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Upper and Lower $\tau^\star\beta(\sigma_1, \sigma_2)$-Continuity

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Abstract. A new class of continuous multifunctions between an ideal topological space and a bitopological space, called upper (lower) $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions, has been defined and studied. Furthermore, several characterizations and some properties concerning upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

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1. Introduction

The notion of β-continuous functions was introduced by Abd El-Monsef et al. [1]. Borsík and Doboš [2] introduced the concept of almost quasicontinuity which is weaker than that of quasicontinuity [3]. Popa and Noiri [4] investigated several characterizations of β-continuity and shown that almost quasi-continuity is equivalent to β-continuity. The equivalence of almost quasicontinuity and β-continuity is also shown by Borsík [5] and Ewert [6]. In 1996-1997, Popa and Noiri [7] extended the concept of β-continuous functions to multifunctions and presented new classes of multifunctions defined from a topological space into a topological space, namely upper β-continuous multifunctions and lower β-continuous multifunctions. Moreover, Popa and Noiri [7] investigated several characterizations and some properties concerning upper β-continuous multifunctions and lower β-continuous multifunctions. On the other hand, the present author introduced and investigated four classes of multifunctions defined from an ideal topological space into an ideal topological space, namely upper $\star$-continuous multifunctions [8], lower $\star$-continuous

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multifunctions [8], upper $\beta(\star)$-continuous multifunctions [9], lower $\beta(\star)$-continuous multifunctions [9], upper $s\beta(\star)$-continuous multifunctions [10], lower $s\beta(\star)$-continuous multifunctions [10], upper α-$\star$-continuous multifunctions [11], lower α-$\star$-continuous multifunctions [11], $\imath^\star$-continuous multifunctions [12] and $p\imath$-continuous multifunctions [13]. Pue-on et al. [14] introduced and studied two classes of multifunctions between bitopological spaces, namely upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [15] investigated several characterizations of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions by utilizing the notions of $(\tau_1, \tau_2)\theta$-closed sets and $(\tau_1, \tau_2)\theta$-open sets. Thongmoon et al. [16] studied some characterizations of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions by using $\tau_1\tau_2$-δ-open sets and $\tau_1\tau_2$-δ-closed sets. Laprom et al. [17] introduced and investigated the notions of upper $\beta(\tau_1, \tau_2)$-continuous multifunctions and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions. In this paper, we introduce the concepts of continuous multifunctions between an ideal topological space and a bitopological space, called upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [18] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [18] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [18] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [18] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [19] (resp. $(\tau_1, \tau_2)s$-*open* [20], $(\tau_1, \tau_2)p$-*open* [20], $(\tau_1, \tau_2)\beta$-*open* [20]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$).

The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-δ-*open* [21] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-δ-*closed* [21]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-δ-*interior* [21] of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ-*closure* [21] of A and is denoted by $\tau_1\tau_2$-δ-Cl(A). Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [19] of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [19] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [19] if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [19] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*nowhere dense* if $\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Int}(A)) = \emptyset$.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [22], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [23] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *R-*$\mathscr{I}^\star$*-open* [8] (resp. $\mathscr{I}^\star$-*preopen* [8], $\tau^\star$-*semi-open* [24] (*semi-*$\mathscr{I}^\star$*-open* [25]), $\tau^\star$-β-*open* [24] (*semi-*$\mathscr{I}^\star$*-preopen* [25])) if $A = \text{Int}^\star(\text{Cl}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))$). The complement of a *R-*$\mathscr{I}^\star$-open (resp. $\mathscr{I}^\star$-preopen, semi-$\mathscr{I}^\star$-open, $\tau^\star$-β-open) set is said to be *R-*$\mathscr{I}^\star$*-closed* (resp. $\mathscr{I}^\star$-*preclosed*, $\tau^\star$-*semi-closed*, $\tau^\star$-β-*closed*). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^\star$-closed sets containing A is called the *semi-*$\mathscr{I}^\star$*-closure* [25] of A and is denoted by $\text{sCl}^\star(A)$ ($s\text{Cl}_{\mathscr{I}^\star}(A)$ [25]). The union of all semi-$\mathscr{I}^\star$-open sets contained in A is called the *semi-*$\mathscr{I}^\star$*-interior* [25] of A and is denoted by $\text{sInt}^\star(A)$ ($s\text{Int}_{\mathscr{I}^\star}(A)$ [25]). The intersection of all β-$\mathscr{I}^\star$-closed sets containing A is called the β-$\mathscr{I}^\star$-*closure* of A and is denoted by $\beta\text{Cl}^\star(A)$. The union of all β-$\mathscr{I}^\star$-open sets contained in A is called the β-$\mathscr{I}^\star$-*interior* of A and is denoted by $\beta\text{Int}^\star(A)$.

Lemma 2. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$ [25].

(2) $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$ [25].

(3) $\beta Cl^\star(A) = A \cup Int^\star(Cl^\star(Int^\star(A)))$.

(4) $\beta Int^\star(A) = A \cap Cl^\star(Int^\star(Cl^\star(A)))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A}F(x)$.

3. Upper and lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau^\star$-β-open set U of X containing x such that $F(U) \subseteq V$. A multifunction*

$$F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$$

is said to be upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if F is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of X.

Theorem 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at $x \in X$ if and only if $x \in \beta Int^\star(F^+(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, there exists a $\tau^\star$-β-open set U of X containing x such that $F(U) \subseteq V$. Then, $U \subseteq F^+(V)$. Since U is $\tau^\star$-β-open, we have $x \in U \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(U))) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(V))))$. Since $x \in F^+(V)$ and by Lemma 2, $x \in F^+(V) \cap \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(V)))) = \beta\mathrm{Int}^\star(F^+(V))$.

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (2), $x \in s\beta\mathrm{Int}^\star(F^+(V))$ and so there exists a $\tau^\star$-β-open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at x.

Definition 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-β-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if F is lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of X.*

Theorem 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at $x \in X$ if and only if $x \in \beta Int^\star(F^-(V))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 1.

Definition 3. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is called $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau^\star$-β-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is called $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if f is $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of X.*

Corollary 1. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at $x \in X$ if and only if $x \in \beta Int^\star(f^{-1}(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.*

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$;

(2) for each $\star$-open neighborhood U of x and each $\sigma_1\sigma_2$-open set V of Y with $x \in F^+(V)$, $F^+(V) \cap U$ is not $\tau_1\tau_2$-nowhere dense;

(3) for each $\star$-open neighborhood U of x and each $\sigma_1\sigma_2$-open set V of Y with $x \in F^+(V)$, there exists a $\star$-open set G of X such that $\emptyset \neq G \subseteq U$ and $G \subseteq Cl^\star(F^+(V))$;

(4) for each $\sigma_1\sigma_2$-open set V of Y with $x \in F^+(V)$, there exists a $\tau^\star$-semi-open set U of X containing x such that $U \subseteq Cl^\star(F^+(V))$;

(5) $x \in Cl^\star(Int^\star(Cl^\star(F^+(V))))$ for every $\sigma_1\sigma_2$-open set V of Y with $x \in F^+(V)$.

Proof. (1) ⇒ (2) and (2) ⇒ (3): The proof are obvious.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By $\mathscr{U}^\star(x)$ we denote the family of all $\star$-open neighborhood of x. For each $U \in \mathscr{U}^\star(x)$, there exists a $\star$-open set G_U of X such that $\emptyset \neq G_U \subseteq U$ and $G_U \subseteq \mathrm{Cl}^\star(F^+(V))$. Put $W = \cup\{G_U \mid U \in \mathscr{U}^\star(x)\}$. Then, W is a $\star$-open set of X, $x \in \mathrm{Cl}^\star(W)$ and $W \subseteq \mathrm{Cl}^\star(F^+(V))$. Moreover, we put $U_0 = W \cup \{x\}$. Then, $W \subseteq U_0 \subseteq \mathrm{Cl}^\star(W)$ and U_0 is a $\tau^\star$-semi-open set of X containing x and also $U_0 \subseteq \mathrm{Cl}^\star(F^+(V))$.

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. There exists a $\tau^\star$-semi-open set U of X containing x such that $U \subseteq \mathrm{Cl}^\star(F^+(V))$. Thus,

$$x \in U \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(U)) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(V)))).$$

(5) ⇒ (1): By utilizing Lemma 2, this can be proved similarly to that of Theorem 1.

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$;*

(2) *for each $\star$-open neighborhood U of x and each $\sigma_1\sigma_2$-open set V of Y with $x \in F^-(V)$, $F^-(V) \cap U$ is not $\tau_1\tau_2$-nowhere dense;*

(3) *for each $\star$-open neighborhood U of x and each $\sigma_1\sigma_2$-open set V of Y with $x \in F^-(V)$, there exists a $\star$-open set G of X such that $\emptyset \neq G \subseteq U$ and $G \subseteq Cl^\star(F^-(V))$;*

(4) *for each $\sigma_1\sigma_2$-open set V of Y with $x \in F^-(V)$, there exists a $\tau^\star$-semi-open set U of X containing x such that $U \subseteq Cl^\star(F^-(V))$;*

(5) *$x \in Cl^\star(Int^\star(Cl^\star(F^-(V))))$ for every $\sigma_1\sigma_2$-open set V of Y with $x \in F^-(V)$.*

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$;*

(2) *for each $\star$-open neighborhood U of x and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, $f^{-1}(V) \cap U$ is not $\tau_1\tau_2$-nowhere dense;*

(3) *for each $\star$-open neighborhood U of x and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\star$-open set G of X such that $\emptyset \neq G \subseteq U$ and $G \subseteq Cl^\star(f^{-1}(V))$;*

(4) *for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau^\star$-semi-open set U of X containing x such that $U \subseteq Cl^\star(f^{-1}(V))$;*

(5) *$x \in Cl^\star(Int^\star(Cl^\star(f^{-1}(V))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.*

Theorem 5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

(2) *$F^+(V)$ is $\tau^\star$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;*

(3) *$F^-(K)$ is $\tau^\star$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;*

(4) *$\beta Cl^\star(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;*

(5) *$Int^\star(Cl^\star(Int^\star(F^-(B)))) \subseteq F^-(\sigma_1\sigma_2 Cl(B))$ for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. There exists a $\tau^\star$-β-open set U of X containing x such that $F(U) \subseteq V$. Thus,

$$x \in U \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(U))) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(V))))$$

and hence $F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(V))))$. This shows that $F^+(V)$ is $\tau^\star$-β-open in X.

(2) ⇒ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): For any subset B of Y, $\sigma_1\sigma_2$-Cl(B) is $\sigma_1\sigma_2$-closed in Y and by (3), we have $F^-(\sigma_1\sigma_2\text{-Cl}(B))$ is $\tau^\star$-β-closed in X. Thus, $\beta\text{Cl}^\star(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. By (4) and Lemma 2,

$$\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-(B)))) \subseteq \beta\text{Cl}^\star(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)).$$

(5) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y and by (5),

$$\begin{aligned} X - F^+(V) = F^-(Y - V) &\supseteq \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-(Y - V)))) \\ &= \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(X - F^+(V)))) \\ &= X - \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+(V)))). \end{aligned}$$

Thus, $F^+(V) \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+(V))))$ and so $F^+(V)$ is $\tau^\star$-β-open in X.

(2) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (2), we have $F^+(V)$ is $\tau^\star$-β-open in X. Put $U = F^+(V)$. Then, U is a $\tau^\star$-β-open set of X containing x such that $F(U) \subseteq V$. This shows that F is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;

(2) $F^-(V)$ is $\tau^\star$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^+(K)$ is $\tau^\star$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\beta Cl^\star(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $Int^\star(Cl^\star(Int^\star(F^+(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(6) $F(Int^\star(Cl^\star(Int^\star(A)))) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ for every subset A of X;

(7) $F(\beta Cl^\star(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ for every subset A of X.

Proof. It is shown similarly to the proof of Theorem 5 that the statements (1), (2), (3), (4) and (5) are equivalent. We shall prove only the following implications.

(5) ⇒ (6): Let A be any subset of X. By (5), we have

$$\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^+(F(A))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(F(A)))$$

and hence $F(\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(A)))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$.

(6) ⇒ (7): Let A be any subset of X. By (6) and Lemma 2, we have

$$F(\beta\text{Cl}^\star(A)) = F(A \cup \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(A))))$$

$$= F(A) \cup F(\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(A))))$$
$$\subseteq \sigma_1\sigma_2\text{-Cl}(F(A)).$$

(7) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Thus by (7), $F(\beta\text{Cl}^\star(F^+(K))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(F^+(K))) \subseteq \sigma_1\sigma_2\text{-Cl}(K) = K$. Thus, $\beta\text{Cl}^\star(F^+(K)) \subseteq F^+(K)$ and hence $F^+(K)$ is $\tau^\star$-β-closed in X.

Corollary 3. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;

(2) $f^{-1}(V)$ is $\tau^\star$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $f^{-1}(K)$ is $\tau^\star$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\beta Cl^\star(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $Int^\star(Cl^\star(Int^\star(f^{-1}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(6) $f(Int^\star(Cl^\star(Int^\star(A)))) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ for every subset A of X;

(7) $f(\beta Cl^\star(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ for every subset A of X.

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