EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 7047
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Weakly $\tau^{\star}\alpha(\sigma_1, \sigma_2)$-Continuous Multifunctions

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Abstract. This paper presents new classes of continuous multifunctions defined between an ideal topological space and a bitopological space, called upper weakly $\tau^{\star}\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^{\star}\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations and some properties concerning upper weakly $\tau^{\star}\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^{\star}\alpha(\sigma_1, \sigma_2)$-continuous multifunctions are considered.

2020 Mathematics Subject Classifications: 54C08, 54C60

Key Words and Phrases: Upper weakly $\tau^{\star}\alpha(\sigma_1, \sigma_2)$-continuous multifunction, lower weakly $\tau^{\star}\alpha(\sigma_1, \sigma_2)$-continuous multifunction

1. Introduction

The notion of weakly α-continuous functions was first introduced by Noiri [1]. Sen and Bhattacharyya [2] investigated several characterizations of weakly α-continuous functions. In 2002, Popa and Noiri [3] extended the concept of α-continuous functions to multifunctions and presented two classes of multifunctions defined between topological spaces, namely upper weakly α-continuous multifunctions and lower weakly α-continuous multifunctions. Furthermore, Popa and Noiri [3] investigated several characterizations and some properties of upper weakly α-continuous multifunctions and lower weakly α-continuous multifunctions. On the other hand, the present author introduced and studied four classes of multifunctions defined from an ideal topological space into an ideal topological space, called upper weakly $\star$-continuous multifunctions [4], lower weakly $\star$-continuous multifunctions [4], upper weakly $\alpha(\star)$-continuous multifunctions [5], lower weakly $\alpha(\star)$-continuous multifunctions [5], upper weakly $s\beta(\star)$-continuous multifunctions [6], lower weakly $s\beta(\star)$-continuous multifunctions [6], weakly $\imath^{\star}$-continuous multifunctions [7] and weakly $p\imath$-continuous multifunctions [8]. Pue-on et al. [9] introduced and investigated two

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.7047

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classes of continuous multifunctions between bitopological spaces, namely upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Thongmoon et al. [10] introduced and studied the notions of upper weakly (τ_1, τ_2)-continuous multifunctions and lower weakly (τ_1, τ_2)-continuous multifunctions. In [11], the present authors introduced and investigated the concepts of upper weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions and lower weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Quite recently, Pue-on et al. [12] presented new classes of continuous multifunctions defined from an ideal topological space into a bitopological space, namely upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. In this paper, we introduce the concepts of continuous multifunctions between an ideal topological space and a bitopological space, called upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called *$\tau_1\tau_2$-closed* [13] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called *$\tau_1\tau_2$-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the *$\tau_1\tau_2$-closure* [13] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [13] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [13] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)r$-open* [11] (resp. *$(\tau_1, \tau_2)s$-open* [14], *$(\tau_1, \tau_2)p$-open* [14], *$(\tau_1, \tau_2)\beta$-open* [14]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be *$(\tau_1, \tau_2)r$-closed* (resp. *$(\tau_1, \tau_2)s$-closed*, *$(\tau_1, \tau_2)p$-closed*, *$(\tau_1, \tau_2)\beta$-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$\tau_1\tau_2$-δ-open* [15] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called *$\tau_1\tau_2$-δ-closed* [15]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the *$\tau_1\tau_2$-δ-interior* [15] of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the *$\tau_1\tau_2$-δ-closure* [15] of A and is denoted by $\tau_1\tau_2$-δ-Cl(A). Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a *$(\tau_1, \tau_2)\theta$-cluster point* [11] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the *$(\tau_1, \tau_2)\theta$-closure* [11] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)\theta$-closed* [11] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be *$(\tau_1, \tau_2)\theta$-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the *$(\tau_1, \tau_2)\theta$-interior* [11] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 2. [11] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [16], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [17] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *R-$\mathscr{I}^\star$-open* [4] (resp. *$\mathscr{I}^\star$-preopen* [4], *semi-$\mathscr{I}^\star$-open* [18], *semi-$\mathscr{I}^\star$-preopen* [18]) if $A = \text{Int}^\star(\text{Cl}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))$). The complement of a R-$\mathscr{I}^\star$-open (resp. $\mathscr{I}^\star$-preopen, semi-$\mathscr{I}^\star$-open, semi-$\mathscr{I}^\star$-preopen) set is said to be *R-$\mathscr{I}^\star$-closed* (resp. *$\mathscr{I}^\star$-preclosed*, *semi-$\mathscr{I}^\star$-closed*, *semi-$\mathscr{I}^\star$-preclosed*). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^\star$-closed sets containing A is called the *semi-$\mathscr{I}^\star$-closure* [18] of A and is denoted by $\text{sCl}^\star(A)$ ($s\text{Cl}_{\mathscr{I}^\star}(A)$ [18]). The union of all semi-$\mathscr{I}^\star$-open sets contained in A is called the *semi-$\mathscr{I}^\star$-interior* [18] of A and is denoted by $\text{sInt}^\star(A)$ ($s\text{Int}_{\mathscr{I}^\star}(A)$ [18]).

Lemma 3. [18] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$.

(2) $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called $\tau^\star$-α-*open* [19] (α-$\mathscr{I}^\star$-*open* [20]) if $A \subseteq \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(A)))$. The complement of a $\tau^\star$-α-open set is called $\tau^\star$-α-*closed.*

Lemma 4. [20] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) A is α-$\mathscr{I}^\star$-open in X.

(2) $G \subseteq A \subseteq Int^\star(Cl^\star(G))$ for some $\star$-open set G.

(3) $G \subseteq A \subseteq sCl^\star(G)$ for some $\star$-open set G.

(4) $A \subseteq sCl^\star(Int^\star(A))$.

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all α-$\mathscr{I}^\star$-closed sets containing A is called the α-$\mathscr{I}^\star$-*closure* [20] of A and is denoted by $\alpha\text{Cl}^\star(A)$ ($\alpha\text{Cl}_{\mathscr{I}^\star}(A)$ [20]). The α-$\mathscr{I}^\star$-*interior* [20] of A is defined by the union of all α-$\mathscr{I}^\star$-open sets contained in A and is denoted by $\alpha\text{Int}^\star(A)$ ($\alpha\text{Int}_{\mathscr{I}^\star}(A)$ [20]).

Lemma 5. [20] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) A is α-$\mathscr{I}^\star$-closed in X if and only if $sInt^\star(Cl^\star(A)) \subseteq A$.

(2) $sInt^\star(Cl^\star(A)) = Cl^\star(Int^\star(Cl^\star(A)))$.

(3) $\alpha Cl^\star(A) = A \cup Cl^\star(Int^\star(Cl^\star(A)))$.

(4) $\alpha Int^\star(A) = A \cap Int^\star(Cl^\star(Int^\star(A)))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous multifunctions discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-$Cl(V)$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if F is upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at each point of X.*

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in \alpha Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(3) $x \in Int^\star(Cl^\star(Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x \in \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(2) ⇒ (3): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Thus by (2), we have $x \in \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and by Lemma 5, $x \in \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))))$.

(3) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (3), we have

$$x \in \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))))$$

and by Lemma 5, $x \in \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. Therefore, there exists a $\tau^\star$-α-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$; hence $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that F is upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at x.

Definition 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is called lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-α-open set U of X containing x such that $\sigma_1\sigma_2\text{-}Cl(V) \cap F(z) \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is called lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if F is lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at each point of X.*

Theorem 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in \alpha Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$;

(3) $x \in Int^\star(Cl^\star(Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(V) \subseteq Int^\star(Cl^\star(Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $Cl^{\star}(Int^{\star}(Cl^{\star}(F^{-}(\sigma_1\sigma_2\text{-}Int(K))))) \subseteq F^{-}(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\alpha Cl^{\star}(F^{-}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^{-}(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(5) $\alpha Cl^{\star}(F^{-}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^{-}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(6) $F^{+}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \alpha Int^{\star}(F^{+}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ *for every subset* B *of* Y*;*

(7) $F^{+}(V) \subseteq \alpha Int^{\star}(F^{+}(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(8) $\alpha Cl^{\star}(F^{-}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^{-}(K)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*;*

(9) $\alpha Cl^{\star}(F^{-}(V)) \subseteq F^{-}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(10) $\alpha Cl^{\star}(F^{-}(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^{-}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ *for every subset* B *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^{+}(V)$. Then, $F(x) \subseteq V$ and there exists a $\tau^{\star}$-α-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence $U \subseteq F^{+}(\sigma_1\sigma_2\text{-Cl}(V))$ and so $x \in U \subseteq \text{Int}^{\star}(\text{Cl}^{\star}(\text{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-Cl}(V)))))$. This shows that $F^{+}(V) \subseteq \text{Int}^{\star}(\text{Cl}^{\star}(\text{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-Cl}(V)))))$.

(2) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2), we have

$$\begin{aligned} X - F^{-}(K) = F^{+}(Y - K) &\subseteq \text{Int}^{\star}(\text{Cl}^{\star}(\text{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-Cl}(Y - K))))) \\ &= \text{Int}^{\star}(\text{Cl}^{\star}(\text{Int}^{\star}(F^{+}(Y - \sigma_1\sigma_2\text{-Int}(K))))) \\ &= \text{Int}^{\star}(\text{Cl}^{\star}(\text{Int}^{\star}(X - F^{-}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &= \text{Int}^{\star}(\text{Cl}^{\star}(X - \text{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &= \text{Int}^{\star}(X - \text{Int}^{\star}(\text{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &= X - \text{Cl}^{\star}(\text{Int}^{\star}(\text{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-Int}(K))))) \end{aligned}$$

and hence $\text{Cl}^{\star}(\text{Int}^{\star}(\text{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-Int}(K))))) \subseteq F^{-}(K)$.

(3) ⇒ (4): Let K be any $\sigma_1\sigma_2$-closed set of Y. By (3), we have

$$\text{Cl}^{\star}(\text{Int}^{\star}(\text{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-Int}(K))))) \subseteq F^{-}(K)$$

and hence $\alpha\text{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^{-}(K)$ by Lemma 5.

(4) ⇒ (5): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by (4), we have $\alpha\text{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^{-}(\sigma_1\sigma_2\text{-Cl}(B))$.

(5) ⇒ (6): Let B be any subset of Y. By (5),

$$\begin{aligned} F^{+}(\sigma_1\sigma_2\text{-Int}(B)) &= X - F^{-}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \alpha\text{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= X - \alpha\text{Cl}^{\star}(F^{-}(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= X - \alpha\text{Cl}^{\star}(X - F^{+}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \end{aligned}$$

$$= \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))).$$

(6) ⇒ (7): The proof is obvious.

(7) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. It follows from Lemma 5 that

$$x \in F^+(V) \subseteq \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))))$$

and hence F is upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at x by Theorem 1. This shows that F is upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous.

(4) ⇒ (8): The proof is obvious.

(8) ⇒ (9): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, we have $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y and by (8),

$$\alpha\text{Cl}^\star(F^-(V)) \subseteq \alpha\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(9) ⇒ (7): Let V be any $\sigma_1\sigma_2$-open set of Y. Thus by (9), we have

$$\begin{aligned} X - \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V))) &= \alpha\text{Cl}^\star(X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \alpha\text{Cl}^\star(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \end{aligned}$$

and hence $F^+(V) \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \alpha\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(9) ⇒ (10): Let B be any subset of Y. Then, $\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B))$ is $\sigma_1\sigma_2$-open in Y. By (9) and Lemma 2,

$$\begin{aligned} \alpha\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \\ &\subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \\ &\subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B)). \end{aligned}$$

(10) ⇒ (8): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (10) and Lemma 2, we have

$$\begin{aligned} \alpha\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \alpha\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &= \alpha\text{Cl}^\star(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \\ &\subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= F^-(K). \end{aligned}$$

Definition 3. [21] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau^\star$-α-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if F is upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at each point of X.*

Definition 4. [21] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-α-open set U of X containing x such that*

$$F(z) \cap \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)) \neq \emptyset$$

for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if F is lower almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous at each point of X.

Remark 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuity $\Rightarrow$ upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, X\}$ and an ideal $\mathscr{I} = \{\emptyset\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Define a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ as follows: $F(1) = \{a\}$, $F(2) = \{b\}$ and $F(3) = \{a, c\}$. Then F is upper weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous but F is not upper almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous.*

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) $F^-(V) \subseteq Int^\star(Cl^\star(Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $Cl^\star(Int^\star(Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\alpha Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(5) $\alpha Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(6) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \alpha Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y;

(7) $F^-(V) \subseteq \alpha Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $\alpha Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(9) $\alpha Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(10) $\alpha Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 3.

Definition 5. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau^\star$-α-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-$Cl(V)$.*

Corollary 1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) $f^{-1}(V) \subseteq \alpha Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\alpha Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(4) $\alpha Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $\alpha Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(6) $Cl^\star(Int^\star(Cl^\star(f^{-1}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $f^{-1}(V) \subseteq Int^\star(Cl^\star(Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $f(Cl^\star(Int^\star(Cl^\star(A)))) \subseteq (\sigma_1, \sigma_2)\theta\text{-}Cl(f(A))$ for every subset A of X;

(9) $Cl^\star(Int^\star(Cl^\star(f^{-1}(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y.

Definition 6. [22] *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if for each $\sigma_1\sigma_2$-open set V of Y, $f^{-1}(V)$ is $\tau^\star$-α-open in X.*

Definition 7. [21] *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous if $f^{-1}(V)$ is $\tau^\star$-α-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y.*

Theorem 5. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ such that $\alpha Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))) \subseteq \alpha Int^\star(f^{-1}(V))$ for every $\sigma_1\sigma_2$-open set V of Y, the following properties are equivalent:*

(1) f is $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(2) f is almost $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous;

(3) f is weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous.

Proof. We prove only the implication (3) ⇒ (1). Suppose that f is weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous. Let V be any $\sigma_1\sigma_2$-open set of Y. Since f is weakly $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous, by Corollary 1, we have $f^{-1}(V) \subseteq \alpha\text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence

$$f^{-1}(V) \subseteq \alpha\text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \alpha\text{Int}^\star(f^{-1}(V)).$$

Thus, $f^{-1}(V)$ is $\tau^\star$-α-open in X. This shows that f is $\tau^\star\alpha(\sigma_1, \sigma_2)$-continuous.

Acknowledgements

This research project was financially supported by Mahasarakham University.

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