{3} - 0.500002x^{2} + 1
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{u}(x, C_1, C_2, C_3) \quad = \quad & 1 - 0.499725609x^2 - 0.003203684x^3\\
&+0.056106703x^4 - 0.03141667x^5\\
&+0.031943943x^6 - 0.0128500196x^7\\
&-0.0035950170x^8 + 0.002988065x^9\\
&+0.000183446x^{10} - 0.000126185x^{11}\\
&-4.018124199 \times 10^{-6}x^{12} + 2.579615120 \times 10^{-6}x^{13}\\
&+4.9331251480 \times 10^{-8}x^{14} - 3.177894121 \times 10^{-8}x^{15}\\
&+1.93944674 \times 10^{-12}x^{16}.
\end{aligned}
$$

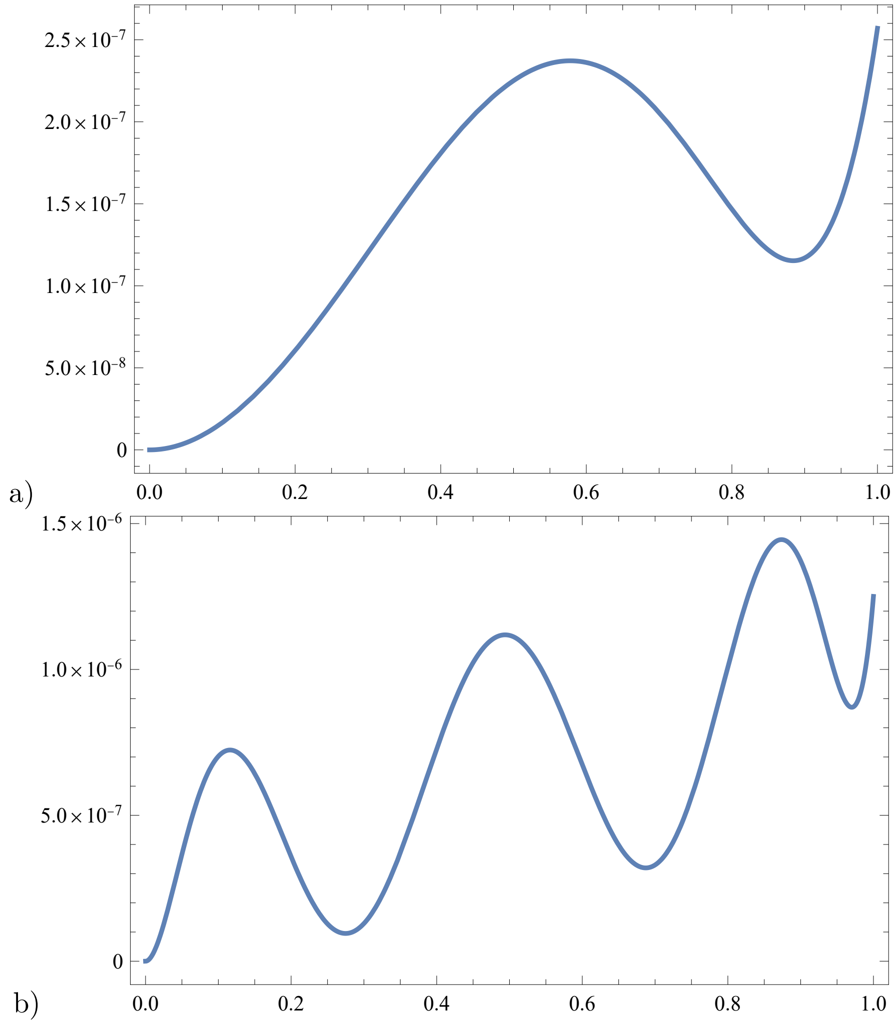


Figure 2: (a) OHAM absolute error resulted from fourth -order of approximation. (b) MOHAM absolute error resulted from asingle -order of approximation foe example 2

The results of Example 2 further confirm the robustness of MOHAM in comparison with OHAM. As shown in Table 2, the fourth–order OHAM approximation attains errors in the order of 10^{-8} to 10^{-7}, while the first–order MOHAM already achieves comparable accuracy, with errors ranging from 10^{-7} to 10^{-6}. Although OHAM demonstrates slightly higher precision at certain points (e.g., 6.06×10^{-8} at $x = 0.2$), MOHAM reaches this level of accuracy with far fewer terms, thus reducing the computational effort. The error profiles illustrated in Figure 2 indicate that OHAM's residual exhibits oscillatory behavior, while MOHAM maintains a smoother distribution of error across the interval. This comparison highlights the advantage of MOHAM in producing reliable results at lower approximation orders, striking a balance between computational efficiency and accuracy.

The Modified Optimal Homotopy Asymptotic Method (MOHAM) offers remarkable advantages over the standard OHAM, especially in terms of the order of approximation.

Unlike the standard OHAM, which generally requires more than two approximation terms to achieve acceptable accuracy, MOHAM attains highly accurate solutions with only a single approximation term. This efficiency not only reduces the analytical effort but also saves considerable time in the solution process. In addition, MOHAM minimizes computational calculations, making it more practical for solving complex nonlinear problems. These benefits are clearly demonstrated in the numerical results displayed in Tables 1 and 2, where MOHAM consistently provides accurate solutions with fewer terms and reduced computational effort compared to the standard OHAM.

3. Conclusions

In this research, we conducted a comparative analysis between the conventional OHAM and a newly suggested modification, which was applied for the first time to derive an analytic approximate solution for VDIDEs. The modified approach introduced a new series-based formulation for the auxiliary function $H(q)$, specifically designed to improve convergence, simplify implementation, and reduce computational complexity and cost, while ensuring high accuracy and reliability. A key advantage of the modified technique is its ability to deliver accurate results using only one order of approximation, in contrast to the standard OHAM, which typically requires several terms to achieve similar levels of accuracy. This highlights the improved efficiency, robustness, and practical applicability of the modified version in addressing this class of VDIDEs.

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