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Almost $\mu(\sigma_1, \sigma_2)$-Continuous Multifunctions

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Abstract. This paper introduces new concepts of continuous multifunctions defined between a generalized topological space and a bitopological space, namely upper almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations and some properties concerning upper almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions are investigated. Furthermore, the relationships between $\mu(\sigma_1, \sigma_2)$-continuity and almost $\mu(\sigma_1, \sigma_2)$-continuity are considered.

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1. Introduction

The concept of almost continuous functions was introduced by Singal and Singal [1]. Munshi and Bassan [2] studied the notion of almost semi-continuous functions. Noiri [3] introduced and investigated the concept of almost α-continuous functions. Nasef and Noiri [4] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions. The class of almost precontinuity is a generalization of almost α-continuity. The class of almost β-continuity is a generalization of almost semi-continuity. The concepts of generalized topological spaces and generalized neighborhood systems were introduced by Császár [5]. The classes of topological spaces and neighborhood systems are contained in the classes of generalized topological spaces and generalized neighborhood systems, respectively. Moreover, Császár [5] introduced two kinds of generalized continuous functions by utilizing the concepts of generalized topological spaces and generalized neighborhood systems. Kanibir and Reilly [6] extended the concept of generalized continuous functions to multifunctions and introduced generalized continuous

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multifunctions between generalized topological spaces. On the other hand, the present authors introduced and investigated four classes of multifunctions defined from a generalized topological space into a generalized topological space, namely upper almost $\beta(\mu_X, \mu_Y)$-continuous multifunctions [7], lower almost $\beta(\mu_X, \mu_Y)$-continuous multifunctions [7], upper $\alpha(\mu_X, \mu_Y)$-continuous multifunctions [8] and lower $\alpha(\mu_X, \mu_Y)$-continuous multifunctions [8]. Pue-on et al. [9] introduced and studied the concepts of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [10] introduced and investigated the notions of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. Quite recently, Viriyapong et al. [11] presented new classes of continuous multifunctions between an ideal topological space and a bitopological space, namely upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations and some properties of upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions were discussed in [11]. In this paper, we introduce new classes of multifunctions between a generalized topological space and a bitopological space, namely upper almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [12] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [12] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [12] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [12] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$*-open* [13] (resp. $(\tau_1, \tau_2)s$*-open* [14], $(\tau_1, \tau_2)p$*-open* [14], $(\tau_1, \tau_2)\beta$*-open* [14]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$*-open* [15] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$*-closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed (resp. $(\tau_1, \tau_2)s$-closed, $\alpha(\tau_1, \tau_2)$-closed) sets of X containing A is called the $(\tau_1, \tau_2)p$*-closure* [16] (resp. $(\tau_1, \tau_2)s$*-closure* [14], $\alpha(\tau_1, \tau_2)$*-closure* [17]) of A and is denoted by $(\tau_1, \tau_2)\text{-pCl}(A)$ (resp. $(\tau_1, \tau_2)\text{-sCl}(A)$, $\alpha(\tau_1, \tau_2)\text{-Cl}(A)$). The union of all $(\tau_1, \tau_2)p$-open (resp. $(\tau_1, \tau_2)s$-open, $\alpha(\tau_1, \tau_2)$-open) sets of X contained in A is called the $(\tau_1, \tau_2)p$*-interior* [16] (resp. $(\tau_1, \tau_2)s$*-interior* [14], $\alpha(\tau_1, \tau_2)$*-interior* [17]) of A and is denoted by $(\tau_1, \tau_2)\text{-pInt}(A)$ (resp. $(\tau_1, \tau_2)\text{-sInt}(A)$, $\alpha(\tau_1, \tau_2)\text{-Int}(A)$).

Lemma 2. [10] *Let A be a subset of a bitopological space (X, τ_1, τ_2). If A is $\tau_1\tau_2$-open in X, then $(\tau_1, \tau_2)\text{-sCl}(A) = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-δ*-open* [18] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-δ*-closed* [18]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-δ*-interior* [18] of A and is denoted by $\tau_1\tau_2\text{-}\delta\text{-Int}(A)$. The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ*-closure* [18] of A and is denoted by $\tau_1\tau_2\text{-}\delta\text{-Cl}(A)$. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* [13] of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* [13] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* [13] if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* [13] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$.

Lemma 3. [13] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2\text{-Cl}(A) = (\tau_1, \tau_2)\theta\text{-Cl}(A)$.

(2) $(\tau_1, \tau_2)\theta\text{-Cl}(A)$ is $\tau_1\tau_2$-closed in X.

Let X be a nonempty set, and denote $\mathcal{P}(X)$ the power set of X. We call a class $\mu \subseteq \mathcal{P}(X)$ a *generalized topology* (briefly, GT) if $\emptyset \in \mu$, and an arbitrary union of elements of μ belongs to μ [5]. A set X with a GT μ on it is said to be a *generalized topological space* (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ*-open sets* and the complements of μ-open sets are called μ*-closed sets*. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A. Then, we have $i_\mu(i_\mu(A)) = i_\mu(A)$, $c_\mu(c_\mu(A)) = c_\mu(A)$, and $i_\mu(A) = X - c_\mu(X - A)$. According to [19], for $A \subseteq X$ and $x \in X$, we have $x \in c_\mu(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$.

3. Upper and lower almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the concepts of upper almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations of upper almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\mu(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a μ-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\mu(\sigma_1, \sigma_2)$-continuous if F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in i_\mu(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(3) $x \in i_\mu(F^+((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(4) $x \in i_\mu(F^+(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$;

(5) for each $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$, there exists a μ-open set U of X containing x such that $F(U) \subseteq V$.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Thus by (1), there exists a μ-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. Therefore, $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ and so $x \in i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(2) ⇒ (3): This follows from Lemma 2.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. It follows from Lemma 2 that $V = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) = (\sigma_1, \sigma_2)\text{-sCl}(V)$.

(4) ⇒ (5): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. Then by (4), we have $x \in i_\mu(F^+(V))$ and there exists a μ-open set U of X containing x such that $x \in U \subseteq F^+(V)$; hence $F(U) \subseteq V$.

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since $\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(V)) is $(\sigma_1, \sigma_2)r$-open, there exists a μ-open set U of X containing x such that

$$F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)).$$

This shows that F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$.

Definition 2. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is called lower almost $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists a μ-open set U of X containing x such that $\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(V)) $\cap F(z) \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is called lower almost $\mu(\sigma_1, \sigma_2)$-continuous if F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 2. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in i_\mu(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y such that

$$V \cap F(x) \neq \emptyset;$$

(3) $x \in i_\mu(F^-((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $V \cap F(x) \neq \emptyset$;

(4) $x \in i_\mu(F^-(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y such that $V \cap F(x) \neq \emptyset$;

(5) for each $(\sigma_1, \sigma_2)r$-open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists a μ-open set U of X containing x such that $U \subseteq F^-(V)$.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(V) \subseteq i_\mu(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $c_\mu(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $c_\mu(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq i_\mu(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ for every subset B of Y;

(6) $F^+(V)$ is μ-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;

(7) $F^-(K)$ is μ-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$. Thus by Theorem 1, we have $x \in i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ and hence

$$F^+(V) \subseteq i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

(2) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2),

$$\begin{aligned} X - F^-(K) = F^+(Y - K) &\subseteq i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)))) \\ &= i_\mu(X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \\ &= X - c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))). \end{aligned}$$

Thus, $c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-closed set of Y and by (3), $c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then, we have

$$\begin{aligned} F^+(\sigma_1\sigma_2\text{-Int}(B)) &= X - F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B))))) \\ &= X - c_\mu(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))) \\ &= i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))). \end{aligned}$$

(5) $\Rightarrow$ (6): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. By (5), we have $F^+(V) \subseteq i_\mu(F^+(V))$ and hence $F^+(V)$ is μ-open in X.

(6) $\Rightarrow$ (7): The proof is obvious.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. Since $Y - V$ is $(\sigma_1, \sigma_2)r$-closed and by (7), $X - F^+(V) = F^-(Y - V)$ is μ-closed in X. Thus, $F^+(V)$ is μ-open and hence $x \in i_\mu(F^+(V))$. Then, there exists a μ-open set U of X containing x such that $F(U) \subseteq V$. It follows from Theorem 1 that F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous.

Theorem 4. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $F^-(V) \subseteq i_\mu(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $c_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $c_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;

(5) $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq i_\mu(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))))$ for every subset B of Y;

(6) $F^-(V)$ is μ-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;

(7) $F^+(K)$ is μ-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $c_\mu(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(V)$ is a $(\sigma_1, \sigma_2)r$-closed set of Y. Since F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous and by Theorem 3, $F^-(\sigma_1\sigma_2\text{-Cl}(V))$ is μ-closed in X. Thus, $c_\mu(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, K is $(\sigma_1, \sigma_2)s$-open in Y. Then by (3), $c_\mu(F^-(K)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K)$ and hence $F^-(K)$ is μ-closed in X. By Theorem 3, F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $c_\mu(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. The proof is similar to that of Theorem 5.

Lemma 4. [20] *For a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $\alpha(\tau_1, \tau_2)\text{-}Cl(V) = \tau_1\tau_2\text{-}Cl(V)$ for every $(\tau_1, \tau_2)\beta$-open set V of X;

(2) $(\tau_1, \tau_2)\text{-}pCl(V) = \tau_1\tau_2\text{-}Cl(V)$ for every $(\tau_1, \tau_2)s$-open set V of X.

Corollary 1. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^-(V)) \subseteq F^-(\alpha(\sigma_1, \sigma_2)\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $c_\mu(F^-(V)) \subseteq F^-((\sigma_1, \sigma_2)\text{-}pCl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Corollary 2. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^+(V)) \subseteq F^+(\alpha(\sigma_1, \sigma_2)\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $c_\mu(F^+(V)) \subseteq F^+((\sigma_1, \sigma_2)\text{-}pCl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Theorem 7. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $c_\mu(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^+(V) \subseteq i_\mu(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is $\sigma_1\sigma_2$-closed in Y and by Theorem 3, we have

$$c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) $\Rightarrow$ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2),

$$\begin{aligned} c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V)))) &\subseteq c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned}$$

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Thus by (3), we have

$$\begin{aligned} X - i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &= c_\mu(X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= c_\mu(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))) \\ &= c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Cl}(V))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^-(Y - V) \\ &= X - F^+(V) \end{aligned}$$

and hence $F^+(V) \subseteq i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(4) $\Rightarrow$ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), $F^+(V) \subseteq i_\mu(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = i_\mu(F^+(V))$. Thus, $F^+(V)$ is μ-open in X. It follows from Theorem 3 that F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous.

Theorem 8. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $c_\mu(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^-(V) \subseteq i_\mu(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 7.

Lemma 5. [21] *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then, the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2\text{-}Cl(A) = \tau_1\tau_2\text{-}\delta\text{-}Cl(A)$.

(2) $\tau_1\tau_2\text{-}\delta\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

Theorem 9. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y;

(3) $c_\mu(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 5, $\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by Theorem 3, $c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(B) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)$.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (3), we have

$$\begin{aligned} c_\mu(F^-(K)) &= c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \\ &= c_\mu(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(K)) \\ &= F^-(K) \end{aligned}$$

and hence $F^-(K)$ is μ-closed in X. By Theorem 3, F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous.

Theorem 10. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y;

(3) $c_\mu(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 9.

Lemma 6. *If $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is lower almost $\mu(\sigma_1, \sigma_2)$-continuous, then for each $x \in X$ and each subset B of Y with $\sigma_1\sigma_2\text{-}\delta\text{-}Int(B) \cap F(x) \neq \emptyset$, there exists a μ-open set U of X containing x such that $U \subseteq F^-(B)$.*

Proof. Let $x \in X$ and B be a subset of Y with $\sigma_1\sigma_2\text{-}\delta\text{-Int}(B) \cap F(x) \neq \emptyset$. Since $\sigma_1\sigma_2\text{-}\delta\text{-Int}(B) \cap F(x) \neq \emptyset$, there exists a nonempty $(\sigma_1, \sigma_2)r$-open set V of Y such that $V \subseteq B$ and $V \cap F(x) \neq \emptyset$. Since F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous, there exists a μ-open set U of X containing x such that $V \cap F(z) \neq \emptyset$ for each $z \in U$; hence $U \subseteq F^-(B)$.

Theorem 11. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_\mu(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y;

(3) $F(c_\mu(A)) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-}Cl(F(A))$ for every subset A of X;

(4) $F^+(K)$ is μ-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;

(5) $F^-(V)$ is μ-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y;

(6) $F^-(\sigma_1\sigma_2\text{-}\delta\text{-}Int(B)) \subseteq i_\mu(F^-(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin F^+(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$. Then, we have $x \in F^-(Y - \sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)) = F^-(\sigma_1\sigma_2\text{-}\delta\text{-Int}(Y - B))$. There exists a μ-open set U of X containing x such that $U \subseteq F^-(Y - B) = X - F^+(B)$. Thus, $U \cap F^+(B) = \emptyset$ and hence $x \in X - c_\mu(F^+(B))$. This shows that $c_\mu(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$.

(2) $\Rightarrow$ (3): Let A be any subset of X. By (2), we have

$$c_\mu(A) \subseteq c_\mu(F^+(F(A))) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(F(A)))$$

and hence $F(c_\mu(A)) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-Cl}(F(A))$.

(3) $\Rightarrow$ (1): Let B be any subset of Y. Then, by the hypothesis and Lemma 5,

$$\begin{aligned} &F(c_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))))) \\ &\subseteq \tau_1\tau_2\text{-}\delta\text{-Cl}(F(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))))) \end{aligned}$$

$$\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))) \subseteq \sigma_1\sigma_2\text{-Cl}(B)$$

and hence $c_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$. By Theorem 4, F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous.

(2) $\Rightarrow$ (4): Let K be any $\sigma_1\sigma_2$-δ-closed set of Y. Then, $\sigma_1\sigma_2\text{-}\delta\text{-Cl}(K) = K$. By (2), we have $c_\mu(F^+(K)) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(K)) = F^+(K)$ and so $F^+(K)$ is μ-closed in X.

(4) $\Rightarrow$ (5): The proof is obvious.

(5) $\Rightarrow$ (6): Let B be any subset of Y. Then by (5), we have

$$F^-(\sigma_1\sigma_2\text{-}\delta\text{-Int}(B)) = i_\mu(F^-(\sigma_1\sigma_2\text{-}\delta\text{-Int}(B))) \subseteq i_\mu(F^-(B)).$$

(6) $\Rightarrow$ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, we have V is $\sigma_1\sigma_2$-δ-open and $\sigma_1\sigma_2\text{-}\delta\text{-Int}(V) = V$. Thus by (6), $F^-(V) \subseteq i_\mu(F^-(V))$ and hence $F^-(V)$ is μ-open in X. By Theorem 4, F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous.

Definition 3. [22] *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a μ-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\mu(\sigma_1, \sigma_2)$-continuous if F is upper $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Definition 4. [22] *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\mu(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a μ-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\mu(\sigma_1, \sigma_2)$-continuous if F is lower $\mu(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Remark 1. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

upper $\mu(\sigma_1, \sigma_2)$-continuity $\Rightarrow$ upper almost $\mu(\sigma_1, \sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let $X = \{1, 2, 3\}$ with a generalized topology $\mu = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$. Let $Y = \{p, q, r\}$ with topologies $\sigma_1 = \{\emptyset, \{p\}, \{p, q\}, Y\}$ and $\sigma_2 = \{\emptyset, \{p\}, \{q\}, \{p, q\}, Y\}$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(1) = \{r\}$ and $F(2) = F(3) = \{p, q\}$. Then, F is upper almost $\mu(\sigma_1, \sigma_2)$-continuous but F is not upper $\mu(\sigma_1, \sigma_2)$-continuous.*

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)*s-regular* [23] if for each $(\tau_1, \tau_2)s$-closed set F and each $x \notin F$, there exist disjoint $(\tau_1, \tau_2)s$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 7. [23] *Let (X, τ_1, τ_2) be a $(\tau_1, \tau_2)s$-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2$-$Cl(A) = \tau_1\tau_2$-δ-$Cl(A)$ *for every subset* A *of* X*.*

(2) Every $\tau_1\tau_2$*-open set is* $\tau_1\tau_2$*-*δ*-open.*

Lemma 8. [22] *For a multifunction* $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$*, where* (Y, σ_1, σ_2) *is a* $(\sigma_1, \sigma_2)s$*-regular space, the following properties are equivalent:*

(1) F *is lower* $\mu(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^+(\sigma_1\sigma_2$-δ-$Cl(B))$ *is* μ*-closed in* X *for every subset* B *of* Y*;*

(3) $F^+(K)$ *is* μ*-closed in* X *for every* $\sigma_1\sigma_2$*-*δ*-closed set* K *of* Y*;*

(4) $F^-(V)$ *is* μ*-open in* X *for every* $\sigma_1\sigma_2$*-*δ*-open set* V *of* Y*.*

Theorem 12. *For a multifunction* $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$*, where* (Y, σ_1, σ_2) *is a* $(\sigma_1, \sigma_2)s$*-regular space, the following properties are equivalent:*

(1) F *is lower* $\mu(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^+(\sigma_1\sigma_2$-δ-$Cl(B))$ *is* μ*-closed in* X *for every subset* B *of* Y*;*

(3) $F^+(K)$ *is* μ*-closed in* X *for every* $\sigma_1\sigma_2$*-*δ*-closed set* K *of* Y*;*

(4) $F^-(V)$ *is* μ*-open in* X *for every* $\sigma_1\sigma_2$*-*δ*-open set* V *of* Y*;*

(5) F *is lower almost* $\mu(\sigma_1, \sigma_2)$*-continuous.*

Proof. The proofs of the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are similar as in Lemma 8.

(4) ⇒ (5): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $\sigma_1\sigma_2$-open in Y and by Lemma 7, V is $\sigma_1\sigma_2$-δ-open in Y. By (4), we have $F^-(V)$ is μ-open in X. Thus by Theorem 4, F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $V \cap F(x) \neq \emptyset$. Since (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)s$-regular, there exists a $(\sigma_1, \sigma_2)r$-open set W such that $W \cap F(x) \neq \emptyset$ and $W \subseteq V$. Since F is lower almost $\mu(\sigma_1, \sigma_2)$-continuous, there exists a μ-open set U of X containing x such that $W \cap F(z) \neq \emptyset$ for every $z \in U$. Thus, $F(z) \cap V \neq \emptyset$ for every $z \in U$. This shows that F is lower $\mu(\sigma_1, \sigma_2)$-continuous.

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