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Upper and Lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-Continuity

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Abstract. A new class of continuous multifunctions between an ideal topological space and a bitopological space, called upper (lower) $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions, has been defined and studied. Moreover, several characterizations and some properties concerning upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions are established.

2020 Mathematics Subject Classifications: 54C08, 54C60

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1. Introduction

In 1982, Noiri [1] introduced a class of functions defined between topological spaces, namely strongly semi-continuous functions. Mashhour et al. [2] called strongly semi-continuous functions α-continuous functions and investigated some characterizations of such functions. In 1986, Neubrunn [3] extended the concept of α-continuous functions to multifunctions and presented two classes of multifunctions defined from a topological space into a topological space, called upper α-continuous multifunctions and lower α-continuous multifunctions. In 1993, Popa and Noiri [4] obtained several characterizations and some basic properties of upper α-continuous multifunctions and lower α-continuous multifunctions. On the other hand, the present author introduced and investigated four classes of multifunctions defined from an ideal topological space into an ideal topological space, namely upper $\star$-continuous multifunctions [5], lower $\star$-continuous multifunctions [5], upper $\alpha(\star)$-continuous multifunctions [6], lower $\alpha(\star)$-continuous multifunctions [6], upper $\beta(\star)$-continuous multifunctions [7], lower $\beta(\star)$-continuous multifunctions [7], upper

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$s\beta(\star)$-continuous multifunctions [8], lower $s\beta(\star)$-continuous multifunctions [8], upper α-$\star$-continuous multifunctions [9], lower α-$\star$-continuous multifunctions [9], $\imath^\star$-continuous multifunctions [10] and $p\imath$-continuous multifunctions [11]. Pue-on et al. [12] introduced and studied two classes of multifunctions between bitopological spaces, namely upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [13] investigated several characterizations of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions by utilizing the notions of $(\tau_1, \tau_2)\theta$-closed sets and $(\tau_1, \tau_2)\theta$-open sets. Thongmoon et al. [14] studied some characterizations of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions by using $\tau_1\tau_2$-δ-open sets and $\tau_1\tau_2$-δ-closed sets. In [15], the present authors introduced and investigated the concepts of upper $(\tau_1, \tau_2)\alpha$-continuous multifunctions and lower $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Quite recently, Khampakdee et al. [16] presented new classes of continuous multifunctions defined from an ideal topological space into a bitopological space, namely upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. In this paper, we introduce the concepts of multifunctions between an ideal topological space and a bitopological space, called upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [17] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [17] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [17] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [17] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$*-open* [15] (resp. $(\tau_1, \tau_2)s$*-open* [18], $(\tau_1, \tau_2)p$*-open* [18], $(\tau_1, \tau_2)\beta$*-open* [18]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-δ-open* [19] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$*-δ-closed* [19]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$*-δ-interior* [19] of A and is denoted by $\tau_1\tau_2\text{-}\delta\text{-Int}(A)$. The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$*-δ-closure* [19] of A and is denoted by $\tau_1\tau_2\text{-}\delta\text{-Cl}(A)$. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* [15] of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* [15] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* [15] if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* [15] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [20], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [21] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *R-$\mathscr{I}^\star$-open* [5] (resp. *$\mathscr{I}^\star$-preopen* [5], *semi-$\mathscr{I}^\star$-open* [22], *semi-$\mathscr{I}^\star$-preopen* [22]) if $A = \text{Int}^\star(\text{Cl}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))$). The complement of a R-$\mathscr{I}^\star$-open (resp. $\mathscr{I}^\star$-preopen, semi-$\mathscr{I}^\star$-open, semi-$\mathscr{I}^\star$-preopen) set is said to be *R-$\mathscr{I}^\star$-closed* (resp. *$\mathscr{I}^\star$-preclosed*, *semi-$\mathscr{I}^\star$-closed*, *semi-$\mathscr{I}^\star$-preclosed*). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^\star$-closed sets containing A is called the *semi-$\mathscr{I}^\star$-closure* [22] of A and is denoted by $\text{sCl}^\star(A)$ ($s\text{Cl}_{\mathscr{I}^\star}(A)$ [22]). The union of all semi-$\mathscr{I}^\star$-open sets contained in A is called the *semi-$\mathscr{I}^\star$-interior* [22] of A and is denoted by $\text{sInt}^\star(A)$ ($s\text{Int}_{\mathscr{I}^\star}(A)$ [22]).

Lemma 2. [22] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$.

(2) $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called *$\tau^\star$-α-open* [23] (*α-$\mathscr{I}^\star$-open* [24]) if $A \subseteq \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(A)))$. The complement of a $\tau^\star$-α-open set is called *$\tau^\star$-α-closed.*

Lemma 3. [24] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) A is α-$\mathscr{I}^\star$-open in X.

(2) $G \subseteq A \subseteq Int^\star(Cl^\star(G))$ for some $\star$-open set G.

(3) $G \subseteq A \subseteq sCl^\star(G)$ for some $\star$-open set G.

(4) $A \subseteq sCl^\star(Int^\star(A))$.

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all α-$\mathscr{I}^\star$-closed sets containing A is called the *α-$\mathscr{I}^\star$-closure* [24] of A and is denoted by $\alpha\text{Cl}^\star(A)$ ($\alpha\text{Cl}_{\mathscr{I}^\star}(A)$ [24]). The *α-$\mathscr{I}^\star$-interior* [24] of A is defined by the union of all α-$\mathscr{I}^\star$-open sets contained in A and is denoted by $\alpha\text{Int}^\star(A)$ ($\alpha\text{Int}_{\mathscr{I}^\star}(A)$ [24]).

Lemma 4. [24] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) A is α-$\mathscr{I}^\star$-closed in X if and only if $sInt^\star(Cl^\star(A)) \subseteq A$.

(2) $sInt^\star(Cl^\star(A)) = Cl^\star(Int^\star(Cl^\star(A)))$.

(3) $\alpha Cl^\star(A) = A \cup Cl^\star(Int^\star(Cl^\star(A)))$.

(4) $\alpha Int^\star(A) = A \cap Int^\star(Cl^\star(Int^\star(A)))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V such that $F(x) \subseteq V$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq V$. A multifunction*

$$F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$$

is said to be upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if F is upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at each point of X.

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in sCl^\star(Int^\star(F^+(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(3) $x \in \alpha Int^\star(F^+(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq V$; hence $x \in U \subseteq F^+(V)$. Since U is $\tau^\star$-α-open, by Lemma 3 we have $x \in U \subseteq \text{sCl}^\star(\text{Int}^\star(U)) \subseteq s\text{Cl}^\star(\text{Int}^\star(F^+(V)))$.

(2) $\Rightarrow$ (3): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then by (2), we have $x \in \text{sCl}^\star(\text{Int}^\star(F^+(V)))$ and by Lemma 2, $x \in \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^+(V))))$. Therefore, $x \in \alpha\text{Int}^\star(F^+(V))$ by Lemma 4.

(3) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (3), we have $x \in \alpha\text{Int}^\star(F^+(V))$ and so there exists a $\tau^\star$-α-open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at x.

Definition 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-α-open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if F is lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at each point of X.*

Theorem 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in sCl^\star(Int^\star(F^-(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$;

(3) $x \in \alpha Int^\star(F^-(V))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. The proof is similar to that of Theorem 1.

Definition 3. *A subset N of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be a $\tau^\star$-α-neighbourhood of $x \in X$ if there exists a $\tau^\star$-α-open set V of X such that $x \in V \subseteq N$.*

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(V)$ is $\tau^\star$-α-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is $\tau^\star$-α-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $sInt^\star(Cl^\star(F^-(B))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $\alpha Cl^\star(F^-((B)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(6) for each $x \in X$ and each $\sigma_1\sigma_2$-neighbourhood V of $F(x)$, $F^+(V)$ is a $\tau^\star$-α-neighbourhood of x;

(7) for each $x \in X$ and each $\sigma_1\sigma_2$-neighbourhood V of $F(x)$, there exists a $\tau^\star$-α-neighbourhood U of x such that $F(U) \subseteq V$.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$. Since F is upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at x, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq V$; hence $x \in U \subseteq F^+(V)$. By Lemma 3, we have $x \in U \subseteq \text{sCl}^\star(\text{Int}^\star(U)) \subseteq \text{sCl}^\star(\text{Int}^\star(F^+(V)))$. Thus, $F^+(V) \subseteq \text{sCl}^\star(\text{Int}^\star(F^+(V)))$. It follows from Lemma 3 that $F^+(V)$ is $\tau^\star$-α-open in X.

(2) ⇔ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2$-Cl(B) is $\sigma_1\sigma_2$-closed in Y and by (3), $F^-(\sigma_1\sigma_2\text{-Cl}(B))$ is $\tau^\star$-α-closed in X. By Lemma 4, we have

$$\text{sInt}^\star(\text{Cl}^\star(F^-(B))) \subseteq \text{sInt}^\star(\text{Cl}^\star(F^-(\text{Cl}^\star(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)).$$

(4) ⇒ (5): Let B be any subset of Y. By (4) and Lemma 4,

$$\alpha\text{Cl}^\star(F^-(B)) = F^-(B) \cup \text{sInt}^\star(\text{Cl}^\star(F^-(B))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)).$$

(5) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Thus by (5), we have

$$\alpha\text{Cl}^\star(F^-(K)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K)$$

and hence $F^-(K)$ is $\tau^\star$-α-closed in X.

(2) ⇒ (6): Let $x \in X$ and V be a $\sigma_1\sigma_2$-neighbourhood of $F(x)$. Then, there exists a $\sigma_1\sigma_2$-open set G of Y such that $F(x) \subseteq G \subseteq V$. Thus, $x \in F^+(G) \subseteq F^+(V)$. By (2), $F^+(G)$ is $\tau^\star$-α-open in X and so $F^+(V)$ is a $\tau^\star$-α-neighbourhood of x.

(6) ⇒ (7): Let $x \in X$ and V be a $\star$-neighbourhood of $F(x)$. By (6), we have $F^+(V)$ is a $\tau^\star$-α-neighbourhood of x. Put $U = F^+(V)$, then U is a $\tau^\star$-α-neighbourhood of x such that $F(U) \subseteq V$.

(7) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then, V is a $\sigma_1\sigma_2$-neighbourhood of $F(x)$ and so there exists a $\tau^\star$-α-neighbourhood U of x such that $F(U) \subseteq V$. Since U is a $\tau^\star$-α-neighbourhood of x, there exists a $\tau^\star$-α-open set G of X such that $x \in G \subseteq U$; hence $F(G) \subseteq V$. This shows that F is upper $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous.

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) $F^-(V)$ is $\tau^\star$-α-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^+(K)$ is $\tau^\star$-α-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $sInt^\star(Cl^\star(F^+(B))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $\alpha Cl^\star(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(6) $F(\alpha Cl^\star(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ for every subset A of X;

(7) $F(sInt^\star(Cl^\star(A))) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ for every subset A of X;

(8) $F(Cl^\star(Int^\star(Cl^\star(A)))) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ for every subset A of X.

Proof. The proofs except for the following are similar to the proof of Theorem 3.
(5) ⇒ (6): Let A be any subset of X. Since $A \subseteq F^+(F(A))$, we have

$$\alpha\text{Cl}^\star(A) \subseteq \alpha\text{Cl}^\star(F^+(F(A))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(F(A)))$$

and so $F(\alpha\text{Cl}^\star(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$.
(6) ⇒ (7): Let A be any subset of X. By (6) and Lemma 4,

$$\begin{aligned} F(\text{sInt}^\star(\text{Cl}^\star(A))) &= F(\text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))) \\ &\subseteq F(A \cup \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))) \\ &= F(\alpha\text{Cl}^\star(A)) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(F(A)). \end{aligned}$$

(7) ⇒ (8): Let A be any subset of X. By (7) and Lemma 4, we have

$$F(\text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))) = F(\text{sInt}^\star(\text{Cl}^\star(A))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A)).$$

(8) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set such that $F(x) \cap V \neq \emptyset$. Then, we have $x \in F^-(V)$. We shall show that $F^-(V)$ is $\tau^\star$-α-open in X. By the hypothesis, $F(\text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+(Y - V))))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(F^+(Y - V))) \subseteq Y - V$ and hence $\text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+(Y - V)))) \subseteq F^+(Y - V) = X - F^-(V)$. Thus,

$$F^-(V) \subseteq \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-(V))))$$

and so $F^-(V)$ is $\tau^\star$-α-open in X. Put $U = F^-(V)$. Then, U is a $\tau^\star$-α-open set of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. This shows that F is lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous.

Definition 4. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if for every $\sigma_1\sigma_2$-open set V of Y, $f^{-1}(V)$ is $\tau^\star$-α-open in X.*

Corollary 1. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is* $\tau^\star\alpha(\sigma_1, \sigma_2)$*-continuous;*

(2) $f^{-1}(K)$ *is* $\tau^\star$*-*α*-closed in* X *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(3) $sInt^\star(Cl^\star(f^{-1}(B))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(4) $\alpha Cl^\star(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) for each $x \in X$ *and each* $\sigma_1\sigma_2$*-neighbourhood* V *of* $f(x)$*,* $f^{-1}(V)$ *is a* $\tau^\star$*-*α*-neighbourhood of* x*;*

(6) for each $x \in X$ *and each* $\sigma_1\sigma_2$*-neighbourhood* V *of* $f(x)$*, there exists a* $\tau^\star$*-*α*-neighbourhood* U *of* x *such that* $f(U) \subseteq V$*;*

(7) $f(\alpha Cl^\star(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ *for every subset* A *of* X*;*

(8) $f(sInt^\star(Cl^\star(A))) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ *for every subset* A *of* X*;*

(9) $f(Cl^\star(Int^\star(Cl^\star(A)))) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ *for every subset* A *of* X*.*

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