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Almost $\tau^\star\beta(\sigma_1, \sigma_2)$-Continuity for Multifunctions

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Abstract. This paper introduces new classes of continuous multifunctions defined between an ideal topological space and a bitopological space, called upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations and some properties concerning upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions are established.

2020 Mathematics Subject Classifications: 54C08, 54C60

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1. Introduction

In 1997, Nasef and Noiri [1] introduced and investigated two classes of functions defined between topological spaces, namely almost precontinuous functions and almost β-continuous functions by utilizing the notions of preopen sets and β-open sets due to Mashhour et al. [2] and Abd El-Monsef et al. [3], respectively. In 1998, Noiri and Popa [4] investigated several characterizations and some properties of almost β-continuous functions. Noiri [5] introduced the concept of almost α-continuous functions and proved that the notions of almost feeble continuity [6] and almost α-continuity are equivalent. The class of almost precontinuity is a generalization of almost α-continuity and almost feeble continuity. The class of almost β-continuity is a generalization of almost quasi-continuity [7]. In 1999, Noiri and Popa [8] extended the concept of almost β-continuous functions to multifunctions and introduced new classes of multifunctions defined between topological spaces, namely upper almost β-continuous multifunctions and lower almost β-continuous multifunctions. Furthermore, Noiri and Popa [8] investigated several characterizations and some properties concerning upper almost β-continuous multifunctions and lower almost

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β-continuous multifunctions. On the other hand, the present author introduced and investigated classes of continuous multifunctions defined from an ideal topological space into an ideal topological space, namely upper almost $\star$-continuous multifunctions [9], lower almost $\star$-continuous multifunctions [9], upper almost α-$\star$-continuous multifunctions [10], lower almost α-$\star$-continuous multifunctions [10], upper almost $\beta(\star)$-continuous multifunctions [11], lower almost $\beta(\star)$-continuous multifunctions [11], upper almost $s\beta(\star)$-continuous multifunctions [12], lower almost $s\beta(\star)$-continuous multifunctions [12] and almost $\imath^\star$-continuous multifunctions [13]. Pue-on et al. [14] introduced and studied two classes of multifunctions between bitopological spaces, called upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Moreover, Boonpok and Pue-on [15] introduced and investigated the concepts of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. Laprom et al. [16] introduced and studied the notions of upper almost $\beta(\tau_1, \tau_2)$-continuous multifunctions and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. In this paper, we introduce the concepts of continuous multifunctions between an ideal topological space and a bitopological space, called upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [17] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [17] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [17] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [17] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-Cl}(A)$ and $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\tau_2\text{-Cl}(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-Cl}(A)$.

(5) $\tau_1\tau_2\text{-Cl}(X - A) = X - \tau_1\tau_2\text{-Int}(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* [18] (resp. $(\tau_1, \tau_2)s$-*open* [19], $(\tau_1, \tau_2)p$-*open* [19], $(\tau_1, \tau_2)\beta$-*open* [19]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq$

$\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A)), $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A)), $A \subseteq \tau_1\tau_2$-Cl($\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A)))). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the $(\tau_1, \tau_2)s$-*closure* [19] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the $(\tau_1, \tau_2)s$-*interior* [19] of A and is denoted by (τ_1, τ_2)-sInt(A).

Lemma 2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) (τ_1, τ_2)-*sCl*$(A) = \tau_1\tau_2$-*Int*$(\tau_1\tau_2$-*Cl*$(A)) \cup A$ [19];

(2) (τ_1, τ_2)-*sInt*$(A) = \tau_1\tau_2$-*Cl*$(\tau_1\tau_2$-*Int*$(A)) \cap A$ [20].

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [21], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [22] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *R-$\mathscr{I}^\star$-open* [9] (resp. *$\mathscr{I}^\star$-preopen* [9], *$\tau^\star$-semi-open* [23] (*semi-$\mathscr{I}^\star$-open* [15]), *$\tau^\star$-β-open* [23] (*semi-$\mathscr{I}^\star$-preopen* [15])) if $A = \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$ (resp. $A \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$, $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(A))$, $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(A)))$). The complement of a R-$\mathscr{I}^\star$-open (resp. $\mathscr{I}^\star$-preopen, $\tau^\star$-semi-open, $\tau^\star$-β-open) set is said to be *R-$\mathscr{I}^\star$-closed* (resp. *$\mathscr{I}^\star$-preclosed*, *$\tau^\star$-semi-closed*, *$\tau^\star$-β-closed*). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^\star$-closed sets containing A is called the *semi-$\mathscr{I}^\star$-closure* [15] of A and is denoted by $\mathrm{sCl}^\star(A)$ ($s\mathrm{Cl}_{\mathscr{I}^\star}(A)$ [15]). The union of all semi-$\mathscr{I}^\star$-open sets contained in A is called the *semi-$\mathscr{I}^\star$-interior* [15] of A and is denoted by $\mathrm{sInt}^\star(A)$ ($s\mathrm{Int}_{\mathscr{I}^\star}(A)$ [15]). The intersection of all β-$\mathscr{I}^\star$-closed sets containing A is called the *β-$\mathscr{I}^\star$-closure* of A and is denoted by $\beta\mathrm{Cl}^\star(A)$. The union of all β-$\mathscr{I}^\star$-open sets contained in A is called the *β-$\mathscr{I}^\star$-interior* of A and is denoted by $\beta\mathrm{Int}^\star(A)$.

Lemma 3. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$ [15].

(2) $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$ [15].

(3) $\beta Cl^\star(A) = A \cup Int^\star(Cl^\star(Int^\star(A)))$.

(4) $\beta Int^\star(A) = A \cap Cl^\star(Int^\star(Cl^\star(A)))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau^\star$-β-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)). A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if F is upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of X.*

Theorem 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at $x \in X$ if and only if $x \in \beta Int^\star(F^+((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, there exists a $\tau^\star$-β-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) = (\sigma_1, \sigma_2)\text{-sCl}(V)$; hence $U \subseteq F^+((\sigma_1, \sigma_2)\text{-sCl}(V))$. Since U is $\tau^\star$-β-open, we have

$$x \in U \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(U))) \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))))).$$

Since $x \in F^+(V) \subseteq F^+((\sigma_1, \sigma_2)\text{-sCl}(V))$ and by Lemma 3,

$$x \in F^+((\sigma_1, \sigma_2)\text{-sCl}(V)) \cap \text{Cl}^\star(\text{Int}(\text{Cl}^\star((\sigma_1, \sigma_2)\text{-sCl}(V)))) = \beta\text{Int}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))).$$

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, we have

$$x \in \beta\text{Int}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))$$

and so there exists a $\tau^\star$-β-open set U of X containing x such that $U \subseteq F^+((\sigma_1, \sigma_2)\text{-sCl}(V))$; hence $F(U) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V) = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. This shows that F is upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at x.

Definition 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-β-open set U of X containing x such that*

$$F(z) \cap \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)) \neq \emptyset$$

for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if F is lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of X.

Theorem 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at $x \in X$ if and only if $x \in \beta Int^\star(F^-((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 1.

Definition 3. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau^\star$-β-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if f is $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of X.*

Corollary 1. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at $x \in X$ if and only if $x \in \beta Int^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.*

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau^\star$-β-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)\text{-}sCl(V)$;

(3) for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$, there exists a $\tau^\star$-β-open set U of X containing x such that $F(U) \subseteq V$;

(4) $F^+(V)$ is $\tau^\star$-β-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;

(5) $F^-(K)$ is $\tau^\star$-β-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(6) $F^+(V) \subseteq \beta Int^\star(F^+((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\beta Cl^\star(F^-((\sigma_1, \sigma_2)\text{-}sInt(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(8) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(9) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(10) $Int^\star(Cl^\star(Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(11) $Int^\star(Cl^\star(Int^\star(F^-((\sigma_1,\sigma_2)\text{-}sInt(K))))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(12) $F^+(V) \subseteq Cl^\star(Int^\star(Cl^\star(F^+((\sigma_1,\sigma_2)\text{-}sCl(V)))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2) and (2) ⇒ (3): The proofs are obvious.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and so there exists a $\tau^\star$-β-open set U_x of X containing x such that $F(U_x) \subseteq V$. Thus, $x \in U_x \subseteq F^+(V)$ and hence $F^+(V) = \cup_{x\in F^+(V)} U_x$. This shows that $F^+(V)$ is $\tau^\star$-β-open in X.

(4) ⇒ (5): This follows from the fact that $F^+(Y - B) = Y - F^-(B)$ for every subset B of Y.

(5) ⇒ (6): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V \subseteq (\sigma_1,\sigma_2)\text{-sCl}(V)$ and hence $x \in F^+((\sigma_1,\sigma_2)\text{-sCl}(V)) = X - F^-(Y - (\sigma_1,\sigma_2)\text{-sCl}(V))$. Since $Y - (\sigma_1,\sigma_2)\text{-sCl}(V)$ is $(\sigma_1,\sigma_2)r$-closed, we have $F^-(Y - (\sigma_1,\sigma_2)\text{-sCl}(V))$ is $\tau^\star$-β-closed in X. Thus, $F^+((\sigma_1,\sigma_2)\text{-sCl}(V))$ is a $\tau^\star$-β-open set of X containing x and so $x \in \beta\text{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))$. This shows that $F^+(V) \subseteq \beta\text{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))$.

(6) ⇒ (7): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, since $Y - K$ is $\sigma_1\sigma_2$-open and by (6),

$$\begin{aligned} X - F^-(K) = F^+(Y - K) &\subseteq \beta\text{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(Y - K))) \\ &= \beta\text{Int}^\star(F^+(Y - (\sigma_1,\sigma_2)\text{-sInt}(K))) \\ &= \beta\text{Int}^\star(X - F^-((\sigma_1,\sigma_2)\text{-sInt}(K))) \\ &= X - \beta\text{Cl}^\star(F^-((\sigma_1,\sigma_2)\text{-sInt}(K))). \end{aligned}$$

Thus, $\beta\text{Cl}^\star(F^-((\sigma_1,\sigma_2)\text{-sInt}(K))) \subseteq F^-(K)$.

(7) ⇒ (8): The proof is obvious since $(\sigma_1,\sigma_2)\text{-sInt}(K) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ for every $\sigma_1\sigma_2$-closed set K of Y.

(8) ⇒ (9): The proof is obvious.

(9) ⇒ (10): By (9) and Lemma 3,

$$\begin{aligned} \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))))) &\subseteq \beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \\ &\subseteq \beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K). \end{aligned}$$

(10) ⇒ (11): The proof is obvious since $(\sigma_1,\sigma_2)\text{-sInt}(K) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ for every $\sigma_1\sigma_2$-closed set K of Y.

(11) ⇒ (12): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y and by (11),

$$\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-((\sigma_1,\sigma_2)\text{-sInt}(Y - V))))) \subseteq F^-(Y - V) = X - F^+(V).$$

Moreover, we have

$$\begin{aligned}\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^-((\sigma_1,\sigma_2)\text{-sInt}(Y-V))))) &= \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^-(Y-(\sigma_1,\sigma_2)\text{-sCl}(V)))))\\ &= \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(X-F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))))\\ &= X-\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V))))).\end{aligned}$$

Thus, $F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))))$.

(12) ⇒ (1): Let x be any point of X and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, we have $x \in F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))))$ and hence

$$x \in \beta\mathrm{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sCl}(V))).$$

Thus, F is upper almost $\tau^\star\beta(\sigma_1,\sigma_2)$-continuous at x by Theorem 1.

Theorem 4. *For a multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^\star\beta(\sigma_1,\sigma_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-β-open set U of X containing x such that $U \subseteq F^-((\sigma_1,\sigma_2)\text{-sCl}(V))$;

(3) for each $x \in X$ and each $(\sigma_1,\sigma_2)r$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-β-open set U of X containing x such that $U \subseteq F^-(V)$;

(4) $F^-(V)$ is $\tau^\star$-β-open in X for every $(\sigma_1,\sigma_2)r$-open set V of Y;

(5) $F^+(K)$ is $\tau^\star$-β-closed in X for every $(\sigma_1,\sigma_2)r$-closed set K of Y;

(6) $F^-(V) \subseteq \beta\mathrm{Int}^\star(F^-((\sigma_1,\sigma_2)\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\beta\mathrm{Cl}^\star(F^+((\sigma_1,\sigma_2)\text{-sInt}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(8) $\beta\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(9) $\beta\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;

(10) $\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(11) $\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1,\sigma_2)\text{-sInt}(K))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(12) $F^-(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^-((\sigma_1,\sigma_2)\text{-sCl}(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function $f : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) f is almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;

(2) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau^\star$-β-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$;

(3) for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\tau^\star$-β-open set U of X containing x such that $f(U) \subseteq V$;

(4) $f^{-1}(V)$ is $\tau^\star$-β-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;

(5) $f^{-1}(K)$ is $\tau^\star$-β-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(6) $f^{-1}(V) \subseteq \beta Int^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\beta Cl^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sInt(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(8) $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(9) $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(10) $Int^\star(Cl^\star(Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(11) $Int^\star(Cl^\star(Int^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sInt(K))))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(12) $f^{-1}(V) \subseteq Cl^\star(Int^\star(Cl^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))))$ for every $\sigma_1\sigma_2$-open set V of Y.

Definition 4. [24] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau^\star$-β-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if F is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of X.*

Definition 5. [24] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point x of X if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-β-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if F is lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of X.*

Remark 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuity $\Rightarrow$ upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let* $X = \{1, 2, 3\}$ *with a topology* $\tau = \{\emptyset, X\}$ *and an ideal* $\mathscr{I} = \{\emptyset\}$*. Let* $Y = \{a, b, c\}$ *with topologies* $\sigma_1 = \{\emptyset, \{b\}, Y\}$ *and* $\sigma_2 = \{\emptyset, \{b\}, \{a, b\}, Y\}$*. A multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is defined as follows:* $F(1) = \{b\}$ *and* $F(2) = F(3) = \{a, c\}$*. Then,* F *is upper almost* $\tau^\star\beta(\sigma_1, \sigma_2)$*-continuous but* F *is not upper* $\tau^\star\beta(\sigma_1, \sigma_2)$*-continuous, since* $\{a, c\}$ *is* $\sigma_1\sigma_2$*-open in* Y *but* $F^+(\{a, c\})$ *is not* $\tau^\star$*-*β*-open in* X*.*

Theorem 5. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper almost* $\tau^\star\beta(\sigma_1, \sigma_2)$*-continuous;*

(2) $\beta Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\beta Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*;*

(4) $F^+(V) \subseteq \beta Int^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Since $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed, by Theorem 3 we have $F^-(\sigma_1\sigma_2\text{-Cl}(V))$ is $\tau^\star$-β-closed in X and hence

$$\beta\text{Cl}^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): This is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and $Y - V \supseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V))$. Since $\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V))$ is $(\sigma_1, \sigma_2)s$-open in Y and by (3),

$$\begin{aligned} X - F^+(V) = F^-(Y - V) &\supseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V))) \\ &\supseteq \beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V)))) \\ &= \beta\text{Cl}^\star(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \beta\text{Cl}^\star(X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= X - \beta\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))). \end{aligned}$$

Thus, $F^+(V) \subseteq \beta\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(4) ⇒ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), $F^+(V) \subseteq \beta\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = \beta\text{Int}^\star(F^+(V))$ and hence $F^+(V)$ is $\tau^\star$-β-open in X. It follows from Theorem 3 that F is upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower almost* $\tau^\star\beta(\sigma_1, \sigma_2)$*-continuous;*

(2) $\beta Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\beta Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*;*

(4) $F^-(V) \subseteq \beta Int^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y.

Proof. The proof is similar to that of Theorem 5.

Corollary 3. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is almost* $\tau^\star\beta(\sigma_1, \sigma_2)$*-continuous;*

(2) $\beta Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\beta Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*;*

(4) $f^{-1}(V) \subseteq \beta Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y.

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