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On Upper and Lower $\mu(\sigma_1, \sigma_2)$-Continuous Multifunctions

Nipaporn Chutiman[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. This paper presents new concepts of continuous multifunctions defined from a generalized topological space into a bitopological space, called upper $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations and some properties concerning upper $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\mu(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

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Key Words and Phrases: Upper $\mu(\sigma_1, \sigma_2)$-continuous multifunction, lower $\mu(\sigma_1, \sigma_2)$-continuous multifunction

1. Introduction

In 2002, Császár [1] introduced the concepts of generalized topological spaces and generalized neighborhood systems. The classes of topological spaces and neighborhood systems are contained in the classes of generalized topological spaces and generalized neighborhood systems, respectively. Moreover, Császár [1] introduced two kinds of generalized continuous functions by utilizing the notions of generalized topological spaces and generalized neighborhood systems. In 2009, Kanibir and Reilly [2] extended the concept of generalized continuous functions to multifunctions and defined upper semi generalized continuous multifunctions and lower semi generalized continuous multifunctions. On the other hand, the present authors introduced and investigated four classes of multifunctions defined from a generalized topological space into a generalized topological space, namely upper $\beta(\mu_X, \mu_Y)$-continuous multifunctions [3], lower $\beta(\mu_X, \mu_Y)$-continuous multifunctions [3], upper $\alpha(\mu_X, \mu_Y)$-continuous multifunctions [4] and lower $\alpha(\mu_X, \mu_Y)$-continuous multifunctions [4]. Pue-on et al. [5] introduced and studied the concepts of

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Email addresses: nipaporn.c@msu.ac.th (N. Chutiman), areeyuth.s@psu.ac.th (A. Sama-Ae), chawalit.b@msu.ac.th (C. Boonpok)

upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [6] investigated several characterizations of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions by utilizing the notions of $(\tau_1, \tau_2)\theta$-closed sets and $(\tau_1, \tau_2)\theta$-open sets. Thongmoon et al. [7] studied some characterizations of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions by using $\tau_1\tau_2$-δ-open sets and $\tau_1\tau_2$-δ-closed sets. Quite recently, Khampakdee et al. [8] presented new classes of continuous multifunctions defined from an ideal topological space into a bitopological space, namely upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations and some properties of upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions were established in [8]. In this paper, we introduce the concepts of upper $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\mu(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [9] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [9] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [9] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [9] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [10] (resp. $(\tau_1, \tau_2)s$-*open* [11], $(\tau_1, \tau_2)p$-*open* [11], $(\tau_1, \tau_2)\beta$-*open* [11]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)p$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-δ-*open* [12] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-δ-*closed* [12]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-δ-*interior* [12] of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ-*closure* [12] of A and is denoted by $\tau_1\tau_2$-δ-Cl(A). Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [10] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [10] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [10] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [10] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 2. [10] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

Let X be a nonempty set, and denote $\mathcal{P}(X)$ the power set of X. We call a class $\mu \subseteq \mathcal{P}(X)$ a *generalized topology* (briefly, GT) if $\emptyset \in \mu$, and an arbitrary union of elements of μ belongs to μ [1]. A set X with a GT μ on it is said to be a *generalized topological space* (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ-*open sets* and the complements of μ-open sets are called μ-*closed sets*. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A. Then, we have $i_\mu(i_\mu(A)) = i_\mu(A)$, $c_\mu(c_\mu(A)) = c_\mu(A)$, and $i_\mu(A) = X - c_\mu(X - A)$. According to [13], for $A \subseteq X$ and $x \in X$, we have $x \in c_\mu(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower $\mu(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the concepts of upper $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations of upper $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\mu(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is called upper $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a μ-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is called upper $\mu(\sigma_1, \sigma_2)$-continuous if F is upper $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(V)$ is μ-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is μ-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $c_\mu(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq i_\mu(F^+(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (1), there exists a μ-open set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $x \in i_\mu(F^+(V))$. Therefore, $F^+(V) \subseteq i_\mu(F^+(V))$. This shows that $F^+(V)$ is μ-open in X.

(2) ⇒ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by (3), $c_\mu(F^-(B)) \subseteq c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(B))) = F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - i_\mu(F^+(B)) = c_\mu(X - F^+(B)) &= c_\mu(F^-(Y - B)) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)) \end{aligned}$$

and so $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq i_\mu(F^+(B))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then, $x \in F^+(V) = i_\mu(F^+(V))$. There exists a μ-open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper $\mu(\sigma_1, \sigma_2)$-continuous.

Definition 2. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is called lower $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a μ-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is called lower $\mu(\sigma_1, \sigma_2)$-continuous if F is lower $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 2. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $F^-(V)$ is μ-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^+(K)$ is μ-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $c_\mu(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $F(c_\mu(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ *for every subset* A *of* X*;*

(6) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq i_\mu(F^-(B))$ *for every subset* B *of* Y*.*

Proof. We prove only the implications (4) ⇒ (5) and (5) ⇒ (6) being the proofs of the other similar to those of Theorem 1.

(4) ⇒ (5): Let A be any subset of X. By (4), we have

$$c_\mu(A) \subseteq c_\mu(F^+(F(A))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(F(A)))$$

and hence $F(c_\mu(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$.

(5) ⇒ (6): Let B be any subset of Y. By (5),

$$F(c_\mu(F^+(Y - B))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(F^+(Y - B))) \subseteq \sigma_1\sigma_2\text{-Cl}(Y - B) = Y - \sigma_1\sigma_2\text{-Int}(B).$$

Since $F(c_\mu(F^+(Y - B))) = F(c_\mu(X - F^-(B))) = F(X - i_\mu(F^-(B)))$, we have

$$X - i_\mu(F^-(B)) \subseteq F^+(Y - \sigma_1\sigma_2\text{-Int}(B)) = X - F^-(\sigma_1\sigma_2\text{-Int}(B))$$

and hence $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq i_\mu(F^-(B))$.

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)*s-regular* [7] if for each $(\tau_1, \tau_2)s$-closed set F and each $x \notin F$, there exist disjoint $(\tau_1, \tau_2)s$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 3. [7] *A bitopological space* (X, τ_1, τ_2) *is* $(\tau_1, \tau_2)s$*-regular if and only if for each* $x \in X$ *and each* $(\tau_1, \tau_2)s$*-open set* U *containing* x*, there exists a* $(\tau_1, \tau_2)s$*-open set* V *such that* $x \in V \subseteq (\tau_1, \tau_2)\text{-}sCl(V) \subseteq U$*.*

Lemma 4. [7] *Let* (X, τ_1, τ_2) *be a* $(\tau_1, \tau_2)s$*-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2\text{-}Cl(A) = \tau_1\tau_2\text{-}\delta\text{-}Cl(A)$ *for every subset* A *of* X*.*

(2) Every $\tau_1\tau_2$*-open set is* $\tau_1\tau_2$*-*δ*-open.*

Theorem 3. *For a multifunction* $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper* $\mu(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^-(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ *is* μ*-closed in* X *for every subset* B *of* Y*;*

(3) $F^-(K)$ *is* μ*-closed in* X *for every* $\sigma_1\sigma_2$*-*δ*-closed set* K *of* Y*;*

(4) $F^+(V)$ *is* μ*-open in* X *for every* $\sigma_1\sigma_2$*-*δ*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 4, $\sigma_1\sigma_2$-δ-Cl(B) is $\sigma_1\sigma_2$-closed in Y. Since F is upper $\mu(\sigma_1,\sigma_2)$-continuous, by Theorem 1 we have $F^-(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$ is μ-closed in X.

(2) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-δ-closed set of Y. Then, $\sigma_1\sigma_2$-δ-Cl$(K) = K$ and by (2), $F^-(K)$ is μ-closed in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)s$-regular, we have V is $\sigma_1\sigma_2$-δ-open in Y and by (4), $F^+(V)$ is μ-open in X. Thus, F is upper $\mu(\sigma_1,\sigma_2)$-continuous by Theorem 1.

Theorem 4. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower $\mu(\sigma_1,\sigma_2)$-continuous;

(2) $F^+(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ is μ-closed in X for every subset B of Y;

(3) $F^+(K)$ is μ-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;

(4) $F^-(V)$ is μ-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-*regular* [14] if for each $\tau_1\tau_2$-closed set F and each $x \notin F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 5. [14] *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-regular if and only if for each $x \in X$ and each $\tau_1\tau_2$-open set U containing x, there exists a $\tau_1\tau_2$-open set V such that $x \in V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.*

Lemma 6. [6] *Let (X, τ_1, τ_2) be a (τ_1, τ_2)-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2\text{-}Cl(A) = (\tau_1, \tau_2)\theta\text{-}Cl(A)$ for every subset A of X.

(2) Every $\tau_1\tau_2$-open set is $(\tau_1, \tau_2)\theta$-open.

Theorem 5. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, the following properties are equivalent:*

(1) F is upper $\mu(\sigma_1,\sigma_2)$-continuous;

(2) $F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ is μ-closed in X for every subset B of Y;

(3) $F^-(K)$ is μ-closed in X for every $(\sigma_1, \sigma_2)\theta$-closed set K of Y;

(4) $F^+(V)$ is μ-open in X for every $(\sigma_1, \sigma_2)\theta$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 6, $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y. Since F is upper $\mu(\sigma_1, \sigma_2)$-continuous, by Theorem 1 $F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ is μ-closed in X.

(2) $\Rightarrow$ (3): Let K be any $(\sigma_1, \sigma_2)\theta$-closed set of Y. Then, $(\sigma_1, \sigma_2)\theta\text{-Cl}(K) = K$ and by (2), we have $F^-(K)$ is μ-closed in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, we have V is $(\sigma_1, \sigma_2)\theta$-open in Y and by (4), $F^+(V)$ is μ-open in X. Thus, F is upper $\mu(\sigma_1, \sigma_2)$-continuous by Theorem 1.

Theorem 6. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, the following properties are equivalent:*

(1) F is lower $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ is μ-closed in X for every subset B of Y;

(3) $F^+(K)$ is μ-closed in X for every $(\sigma_1, \sigma_2)\theta$-closed set K of Y;

(4) $F^-(V)$ is μ-open in X for every $(\sigma_1, \sigma_2)\theta$-open set V of Y.

Proof. The proof is similar to that of Theorem 5.

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