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# On Upper and Lower Weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-Continuous Multifunctions

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**Abstract.** A new class of continuous multifunctions between an ideal topological space and a bitopological space, called upper (lower) weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunctions, has been defined and studied. Furthermore, several characterizations and some properties concerning upper weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

**2020 Mathematics Subject Classifications**: 54C08, 54C60

**Key Words and Phrases**: Upper weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunction, lower weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunction

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## 1. Introduction

In 1983, Abd El-Monsef et al. [1] introduced and studied the notion of $\beta$-continuous functions. Nasef and Noiri [2] defined almost $\beta$-continuous functions by utilizing the notion of $\beta$-open sets due to Abd El-Monsef et al. [1]. In 1994, Popa and Noiri [3] introduced the concept of weakly $\beta$-continuous functions and obtained some characterizations of such functions. The class of almost $\beta$-continuity is a generalization of $\beta$-continuity and the class of weak $\beta$-continuity is a generalization of almost $\beta$-continuity. In 1999, Popa and Noiri [4] introduced new classes of multifunctions defined from a topological space into a topological space, namely upper weakly $\beta$-continuous multifunctions and lower weakly $\beta$-continuous multifunctions. Furthermore, Popa and Noiri [5] investigated several characterizations and some properties of upper weakly $\beta$-continuous multifunctions and lower weakly $\beta$-continuous multifunctions. On the other hand, the present author introduced and investigated four classes of multifunctions defined from an ideal topological space into an ideal topological space, namely upper weakly $\star$-continuous multifunctions [6],

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lower weakly $\star$-continuous multifunctions [6], upper weakly $s\beta(\star)$-continuous multifunctions [7], lower weakly $s\beta(\star)$-continuous multifunctions [7], upper weakly $\alpha$-$\star$-continuous multifunctions [8], lower weakly $\alpha$-$\star$-continuous multifunctions [8], weakly $\imath^{\star}$-continuous multifunctions [9] and weakly $p\imath$-continuous multifunctions [10]. Pue-on et al. [11] introduced and studied two classes of continuous multifunctions between bitopological spaces, namely upper $(\tau_1, \tau_2)$-continuous multifunctions and lower $(\tau_1, \tau_2)$-continuous multifunctions. Thongmoon et al. [12] introduced and studied the notions of upper weakly $(\tau_1, \tau_2)$-continuous multifunctions and lower weakly $(\tau_1, \tau_2)$-continuous multifunctions. In this paper, we introduce the concepts of continuous multifunctions between an ideal topological space and a bitopological space, called upper weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^{\star}\beta(\sigma_1, \sigma_2)$-continuous multifunctions.

## 2. Preliminaries

Throughout the present paper, spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ (or simply $X$ and $Y$) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. The closure of $A$ and the interior of $A$ with respect to $\tau_i$ are denoted by $\tau_i$-Cl$(A)$ and $\tau_i$-Int$(A)$, respectively, for $i = 1, 2$. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-*closed* [13] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of $X$ containing $A$ is called the $\tau_1\tau_2$-*closure* [13] of $A$ and is denoted by $\tau_1\tau_2$-Cl$(A)$. The union of all $\tau_1\tau_2$-open sets of $X$ contained in $A$ is called the $\tau_1\tau_2$-*interior* [13] of $A$ and is denoted by $\tau_1\tau_2$-Int$(A)$.

**Lemma 1.** [13] *Let $A$ and $B$ be subsets of a bitopological space $(X, \tau_1, \tau_2)$. For the $\tau_1\tau_2$-closure, the following properties hold:*

*(1) $A \subseteq \tau_1\tau_2$-Cl$(A)$ and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl$(A)$.*

*(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*

*(3) $\tau_1\tau_2$-Cl$(A)$ is $\tau_1\tau_2$-closed.*

*(4) $A$ is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl$(A)$.*

*(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)r$-*open* [14] (resp. $(\tau_1, \tau_2)s$-*open* [15], $(\tau_1, \tau_2)p$-*open* [15], $(\tau_1, \tau_2)\beta$-*open* [15]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). The intersection of all $(\tau_1, \tau_2)s$-closed sets of $X$ containing $A$ is called the $(\tau_1, \tau_2)s$-*closure* [15] of $A$ and is denoted by $(\tau_1, \tau_2)$-sCl$(A)$. The union of all $(\tau_1, \tau_2)s$-open sets of $X$ contained in $A$ is called the $(\tau_1, \tau_2)s$-*interior* [15] of $A$ and is denoted by $(\tau_1, \tau_2)$-sInt$(A)$.

**Lemma 2.** *For a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$, the following properties hold:*

*(1)* $(\tau_1, \tau_2)\text{-}sCl(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) \cup A$ [15]*;*

*(2)* $(\tau_1, \tau_2)\text{-}sInt(A) = \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \cap A$ [16].

Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* [14] of $A$ if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set $U$ containing $x$. The set of all $(\tau_1, \tau_2)\theta$-cluster points of $A$ is called the $(\tau_1, \tau_2)\theta$*-closure* [14] of $A$ and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(\tau_1, \tau_2)\theta$*-closed* [14] if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of $X$ contained in $A$ is called the $(\tau_1, \tau_2)\theta$*-interior* [14] of $A$ and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$.

**Lemma 3.** [14] *For a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$, the following properties hold:*

*(1) If $A$ is $\tau_1\tau_2$-open in $X$, then $\tau_1\tau_2\text{-}Cl(A) = (\tau_1, \tau_2)\theta\text{-}Cl(A)$.*

*(2) $(\tau_1, \tau_2)\theta\text{-}Cl(A)$ is $\tau_1\tau_2$-closed in $X$.*

An ideal $\mathscr{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space $(X, \tau)$ with an ideal $\mathscr{I}$ on $X$ is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset $A$ of $X$, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [17], $A^\star$ is called the local function of $A$ with respect to $\mathscr{I}$ and $\tau$ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than $\tau$. A subset $A$ is said to be $\star$-closed [18] if $A^\star \subseteq A$. The interior of a subset $A$ in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$R$-$\mathscr{I}^\star$-open* [6] (resp. *$\mathscr{I}^\star$-preopen* [6], *$\tau^\star$-semi-open* [19] (*semi-$\mathscr{I}^\star$-open* [20]), *$\tau^\star$-$\beta$-open* [19] (*semi-$\mathscr{I}^\star$-preopen* [20])) if $A = \text{Int}^\star(\text{Cl}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$, $A \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))$). The complement of a $R$-$\mathscr{I}^\star$-open (resp. $\mathscr{I}^\star$-preopen, $\tau^\star$-semi-open, $\tau^\star$-$\beta$-open) set is said to be *$R$-$\mathscr{I}^\star$-closed* (resp. *$\mathscr{I}^\star$-preclosed*, *$\tau^\star$-semi-closed*, *$\tau^\star$-$\beta$-closed*). For a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^\star$-closed sets containing $A$ is called the *semi-$\mathscr{I}^\star$-closure* [20] of $A$ and is denoted by $\text{sCl}^\star(A)$ ($s\text{Cl}_{\mathscr{I}^\star}(A)$ [20]). The union of all semi-$\mathscr{I}^\star$-open sets contained in $A$ is called the *semi-$\mathscr{I}^\star$-interior* [20] of $A$ and is denoted by $\text{sInt}^\star(A)$ ($s\text{Int}_{\mathscr{I}^\star}(A)$ [20]). The intersection of all $\beta$-$\mathscr{I}^\star$-closed sets containing $A$ is called the *$\beta$-$\mathscr{I}^\star$-closure* of $A$ and is denoted by $\beta\text{Cl}^\star(A)$. The union of all $\beta$-$\mathscr{I}^\star$-open sets contained in $A$ is called the *$\beta$-$\mathscr{I}^\star$-interior* of $A$ and is denoted by $\beta\text{Int}^\star(A)$.

**Lemma 4.** *For a subset $A$ of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

*(1)* $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$ [20].

*(2)* $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$ [20].

*(3)* $\beta Cl^\star(A) = A \cup Int^\star(Cl^\star(Int^\star(A)))$.

*(4)* $\beta Int^\star(A) = A \cap Cl^\star(Int^\star(Cl^\star(A)))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from $X$ into $Y$, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set $B$ of $Y$ by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

## 3. Upper and lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous multifunctions discussed.

**Definition 1.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x$ of $X$ if for each $\sigma_1\sigma_2$-open set $V$ of $Y$ such that $F(x) \subseteq V$, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \sigma_1\sigma_2$-$Cl(V)$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of $X$.*

**Theorem 1.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$;*

*(2) $x \in Cl^\star(Int(Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $F(x)$;*

*(3) $x \in \beta Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $F(x)$.*

*Proof.* (1) $\Rightarrow$ (2): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$. Thus by (1), there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. Then, $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Since $U$ is $\tau^\star$-$\beta$-open, we have

$$x \in U \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(U))) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V))))).$$

(2) $\Rightarrow$ (3): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$. Then by (2), we have $x \in \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))))$. Since $x \in F^+(\sigma_1\sigma_2\text{-Cl}(V))$ and by Lemma 4, $x \in F^+(\sigma_1\sigma_2\text{-Cl}(V)) \cap \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V))))) = \beta\mathrm{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(3) ⇒ (1): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$. By (3), we have

$$x \in \beta\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$$

and so there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$; hence $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at $x$.

**Definition 2.** *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is called lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x$ of $X$ if for each $\sigma_1\sigma_2$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $F(z) \cap \sigma_1\sigma_2\text{-}Cl(V) \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is called lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if $F$ is lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of $X$.*

**Theorem 2.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $F$ is lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$;*

*(2) $x \in Cl^\star(Int^\star(Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ such that*

$$F(x) \cap V \neq \emptyset;$$

*(3) $x \in \beta Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$.*

*Proof.* The proof is similar to that of Theorem 1.

**Definition 3.** *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $f(x)$, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if $f$ is $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of $X$.*

**Corollary 1.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $f$ is weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x \in X$;*

*(2) $x \in Cl^\star(Int^\star(Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $f(x)$;*

*(3) $x \in \beta Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $f(x)$.*

**Theorem 3.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2)* $F^+(V) \subseteq Cl^\star(Int^\star(Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))))$ *for every* $\sigma_1\sigma_2$*-open set* $V$ *of* $Y$*;*

*(3)* $Int^\star(Cl^\star(Int^\star(F^-(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* $V$ *of* $Y$*;*

*(4)* $Int^\star(Cl^\star(Int^\star(F^-(\sigma_1\sigma_2\text{-}Int(K))))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* $K$ *of* $Y$*;*

*(5)* $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* $K$ *of* $Y$*;*

*(6)* $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* $B$ *of* $Y$*;*

*(7)* $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \beta Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ *for every subset* $B$ *of* $Y$*;*

*(8)* $F^+(V) \subseteq \beta Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* $V$ *of* $Y$*;*

*(9)* $\beta Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* $V$ *of* $Y$*.*

*Proof.* (1) ⇒ (2): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by Theorem 1, $x \in \beta\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence

$$F^+(V) \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))))$$

by Lemma 4.

(2) ⇒ (3): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$. Thus by (2), we have

$$\begin{aligned} X - F^-(\sigma_1\sigma_2\text{-Cl}(V)) &= F^+(Y - \sigma_1\sigma_2\text{-Cl}(V)) \\ &\subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+(\text{Cl}^\star(Y - \sigma_1\sigma_2\text{-Cl}(V)))))) \\ &= \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))))) \\ &\subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(F^+(Y - V)))) \\ &= \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(X - F^-(V)))) \\ &= X - \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-(V)))) \end{aligned}$$

and hence $\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(3) ⇒ (4): Let $K$ be any $\sigma_1\sigma_2$-closed set of $Y$. Then, $\sigma_1\sigma_2\text{-Int}(K)$ is $\sigma_1\sigma_2$-open in $Y$ and so

$$\begin{aligned} \text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))))) &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K). \end{aligned}$$

(4) ⇒ (5): Let $K$ be any $\sigma_1\sigma_2$-closed set of $Y$. Then, we have

$$\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))))) \subseteq F^-(K)$$

and $F^-(\sigma_1\sigma_2\text{-Int}(K)) \subseteq F^-(K)$. Thus by Lemma 4, $\beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$.

(5) ⇒ (6): Let $B$ be any subset of $Y$. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in $Y$ and by (5), $\beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(6) ⇒ (7): Let $B$ be any subset of $Y$. By (6),

$$\begin{aligned} F^+(\sigma_1\sigma_2\text{-Int}(B)) &= X - F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= \beta\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))). \end{aligned}$$

(7) ⇒ (8): The proof is obvious.

(8) ⇒ (9): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$. Then by (8), we have

$$\begin{aligned} \beta\text{Cl}^\star(F^-(V)) &\subseteq \beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \beta\text{Cl}^\star(X - F^+(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= X - \beta\text{Int}^\star(F^+(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= X - \beta\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq X - F^+(Y - \sigma_1\sigma_2\text{-Cl}(V)) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned}$$

(9) ⇒ (1): Let $x \in X$ and $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$. By (9),

$$\begin{aligned} x \in F^+(V) &\subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - \beta\text{Cl}^\star(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= \beta\text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V))) \end{aligned}$$

and hence $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous by Theorem 1.

**Theorem 4.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $F$ is lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $F^-(V) \subseteq Cl^\star(Int^\star(Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$;*

*(3) $Int^\star(Cl^\star(Int^\star(F^+(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$;*

*(4) $Int^\star(Cl^\star(Int^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set $K$ of $Y$;*

*(5) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set $K$ of $Y$;*

*(6) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset $B$ of $Y$;*

*(7) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \beta Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset $B$ of $Y$;*

*(8) $F^-(V) \subseteq \beta Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$;*

*(9) $\beta Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$.*

*Proof.* The proof is similar to that of Theorem 3.

**Corollary 2.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $f$ is weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $f^{-1}(V) \subseteq Cl^\star(Int^\star(Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$;*

*(3) $Int^\star(Cl^\star(Int^\star(f^{-1}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$;*

*(4) $Int^\star(Cl^\star(Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set $K$ of $Y$;*

*(5) $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set $K$ of $Y$;*

*(6) $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset $B$ of $Y$;*

*(7) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \beta Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset $B$ of $Y$;*

*(8) $f^{-1}(V) \subseteq \beta Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$;*

*(9) $\beta Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$.*

**Definition 4.** [21] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x$ of $X$ if for each $\sigma_1\sigma_2$-open set $V$ of $Y$ such that $F(x) \subseteq V$, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if $F$ is upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of $X$.*

**Definition 5.** [21] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x$ of $X$ if for each $\sigma_1\sigma_2$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that*

$$F(z) \cap \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)) \neq \emptyset$$

*for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if $F$ is lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of $X$.*

**Remark 1.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

*upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuity $\Rightarrow$ upper weak $\tau^\star\beta(\sigma_1, \sigma_2)$-continuity.*

The converse of the implication is not true in general. We give an example for the implication as follows.

**Example 1.** *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{1\}\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{a, b\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(1) = \{c\}$ and $F(2) = F(3) = \{a, b\}$. Then, $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous but $F$ is not upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous.*

**Theorem 5.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset $B$ of $Y$;*

*(3) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset $B$ of $Y$;*

*(4) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$;*

*(5) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set $V$ of $Y$;*

*(6) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$-closed set $K$ of $Y$;*

*(7) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set $V$ of $Y$;*

*(8) $\beta Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set $V$ of $Y$.*

*Proof.* (1) $\Rightarrow$ (2): Let $B$ be any subset of $Y$. Thus by Lemma 3, $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in $Y$ and by Theorem 3,

$$\beta\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}((\sigma_1, \sigma_2)\theta\text{-}\mathrm{Cl}(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}\mathrm{Cl}(B)).$$

(2) $\Rightarrow$ (3): This is obvious since $\sigma_1\sigma_2$-Cl$(B) \subseteq (\sigma_1, \sigma_2)\theta$-Cl$(B)$ for every subset $B$ of $Y$.

(3) $\Rightarrow$ (4): This is obvious since $\sigma_1\sigma_2$-Cl$(V) = (\sigma_1, \sigma_2)\theta$-Cl$(V)$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$.

(4) $\Rightarrow$ (5): Let $V$ be any $(\sigma_1, \sigma_2)p$-open set of $Y$. Then, $V \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ and so $\sigma_1\sigma_2$-Cl$(V) = \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Now, put $G = \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$, then $G$ is $\sigma_1\sigma_2$-open in $Y$ and $\sigma_1\sigma_2$-Cl$(G) = \sigma_1\sigma_2$-Cl$(V)$. Thus by (4), we have

$$\beta\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)).$$

(5) $\Rightarrow$ (6): Let $K$ be any $(\sigma_1, \sigma_2)r$-closed set of $Y$. Then, $\sigma_1\sigma_2$-Int$(K)$ is $(\sigma_1, \sigma_2)p$-open in $Y$, by (5) we have

$$\beta\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))) = \beta\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))))$$

$$\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^-(K).$$

(6) ⇒ (7): Let $V$ be any $(\sigma_1, \sigma_2)\beta$-open set of $Y$. Then, we have

$$V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Since $\sigma_1\sigma_2$-Cl$(V)$ is $(\sigma_1, \sigma_2)r$-closed in $Y$. Thus by (6),

$$\beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(7) ⇒ (8): This is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(8) ⇒ (1): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$. Then, since $V$ is $(\sigma_1, \sigma_2)s$-open set in $Y$, by (8) we have $\beta\text{Cl}^\star(F^-(V)) \subseteq \beta\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$. By Theorem 3, $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous.

**Theorem 6.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $F$ is lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset $B$ of $Y$;*

*(3) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset $B$ of $Y$;*

*(4) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$;*

*(5) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set $V$ of $Y$;*

*(6) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$-closed set $K$ of $Y$;*

*(7) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set $V$ of $Y$;*

*(8) $\beta Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set $V$ of $Y$.*

*Proof.* The proof is similar to that of Theorem 5.

**Corollary 3.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $f$ is weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset $B$ of $Y$;*

*(3)* $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}((\sigma_1,\sigma_2)\theta\text{-}Cl(B))$ *for every subset* $B$ *of* $Y$*;*

*(4)* $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* $V$ *of* $Y$*;*

*(5)* $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)p$*-open set* $V$ *of* $Y$*;*

*(6)* $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ *for every* $(\sigma_1,\sigma_2)r$*-closed set* $K$ *of* $Y$*;*

*(7)* $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)\beta$*-open set* $V$ *of* $Y$*;*

*(8)* $\beta Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)s$*-open set* $V$ *of* $Y$*.*

**Theorem 7.** *For a multifunction* $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$*, the following properties are equivalent:*

*(1)* $F$ *is upper weakly* $\tau^\star\beta(\sigma_1,\sigma_2)$*-continuous;*

*(2)* $\beta Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)p$*-open set* $V$ *of* $Y$*;*

*(3)* $F^+(V) \subseteq \beta Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $(\sigma_1,\sigma_2)p$*-open set* $V$ *of* $Y$*.*

*Proof.* (1) ⇒ (2): Let $V$ be any $(\sigma_1,\sigma_2)p$-open set of $Y$. Since $F$ is upper weakly $\tau^\star\beta(\sigma_1,\sigma_2)$-continuous, by Theorem 3 we have

$$\beta\mathrm{Cl}^\star(F^-(V)) \subseteq \beta\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)).$$

(2) ⇒ (3): Let $V$ be any $(\sigma_1,\sigma_2)p$-open set of $Y$. Then, $V \subseteq \sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ and $Y - V \supseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(Y-V))$. Thus by (3),

$$\begin{aligned} X - F^+(V) = F^-(Y-V) &\supseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(Y-V))) \\ &\supseteq \beta\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(Y-V))) \\ &= \beta\mathrm{Cl}^\star(F^-(Y-\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &= \beta\mathrm{Cl}^\star(X-F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &= X - \beta\mathrm{Int}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \end{aligned}$$

and hence $F^+(V) \subseteq \beta\mathrm{Int}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$.

(3) ⇒ (1): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$. Then, $V$ is $(\sigma_1,\sigma_2)p$-open in $Y$, by (4) we have $F^+(V) \subseteq \beta\mathrm{Int}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$. Thus, $F$ is upper weakly $\tau^\star\beta(\sigma_1,\sigma_2)$-continuous by Theorem 3.

**Theorem 8.** *For a multifunction* $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$*, the following properties are equivalent:*

*(1) $F$ is lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $\beta Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set $V$ of $Y$;*

*(3) $F^-(V) \subseteq \beta Int^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set $V$ of $Y$.*

*Proof.* The proof is similar to that of Theorem 7.

**Corollary 4.** *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $f$ is weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $\beta Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set $V$ of $Y$;*

*(3) $f^{-1}(V) \subseteq \beta Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set $V$ of $Y$.*

**Definition 6.** [22] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x$ of $X$ if for each $\sigma_1\sigma_2$-open set $V$ of $Y$ such that $F(x) \subseteq V$, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if $F$ is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of $X$.*

**Definition 7.** [22] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at a point $x$ of $X$ if for each $\sigma_1\sigma_2$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous if $F$ is lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous at each point of $X$.*

Recall that a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(\tau_1, \tau_2)$-*normal* [23] if for each pair of disjoint $\tau_1\tau_2$-closed sets $F$ and $F'$, there exist disjoint $\tau_1\tau_2$-open sets $U$ and $V$ such that $F \subseteq U$ and $F' \subseteq V$.

**Theorem 9.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ such that $F(x)$ is $\sigma_1\sigma_2$-closed in $Y$ for each $x \in X$ and $(Y, \sigma_1, \sigma_2)$ is a $(\sigma_1, \sigma_2)$-normal space, the following properties are equivalent:*

*(1) $F$ is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $F$ is upper almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(3) $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous.*

*Proof.* We show only the implication (3) $\Rightarrow$ (1) since the others are obvious. Suppose that $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and $V$ be any $\sigma_1\sigma_2$-open set of $Y$ such that $F(x) \subseteq V$. Since $F(x)$ is $\sigma_1\sigma_2$-closed in $Y$ and $(Y, \sigma_1, \sigma_2)$ is $(\sigma_1, \sigma_2)$-normal, there exists a $\sigma_1\sigma_2$-open set $G$ of $Y$ such that $F(x) \subseteq G \subseteq \sigma_1\sigma_2\text{-Cl}(G) \subseteq V$. Since $F$ is upper weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(G)$; hence $F(U) \subseteq V$. This shows that $F$ is upper $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous.

**Theorem 10.** *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ such that $F(x)$ is $\sigma_1\sigma_2$-open in $Y$ for each $x \in X$, the following properties are equivalent:*

*(1) $F$ is lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(2) $F$ is lower almost $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous;*

*(3) $F$ is lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous.*

*Proof.* (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3): The proofs of these implications are obvious.

(3) $\Rightarrow$ (1): Suppose that $F$ is lower weakly $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and $V$ be any $\sigma_1\sigma_2$-open set of $Y$ such that $F(x) \cap V \neq \emptyset$. Then, there exists a $\tau^\star$-$\beta$-open set $U$ of $X$ containing $x$ such that $F(z) \cap \sigma_1\sigma_2\text{-Cl}(V) \neq \emptyset$ for each $z \in U$. Since $F(z)$ is $\sigma_1\sigma_2$-open, we have $F(z) \cap V \neq \emptyset$ for each $z \in U$ and so $F$ is lower $\tau^\star\beta(\sigma_1, \sigma_2)$-continuous.

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