EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 7045
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Upper and Lower Almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-Continuous Multifunctions

Chokchai Viriyapong[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. This paper introduces new classes of continuous multifunctions defined between an ideal topological space and a bitopological space, called upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations and some properties concerning upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions are investigated.

2020 Mathematics Subject Classifications: 54C08, 54C60

Key Words and Phrases: Upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunction, lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunction

---

1. Introduction

In 1988, Noiri [1] introduced a class of functions between topological spaces, called almost α-continuous functions. Furthermore, Noiri [1] investigated several characterizations and some basic properties of almost α-continuous functions. In 1996, Popa and Noiri [2] extended the concept of almost α-continuous functions to multifunctions and presented classes of multifunctions defined from a topological space into a topological space, namely upper almost α-continuous multifunctions and lower almost α-continuous multifunctions. In particular, several characterizations and some properties concerning upper almost α-continuous multifunctions and lower almost α-continuous multifunctions were established in [2]. On the other hand, the present author introduced and studied four classes of multifunctions defined from an ideal topological space into an ideal topological space, called upper almost $\star$-continuous multifunctions [3], lower almost $\star$-continuous multifunctions [3], upper almost $\alpha(\star)$-continuous multifunctions [4], lower almost $\alpha(\star)$-continuous multifunctions [4], upper almost α-$\star$-continuous multifunctions [5], lower almost α-$\star$-continuous

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.7045

Email addresses: nongluk.h@msu.ac.th (C. Viriyapong), areeyuth.s@psu.ac.th (A. Sama-Ae), chawalit.b@msu.ac.th (C. Boonpok)

multifunctions [5] and almost $\imath^\star$-continuous multifunctions [6]. Pue-on et al. [7] introduced and studied two classes of multifunctions between bitopological spaces, namely upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Moreover, Boonpok and Pue-on [8] introduced and investigated the concepts of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. In [9], the present authors introduced and studied the concepts of upper almost $(\tau_1, \tau_2)\alpha$-continuous multifunctions and lower almost $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Quite recently, Viriyapong et al. [10] presented new classes of continuous multifunctions defined from an ideal topological space into a bitopological space, namely upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. In this paper, we introduce the concepts of continuous multifunctions between an ideal topological space and a bitopological space, called upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [11] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [11] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [11] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [11] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [9] (resp. $(\tau_1, \tau_2)s$-*open* [12], $(\tau_1, \tau_2)p$-*open* [12], $(\tau_1, \tau_2)\beta$-*open* [12]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the (τ_1, τ_2)*s-closure* [12] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the (τ_1, τ_2)*s-interior* [12] of A and is denoted by (τ_1, τ_2)-sInt(A).

Lemma 2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $(\tau_1, \tau_2)\text{-}sCl(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) \cup A$ [12]*;*

(2) $(\tau_1, \tau_2)\text{-}sInt(A) = \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \cap A$ [13].

Lemma 3. [14] *Let A be a subset of a bitopological space (X, τ_1, τ_2). If A is $\tau_1\tau_2$-open in X, then $(\tau_1, \tau_2)\text{-}sCl(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A))$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-δ-*open* [8] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-δ-*closed* [8]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-δ-*interior* [8] of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ-*closure* [8] of A and is denoted by $\tau_1\tau_2$-δ-Cl(A). Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [9] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [9] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [9] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [9] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 4. [9] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2\text{-}Cl(A) = (\tau_1, \tau_2)\theta\text{-}Cl(A)$.

(2) $(\tau_1, \tau_2)\theta\text{-}Cl(A)$ is $\tau_1\tau_2$-closed in X.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [15], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [16] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be R-$\mathscr{I}^\star$-*open* [3] (resp. *$\mathscr{I}^\star$-preopen* [3], *semi-$\mathscr{I}^\star$-open* [17], *semi-$\mathscr{I}^\star$-preopen* [17]) if $A = \text{Int}^\star(\text{Cl}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))$). The complement of a R-$\mathscr{I}^\star$-open (resp. $\mathscr{I}^\star$-preopen, semi-$\mathscr{I}^\star$-open, semi-$\mathscr{I}^\star$-preopen) set is said to be *R-$\mathscr{I}^\star$-closed* (resp. *$\mathscr{I}^\star$-preclosed*, *semi-$\mathscr{I}^\star$-closed*, *semi-$\mathscr{I}^\star$-preclosed*). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^\star$-closed sets containing A is called the *semi-$\mathscr{I}^\star$-closure* [17] of A and is denoted by $\text{sCl}^\star(A)$ ($s\text{Cl}_{\mathscr{I}^\star}(A)$ [17]). The union of all semi-$\mathscr{I}^\star$-open sets contained in A is called the *semi-$\mathscr{I}^\star$-interior* [17] of A and is denoted by $\text{sInt}^\star(A)$ ($s\text{Int}_{\mathscr{I}^\star}(A)$ [17]).

Lemma 5. [17] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$.

(2) $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\tau^\star$-α-open* [18] (*α-$\mathscr{I}^\star$-open* [19]) if $A \subseteq \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(A)))$. The complement of an $\tau^\star$-α-open set is said to be *$\tau^\star$-α-closed*.

Lemma 6. [19] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) A is α-$\mathscr{I}^\star$-open in X.

(2) $G \subseteq A \subseteq Int^\star(Cl^\star(G))$ for some $\star$-open set G.

(3) $G \subseteq A \subseteq sCl^\star(G)$ for some $\star$-open set G.

(4) $A \subseteq sCl^\star(Int^\star(A))$.

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all α-$\mathscr{I}^\star$-closed sets containing A is called the *α-$\mathscr{I}^\star$-closure* [19] of A and is denoted by $\alpha\text{Cl}^\star(A)$ ($\alpha\text{Cl}_{\mathscr{I}^\star}(A)$ [19]). The *α-$\mathscr{I}^\star$-interior* [19] of A is defined by the union of all α-$\mathscr{I}^\star$-open sets contained in A and is denoted by $\alpha\text{Int}^\star(A)$ ($\alpha\text{Int}_{\mathscr{I}^\star}(A)$ [19]).

Lemma 7. [19] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) A is α-$\mathscr{I}^\star$-closed in X if and only if $sInt^\star(Cl^\star(A)) \subseteq A$.

(2) $sInt^\star(Cl^\star(A)) = Cl^\star(Int^\star(Cl^\star(A)))$.

(3) $\alpha Cl^\star(A) = A \cup Cl^\star(Int^\star(Cl^\star(A)))$.

(4) $\alpha Int^\star(A) = A \cap Int^\star(Cl^\star(Int^\star(A)))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)). A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at each point of X.*

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl($V$);

(3) $x \in \alpha$Int*($F^+((\sigma_1, \sigma_2)$-sCl(V))) for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(4) $x \in$ Int*(Cl*(Int*($F^+((\sigma_1, \sigma_2)$-sCl(V))))) for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, there exists a $\tau^\star$-α-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)) and by Lemma 3, we have $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl($V$).

(2) ⇒ (3): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (2), there exists a $\tau^\star$-α-open set U containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl($V$) and hence $U \subseteq F^+((\sigma_1, \sigma_2)$-sCl($V$)). Thus, $x \in \alpha$Int*($F^+((\sigma_1, \sigma_2)$-sCl(V))).

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then by (3), we have $x \in \alpha$Int*($F^+((\sigma_1, \sigma_2)$-sCl(V))) and by Lemma 7,

$$x \in \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))))).$$

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (4), we have

$$x \in \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))))$$

and by Lemma 7, $x \in \alpha$Int*($F^+((\sigma_1, \sigma_2)$-sCl(V))). Therefore, there exists a $\tau^\star$-α-open set U of X containing x such that $U \subseteq F^+((\sigma_1, \sigma_2)$-sCl($V$)); hence $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl($V$). Since V is $\sigma_1\sigma_2$-open, by Lemma 3 we have $F(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)). This shows that F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at x.

Definition 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-α-open set U of X containing x such that*

$$F(z) \cap \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)) \neq \emptyset$$

for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if F is lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at each point of X.

Theorem 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(z) \cap (\sigma_1, \sigma_2)\text{-}sCl(V) \neq \emptyset$;

(3) $x \in \alpha Int^\star(F^-((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$;

(4) $x \in Int^\star(Cl^\star(Int^\star(F^-((\sigma_1, \sigma_2)\text{-}sCl(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)\text{-}sCl(V)$;

(3) for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq V$;

(4) $F^+(V)$ is $\tau^\star$-α-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;

(5) $F^-(K)$ is $\tau^\star$-α-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(6) $F^+(V) \subseteq \alpha Int^\star(F^+((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\alpha Cl^\star(F^-((\sigma_1, \sigma_2)\text{-}sInt(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(8) $\alpha Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(9) $\alpha Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(10) $Cl^\star(Int^\star(Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(11) $Cl^\star(Int^\star(Cl^\star(F^-((\sigma_1,\sigma_2)\text{-}sInt(K))))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(12) $F^+(V) \subseteq Int^\star(Cl^\star(Int^\star(F^+((\sigma_1,\sigma_2)\text{-}sCl(V)))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): The proof follows from Theorem 1.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let V be any $(\sigma_1,\sigma_2)r$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (3), there exists a $\tau^\star$-α-open set U_x of X containing x such that $F(U_x) \subseteq V$. Thus, $x \in U_x \subseteq F^+(V)$ and so $F^+(V) = \cup_{x\in F^+(V)} U_x$ is $\tau^\star$-α-open in X.

(4) ⇒ (5): This follows from the fact that $F^+(Y-B) = X - F^-(B)$ for every subset B of Y.

(5) ⇒ (6): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, we have

$$F(x) \subseteq V \subseteq (\sigma_1,\sigma_2)\text{-sCl}(V)$$

and so $x \in F^+((\sigma_1,\sigma_2)\text{-sCl}(V)) = X - F^-(Y-(\sigma_1,\sigma_2)\text{-sCl}(V))$. Since $Y-(\sigma_1,\sigma_2)\text{-sCl}(V)$ is $(\sigma_1,\sigma_2)r$-closed in Y and by (5), $F^-(Y-(\sigma_1,\sigma_2)\text{-sCl}(V))$ is $\tau^\star$-α-closed in X. This shows that $F^+((\sigma_1,\sigma_2)\text{-sCl}(V))$ is $\tau^\star$-α-open in X. Thus, $x \in \alpha\text{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))$ and hence $F^+(V) \subseteq \alpha\text{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))$.

(6) ⇒ (7): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y-K$ is $\sigma_1\sigma_2$-open and by (6), we have

$$\begin{aligned} X - F^-(K) = F^+(Y-K) &\subseteq \alpha\text{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(Y-K))) \\ &= \alpha\text{Int}^\star(F^+(Y-(\sigma_1,\sigma_2)\text{-sInt}(K))) \\ &= \alpha\text{Int}^\star(X - F^-((\sigma_1,\sigma_2)\text{-sInt}(K))) \\ &= X - \alpha\text{Cl}^\star(F^-((\sigma_1,\sigma_2)\text{-sInt}(K))) \end{aligned}$$

and hence $\alpha\text{Cl}^\star(F^-((\sigma_1,\sigma_2)\text{-sInt}(K))) \subseteq F^-(K)$.

(7) ⇒ (8): The proof is obvious since $(\sigma_1,\sigma_2)\text{-sInt}(K) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ for every $\sigma_1\sigma_2$-closed set K of Y.

(8) ⇒ (9): The proof is obvious.

(9) ⇒ (10): It follows from Lemma 7 that $\text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(B))) \subseteq \alpha\text{Cl}^\star(B)$ for every subset B of Y. Thus, for every $\sigma_1\sigma_2$-closed set K of Y, we have

$$\begin{aligned} \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))))) &\subseteq \alpha\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \\ &= \alpha\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K). \end{aligned}$$

(10) ⇒ (11): The proof is obvious since $(\sigma_1,\sigma_2)\text{-sInt}(K) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ for every $\sigma_1\sigma_2$-closed set K of Y.

(11) ⇒ (12): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y and by (11), $\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^-((\sigma_1,\sigma_2)\mathrm{sInt}(Y-V))))) \subseteq F^-(Y-V) = X - F^+(V)$. Moreover, we have

$$\begin{aligned}\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^-((\sigma_1,\sigma_2)\text{-sInt}(Y-V))))) &= \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^-(Y-(\sigma_1,\sigma_2)\text{-sCl}(V))))) \\ &= \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(X - F^+((\sigma_1,\sigma_2)\text{-sCl}(V))))) \\ &= X - \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V))))).\end{aligned}$$

Thus, $F^+(V) \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))))$.

(12) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (12), we have $x \in F^+(V) \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))))$ and hence F is upper almost $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous at x by Theorem 1. This shows that F is upper almost $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous.

Definition 3. [20] *A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is said to be upper $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is said to be upper $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous if F is upper $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous at each point of X.*

Definition 4. [20] *A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is said to be lower $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x)\cap V \neq \emptyset$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(z)\cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is said to be lower $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous if F is lower $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous at each point of X.*

Remark 1. *For a multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, the following implication holds:*

upper $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuity ⇒ upper almost $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let $X = \{1,2,3\}$ with a topology $\tau = \{\emptyset,\{1,2\},X\}$ and an ideal $\mathscr{I} = \{\emptyset,\{3\}\}$. Let $Y = \{a,b,c\}$ with topologies $\sigma_1 = \{\emptyset,\{a,b\},Y\}$ and $\sigma_2 = \{\emptyset,\{c\},\{a,b\},Y\}$. A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is defined as follows: $F(1) = \{c\}$ and $F(2) = \{a\}$ and $F(3) = \{a,b\}$. Then F is upper almost $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous but F is not upper $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous.*

Theorem 4. *For a multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^\star\alpha(\sigma_1,\sigma_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x)\cap V \neq \emptyset$, there exists a $\tau^\star$-α-open set U of X containing x such that $U \subseteq F^-((\sigma_1,\sigma_2)\text{-}s\mathit{Cl}(V))$;

(3) for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-α-open set U of X containing x such that $U \subseteq F^-(V)$;

(4) $F^-(V)$ is $\tau^\star$-α-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;

(5) $F^+(K)$ is $\tau^\star$-α-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(6) $F^-(V) \subseteq \alpha Int^\star(F^-((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\alpha Cl^\star(F^+((\sigma_1, \sigma_2)\text{-}sInt(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(8) $\alpha Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(9) $\alpha Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(10) $Cl^\star(Int^\star(Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(11) $Cl^\star(Int^\star(Cl^\star(F^+((\sigma_1, \sigma_2)\text{-}sInt(K))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(12) $F^-(V) \subseteq Int^\star(Cl^\star(Int^\star(F^-((\sigma_1, \sigma_2)\text{-}sCl(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) $\alpha Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\alpha Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y;

(4) $F^+(V) \subseteq \alpha Int^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed in Y. Since F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous, by Theorem 3 we have $F^-(\sigma_1\sigma_2\text{-Cl}(V))$ is $\tau^\star$-α-closed in X and hence

$$\alpha\text{Cl}^\star(F^-(V)) \subseteq \alpha\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(V))) = F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): This is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, K is $(\sigma_1, \sigma_2)s$-open in Y and by (3), we have $\alpha\text{Cl}^\star(F^-(K)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K)$. Thus, $F^-(K)$ is $\tau^\star$-α-closed in X and hence F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous by Theorem 3.

(1) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, we have $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open in Y. Since F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous, by Theorem 3 we have $F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ is $\tau^\star$-α-open in X. Thus,

$$F^+(V) \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) = \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

(4) ⇒ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), $F^+(V) \subseteq \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = \alpha\text{Int}^\star(F^+(V))$. This shows that $F^+(V)$ is $\tau^\star$-α-open in X. It follows from Theorem 3 that F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) $\alpha Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\alpha Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y;

(4) $F^-(V) \subseteq \alpha Int^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 5.

Definition 5. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if $f^{-1}(V)$ is $\tau^\star$-α-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y.*

Corollary 1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau^\star$-α-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)\text{-}sCl(V)$;

(3) for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $f(U) \subseteq V$;

(4) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau^\star$-α-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$;

(5) $f^{-1}(K)$ is $\tau^\star$-α-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(6) $f^{-1}(V) \subseteq \alpha Int^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\alpha Cl^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sInt(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(8) $\alpha Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(9) $\alpha Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(10) $Cl^\star(Int^\star(Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(11) $Cl^\star(Int^\star(Cl^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sInt(K))))) \subseteq f^{-1}(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(12) $f^{-1}(V) \subseteq Int^\star(Cl^\star(Int^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Corollary 2. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is almost* $\tau^\star\alpha(\sigma_1, \sigma_2)$*-continuous;*

(2) $\alpha Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\alpha Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*;*

(4) $f^{-1}(V) \subseteq \alpha Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

4. Conclusion

In this paper, we have introduced new classes of continuous multifunctions defined from an ideal topological space into a bitopological space, namely upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. Also, we have discussed the relationships between $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations and some properties concerning upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions are obtained. The ideas and results of this paper may motivate further research.

Acknowledgements

This research project was financially supported by Mahasarakham University.

References

[1] T. Noiri. Almost α-continuous functions. *Kyungpook Mathematical Journal*, 28:71–77, 1988.

[2] V. Popa and T. Noiri. On upper and lower almost α-continuous multifunctions. *Demonstratio Mathematica*, 24(2):381–396, 1996.

[3] C. Boonpok. On continuous multifunctions in ideal topological spaces. *Lobachevskii Journal of Mathematics*, 40(1):24–35, 2019.

[4] C. Boonpok. On some types of continuity for multifunctions in ideal topological spaces. *Advances in Mathematics: Scientific Journal*, 9(3):859–886, 2020.

[5] C. Boonpok and N. Srisarakham. Almost α-$\star$-continuity for multifunctions. *International Journal of Analysis and Applications*, 21:107, 2023.

[6] C. Boonpok and P. Pue-on. Continuity for multifunctions in ideal topological spaces. *WSEAS Transactions on Mathematics*, 19:624–631, 2020.

[7] P. Pue-on, S. Sompong, and C. Boonpok. Upper and lower (τ_1, τ_2)-continuous multifunctions. *International Journal of Mathematics and Computer Science*, 19(4):1305–1310, 2024.

[8] C. Boonpok and P. Pue-on. Characterizations of almost (τ_1, τ_2)-continuous multifunctions. *International Journal of Analysis and Applications*, 22:33, 2024.

[9] C. Viriyapong and C. Boonpok. $(\tau_1, \tau_2)\alpha$-continuity for multifunctions. *Journal of Mathematics*, 2020:6285763, 2020.

[10] C. Viriyapong, A. Sama-Ae, and C. Boonpok. Almost continuity for multifunctions defined from an ideal topological space into a bitopological space. *European Journal of Pure and Applied Mathematics*, 18(3):6566, 2025.

[11] C. Boonpok, C. Viriyapong, and M. Thongmoon. On upper and lower (τ_1, τ_2)-precontinuous multifunctions. *Journal of Mathematics and Computer Science*, 18:282–293, 2018.

[12] C. Boonpok. $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions. *Heliyon*, 6:e05367, 2020.

[13] P. Pue-on, S. Sompong, and C. Boonpok. Almost quasi (τ_1, τ_2)-continuity for multifunctions. *International Journal of Analysis and Applications*, 22:97, 2024.

[14] C. Klanarong, A. Sama-Ae, and C. Boonpok. Upper and lower almost (τ_1, τ_2)-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 17(2):1244–1253, 2024.

[15] K. Kuratowski. *Topology, Vol. I.* Academic Press, New York, 1966.

[16] D. Janković and T. R. Hamlett. New topologies from old via ideals. *The American Mathematical Monthly*, 97:295–310, 1990.

[17] C. Boonpok. Weak quasi continuity for multifunctions in ideal topological spaces. *Advances in Mathematics: Scientific Journal*, 9(3):339–355, 2020.

[18] T. Noiri and V. Popa. On (mI, nJ)-continuous multifunctions. *Romanian Journal of Mathematics and Computer Science*, 15(1):1–8, 2025.

[19] C. Boonpok. A study of some forms of continuity for multifunctions in ideal topological spaces. *Mathematica*, 63(2):186–198, 2021.

[20] J. Khampakdee, A. Sama-Ae, and C. Boonpok. Upper and lower $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuity. (submitted).