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Almost Weak $\mu(\sigma_1, \sigma_2)$-Continuity for Multifunctions

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Abstract. This paper presents new concepts of continuous multifunctions, called upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations and some properties concerning upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions are established.

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1. Introduction

In 1968, Singal and Singal [1] introduced and investigated the concept of almost continuous functions. Munshi and Bassan [2] studied the notion of almost semi-continuous functions. Noiri [3] introduced and investigated the concept of almost α-continuous functions. Nasef and Noiri [4] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions. The class of almost precontinuity is a generalization of almost α-continuity. The class of almost β-continuity is a generalization of almost semi-continuity. Levine [5] introduced and investigated the concept of weakly continuous functions. Husain [6] introduced and studied the notion of almost continuous functions. Noiri [7] investigated several characterizations of almost weakly continuous functions. Rose [8] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. In 1993, Noiri and Popa [9] extended the concept of almost weakly continuous functions to multifunctions and defined upper almost weakly continuous multifunctions and lower almost weakly continuous multifunctions. Popa and Noiri [10] investigated some characterizations and

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several properties concerning upper almost weakly continuous multifunctions and lower almost weakly continuous multifunctions. In 2002, Császár [11] introduced the concepts of generalized topological spaces and generalized neighborhood systems. The classes of topological spaces and neighborhood systems are contained in the classes of generalized topological spaces and generalized neighborhood systems, respectively. Furthermore, Császár [11] introduced two kinds of generalized continuous functions by utilizing the notions of generalized topological spaces and generalized neighborhood systems. In 2009, Kanibir and Reilly [12] extended the concept of generalized continuous functions to multifunctions and defined upper semi generalized continuous multifunctions and lower semi generalized continuous multifunctions. On the other hand, the present authors introduced and investigated four classes of multifunctions defined from a generalized topological space into a generalized topological space, namely upper $\beta(\mu_X, \mu_Y)$-continuous multifunctions [13], lower $\beta(\mu_X, \mu_Y)$-continuous multifunctions [13], upper $\alpha(\mu_X, \mu_Y)$-continuous multifunctions [14] and lower $\alpha(\mu_X, \mu_Y)$-continuous multifunctions [14]. Pue-on et al. [15] introduced and studied the concepts of upper (τ_1, τ_2)-continuous continuous multifunctions and lower (τ_1, τ_2)-continuous continuous multifunctions. Klanarong et al. [16] introduced and investigated the notions of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. Moreover, several characterizations and some properties of weakly (τ_1, τ_2)-continuous multifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions were established in [17] and [18], respectively. Quite recently, Viriyapong et al. [19] presented new classes of continuous multifunctions between an ideal topological space and a bitopological space, namely upper almost weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions. In this paper, we introduce the concepts of upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [20] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [20] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [20] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [21] (resp. $(\tau_1, \tau_2)s$-*open* [22], $(\tau_1, \tau_2)p$-*open* [22], $(\tau_1, \tau_2)\beta$-*open* [22]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-

open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). For a subset A of a bitopological space (X, τ_1, τ_2), a point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [21] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [21] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [21] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [21] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 1. [21] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

Let X be a nonempty set, and denote $\mathcal{P}(X)$ the power set of X. We call a class $\mu \subseteq \mathcal{P}(X)$ a *generalized topology* (briefly, GT) if $\emptyset \in \mu$, and an arbitrary union of elements of μ belongs to μ [11]. A set X with a GT μ on it is said to be a *generalized topological space* (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ-*open sets* and the complements of μ-open sets are called μ-*closed sets*. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A. Then, we have $i_\mu(i_\mu(A)) = i_\mu(A)$, $c_\mu(c_\mu(A)) = c_\mu(A)$, and $i_\mu(A) = X - c_\mu(X - A)$. According to [23], for $A \subseteq X$ and $x \in X$, we have $x \in c_\mu(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$. A subset A of a generalized topological space (X, μ) is called μ-*preopen* [24] if $A \subseteq i_\mu(c_\mu(A))$. The complement of a μ-preopen set is called μ-*preclosed*. For a generalized topological space (X, μ), we will denote the class of μ-preopen sets by $\mu(\pi)$. Let A be a subset of a generalized topological space (X, μ). The intersection of all μ-preclosed sets of X containing A is called the μ-*perclosure* of A and is denoted by $c_{\mu(\pi)}(A)$. The union of all μ-preopen sets of X contained in A is called the μ-*preinterior* of A and is denoted by $i_{\mu(\pi)}(A)$.

Lemma 2. *Let A be a subset of a generalized topological space (X, μ) and $x \in X$. Then, the following properties hold:*

(1) $x \in c_{\mu(\pi)}$ if and only if $U \cap A \neq \emptyset$ for every μ-peropen set U of X containing x;

(2) A is μ-preclosed if and only if $A = c_{\mu(\pi)}(A)$;

(3) $i_{\mu(\pi)}(X - A) = X - c_{\mu(\pi)}(A)$;

(4) $c_{\mu(\pi)}(X - A) = X - i_{\mu(\pi)}(A)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$.

3. Upper and lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, $x \in i_\mu(c_\mu(F^+(\sigma_1\sigma_2\text{-}Cl(V))))$.*

Theorem 1. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(V) \subseteq i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $c_{\mu(\pi)}(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a μ-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (1), we have $x \in i_\mu(c_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(V))))$ and so $x \in i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. Thus, $F^+(V) \subseteq i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(2) ⇒ (3): Let V be any $\sigma_1\sigma_2$-open set of Y. Since $Y - \sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open and by (2), we have

$$\begin{aligned} X - F^-(\sigma_1\sigma_2\text{-Cl}(V)) &= F^+(Y - \sigma_1\sigma_2\text{-Cl}(V)) \\ &\subseteq i_\mu(c_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))))) \\ &\subseteq i_\mu(c_\mu(F^+(Y - V))) \\ &= i_\mu(c_\mu(X - F^-(V))) \\ &= X - c_\mu(i_\mu(F^-(V))) \end{aligned}$$

and hence $c_\mu(i_\mu(F^-(V))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(3) ⇒ (2): The proof is obvious.

(2) ⇒ (4): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (2), $x \in F^+(V) \subseteq i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and there exists a μ-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

(4) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (4), there exists a μ-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x \in U \subseteq i_\mu(c_\mu(U)) \subseteq i_\mu(c_\mu(F^+(\sigma_1\sigma_2\text{-Cl}(V))))$. This shows that F is upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous.

Definition 2. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, $x \in i_\mu(c_\mu(F^-(\sigma_1\sigma_2\text{-}Cl(V))))$.*

Theorem 2. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $F^-(V) \subseteq i_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $c_{\mu(\pi)}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a μ-preopen set U of X containing x such that $F(z) \cap \sigma_1\sigma_2\text{-}Cl(V) \neq \emptyset$ for each $z \in U$.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(3) $c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2 Cl(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(4) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $\sigma_1\sigma_2\text{-Int}(K)$ is $\sigma_1\sigma_2$-open in Y, by Theorem 1 we have

$$c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K).$$

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let B be any subset of Y. By (3), we have

$$\begin{aligned} X - i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) &= c_{\mu(\pi)}(X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= c_{\mu(\pi)}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Thus, $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then by (5), we have

$$F^+(V) \subseteq i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence F is upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous by Theorem 1.

Theorem 4. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(3) $c_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(4) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq i_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\text{-}\theta Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(5) $c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Let $x \in X - F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, $x \in F^+(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ and $(\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y. By Theorem 1, there exists a μ-preopen set U of X containing x such that

$$\begin{aligned} U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B))) &= F^+(Y - \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) \\ &= X - F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))). \end{aligned}$$

Thus, $U \cap F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) = \emptyset$ and hence

$$x \in X - c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))).$$

Therefore, $c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$.

(2) ⇒ (3): The proof is obvious since $(\sigma_1, \sigma_2)\theta\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and by (3), we have

$$\begin{aligned} c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &= c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Int}(V)))) \end{aligned}$$

$$= F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(4) ⇒ (5): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, $\sigma_1\sigma_2$-Int(K) is $(\sigma_1, \sigma_2)p$-open in Y and by (4),

$$\begin{aligned} c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= F^-(K). \end{aligned}$$

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y and by (5),

$$c_{\mu(\pi)}(F^-(V)) \subseteq c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

It follows from Theorem 1 that F is upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous;

(2) $c_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y;

(3) $c_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $c_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(5) $c_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. The proof is similar to that of Theorem 5.

The *μ-prefrontier* of a subset A of a generalized topological space (X, μ), denoted by $\mu(\pi)$fr(A), is defined by $\mu(\pi)\text{pf}(A) = c_{\mu(\pi)}(A) \cap c_{\mu(\pi)}(X - A) = c_{\mu(\pi)}(A) - i_{\mu(\pi)}(A)$.

Theorem 7. *The set of all points x of X at which a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is not upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous is identical with the union of the μ-prefrontier of the upper inverse images of the $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets containing $F(x)$.*

Proof. Let $x \in X$ at which F is not upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous. There exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every μ-preopen set U of X containing x. Therefore, we have

$$x \in c_{\mu(\pi)}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) = X - i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

Since $x \in F^+(V)$, we have $x \in c_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and so $x \in \mu(\pi)\text{fr}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

Conversely, if F is upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous, then for any $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ there exists a μ-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Therefore, $x \in i_{\mu(\pi)}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. This contradicts with the fact that $x \in \mu(\pi)\text{fr}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. Thus, F is not upper almost weakly $\mu(\sigma_1, \sigma_2)$-continuous at x.

Theorem 8. *The set of all points x of X at which a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is not lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous is identical with the union of the μ-prefrontier of the lower inverse images of $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets meeting $F(x)$.*

Proof. The proof is similar to that of Theorem 7.

Definition 3. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\mu(\sigma_1, \sigma_2)$-precontinuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a μ-preopen set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\mu(\sigma_1, \sigma_2)$-precontinuous if F is upper $\mu(\sigma_1, \sigma_2)$-precontinuous at each point x of X.*

Theorem 9. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper $\mu(\sigma_1, \sigma_2)$-precontinuous;

(2) $F^+(V)$ is μ-preopen in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is μ-preclosed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $c_{\mu(\pi)}(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq i_{\mu(\pi)}(F^+(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (1), there exists an μ-preopen set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $x \in i_{\mu(\pi)}(F^+(V))$. Therefore, $F^+(V) \subseteq i_{\mu(\pi)}(F^+(V))$. This shows that $F^+(V)$ is μ-preopen in X.

(2) ⇒ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by (3), $c_{\mu(\pi)}(F^-(B)) \subseteq c_{\mu(\pi)}(F^-(\sigma_1\sigma_2\text{-Cl}(B))) = F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. Thus by (4),

$$\begin{aligned} X - i_{\mu(\pi)}(F^+(B)) &= c_{\mu(\pi)}(X - F^+(B)) \\ &= c_{\mu(\pi)}(F^-(Y - B)) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \end{aligned}$$

$$= F^{-}(Y - \sigma_1\sigma_2\text{-Int}(B))$$
$$= X - F^{+}(\sigma_1\sigma_2\text{-Int}(B))$$

and so $F^{+}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq i_{\mu(\pi)}(F^{+}(B))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then, $x \in F^{+}(V) = i_{\mu(\pi)}(F^{+}(V))$. There exists a μ-preopen set U of X containing x such that $U \subseteq F^{+}(V)$; hence $F(U) \subseteq V$. This shows that F is upper $\mu(\sigma_1, \sigma_2)$-precontinuous.

Definition 4. *A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\mu(\sigma_1, \sigma_2)$-precontinuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an μ-preopen set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$ is called lower $\mu(\sigma_1, \sigma_2)$-precontinuous if F is lower $\mu(\sigma_1, \sigma_2)$-precontinuous at each point x of X.*

Theorem 10. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower $\mu(\sigma_1, \sigma_2)$-precontinuous;

(2) $F^{-}(V)$ is μ-preopen in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^{+}(K)$ is μ-preclosed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $c_{\mu(\pi)}(F^{+}(B)) \subseteq F^{+}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F(c_{\mu(\pi)}(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ for every subset A of X;

(6) $F^{-}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq i_{\mu(\pi)}(F^{-}(B))$ for every subset B of Y.

Proof. We prove only the implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) being the proofs of the other similar to those of Theorem 9.

(4) $\Rightarrow$ (5): Let A be any subset of X. By (4), we have

$$c_{\mu(\pi)}(A) \subseteq c_{\mu(\pi)}(F^{+}(F(A))) \subseteq F^{+}(\sigma_1\sigma_2\text{-Cl}(F(A)))$$

and hence $F(c_{\mu(\pi)}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$.

(5) $\Rightarrow$ (6): Let B be any subset of Y. By (5),

$$F(c_{\mu(\pi)}(F^{+}(Y - B))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(F^{+}(Y - B)))$$
$$\subseteq \sigma_1\sigma_2\text{-Cl}(Y - B) = Y - \sigma_1\sigma_2\text{-Int}(B).$$

Since $F(c_{\mu(\pi)}(F^{+}(Y - B))) = F(c_{\mu(\pi)}(X - F^{-}(B))) = F(X - i_{\mu(\pi)}(F^{-}(B)))$, we have

$$X - i_{\mu(\pi)}(F^{-}(B)) \subseteq F^{+}(Y - \sigma_1\sigma_2\text{-Int}(B))$$
$$= X - F^{-}(\sigma_1\sigma_2\text{-Int}(B))$$

and so $F^{-}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq i_{\mu(\pi)}(F^{-}(B))$.

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-*regular* [25] if for each $\tau_1\tau_2$-closed set F and each $x \notin F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 3. [26] *Let (X, τ_1, τ_2) be a (τ_1, τ_2)-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$ for every subset A of X.

(2) Every $\tau_1\tau_2$-open set is $(\tau_1, \tau_2)\theta$-open.

Theorem 11. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, the following properties are equivalent:*

(1) F is upper $\mu(\sigma_1, \sigma_2)$-precontinuous;

(2) $F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ is μ-preclosed in X for every subset B of Y;

(3) $F^-(K)$ is μ-preclosed in X for every $(\sigma_1, \sigma_2)\theta$-closed set K of Y;

(4) $F^+(V)$ is μ-preopen in X for every $(\sigma_1, \sigma_2)\theta$-open set V of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Then, $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y and by Theorem 9, $F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ is μ-preclosed in X.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)\theta$-open set of Y. By (3), $F^-(Y - V)$ is μ-preclosed in X and $F^-(Y - V) = X - F^+(V)$. Thus, $F^+(V)$ is μ-preopen in X.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, by Lemma 3 we have V is $(\sigma_1, \sigma_2)\theta$-open in Y and by (4), $F^+(V)$ is μ-preopen in X. Thus by Theorem 9, F is upper $\mu(\sigma_1, \sigma_2)$-precontinuous.

Theorem 12. *For a multifunction $F : (X, \mu) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, the following properties are equivalent:*

(1) F is lower $\mu(\sigma_1, \sigma_2)$-precontinuous;

(2) $F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ is μ-preclosed in X for every subset B of Y;

(3) $F^+(K)$ is μ-preclosed in X for every $(\sigma_1, \sigma_2)\theta$-closed set K of Y;

(4) $F^-(V)$ is μ-preopen in X for every $(\sigma_1, \sigma_2)\theta$-open set V of Y;

(5) F is lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous.

Proof. We prove only the implication (5) ⇒ (1), the proof of the other being similar to that of Theprem 11. The proof of the implication (4) ⇒ (5) is obvious.

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^-(V)$. Then, $F(x) \cap V \neq \emptyset$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \cap W \neq \emptyset$ and $\sigma_1\sigma_2$-Cl$(W) \subseteq V$. Since F is lower almost weakly $\mu(\sigma_1, \sigma_2)$-continuous, by Theorem 2 there exists a μ-preopen set U of X containing x such that

$$U \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(W)) \subseteq F^-(V).$$

Thus, $x \in i_{\mu(\pi)}(F^-(V))$ and hence $F^-(V) \subseteq i_{\mu(\pi)}(F^-(V))$. Therefore, $F^-(V)$ is μ-preopen in X and by Theorem 10, F is lower $\mu(\sigma_1, \sigma_2)$-precontinuous.

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