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Exploring Categorical Perspectives on Soft BCK/BCI-Algebras

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Abstract. In this manuscript, we present new ideas concerning the domain of soft BCK/BCI-algebras and outline specific categorical frameworks, including equalizers and finite products. Additionally, we demonstrate that the category of soft BCK/BCI-algebras conforms to a topological construct. Moreover, we establish that the category of soft BCK/BCI-algebras features distinctive elements such as terminal objects, initial objects, and zero objects.

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Key Words and Phrases: BCK/BCI-algebra, soft BCK/BCI-algebra, category of soft BCK/BCI-algebra, soft BCK/BCI-homomorphism

1. Introduction

Imai and Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [1, 2]. It is established that the category of BCK-algebras is a proper subset of the category of BCI-algebras.
In the realm of fuzzy set theory [3, 4], Molodtsov [5] proposed the concept of soft sets as a novel mathematical approach to address uncertainties without the presence of errors found in existing theories. Subsequently, Maji et al. [6, 7] introduced fuzzy soft sets. Ali et al. [8] explored new operations on soft sets, while ongoing research continues to advance soft set theory. In [9], the application of soft set theory is extended to various concepts including (i) filters in R_0-algebras; (ii) positive implicative ideals of BCK-algebras [10]; (iii) decision-making problems using fuzzy soft sets [11]; (iv) fuzzy soft groups [12]; (v) fuzzy soft sets in BCK/BCI-algebras [13]; (vi) Normal Unisoft Filters in R0-algebras [14]. Additionally, Muhiuddin et al. studied the application of soft set theory in areas such as filter theory in MTL-algebras [15], Unisoft Filters in R0-algebras [16], Cubic Soft

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BCK/BCI-Algebras [17, 18], and soft ordered semigroups [19].
Moreover, category theory is broadly defined as a general mathematical theory of structures and systems of structures. For detailed insights into categorical concepts, readers are directed to [20–24].

In recent years, researchers have merged category theory with soft set theory to establish the category of soft sets across various domains. Notable contributions include:

- In 2013, Zahiri [25] introduced the concept of the "category of soft sets."
- In the same year, Sardar and Gupta [26] constructed a soft category and investigated several intriguing properties.
- In 2014, Zhou et al. [27] delved into the categorical properties of soft sets.
- Borzooei et al. [28] explored key concepts related to the category of soft sets in 2015.
- In 2016, Öztunç [29] examined specific properties of soft categories.
- Shirmohammadi and Rasouli [30] presented a categorical approach to soft S-acts in 2017.
- In 2022, Sharma et al. [31] introduced the notion of the category of intuitionistic fuzzy modules.

2. Purpose for conducting this research

The study of soft BCK/BCI-algebras represents a fascinating area of research that offers intriguing insights into algebraic systems. By investigating and elucidating the categorical structures and properties within this domain, we aim to expand the theoretical foundations of soft BCK/BCI-algebras. Our exploration of concepts such as equalizers and finite products sheds light on the organizational principles governing these algebraic systems. Furthermore, our discovery of terminal, initial, and zero objects in the category of soft BCK/BCI-algebras reveals unique elements that contribute to a deeper understanding of their structural nuances. Through this work, we not only enhance our knowledge of categorical properties but also pave the way for future research and applications in the realm of algebraic structures and categorical theory.

3. Targets of the planned technique

- To develop a systematic framework for analyzing soft BCK/BCI-algebras and exploring their categorical structures.
- To investigate the presence of equalizers and finite products within the category of soft BCK/BCI-algebras.

- To demonstrate the adherence of the category of soft BCK/BCI-algebras to a topological construct.
- To identify and characterize special objects such as terminal objects, initial objects, and zero objects in the category of soft BCK/BCI-algebras.
- To contribute to a deeper understanding of the categorical properties and structural nuances inherent in soft BCK/BCI-algebras.
- To lay the groundwork for further exploration and applications of the proposed method in the broader context of algebraic structures and categorical theory.

This paper is structured as follows: Section 2 presents fundamental notions of BCK/BCI-algebras and soft BCK/BCI-algebras. Section 3 entails the construction of the soft BCK/BCI-algebras category and various category-related concepts. Finally, in Section 4, we explore the special objects within the category of soft BCK-algebras.

4. Preliminaries

K. Iséki introduced the significant class of logical algebras known as BCK/BCI-algebras, described as the most important class of logical algebras [1, 2]. A nonempty subset T is referred to as a BCK/BCI-subalgebra of $\widetilde{\mathcal{X}}$ if $\varpi * \varrho \in T$ for all $\varpi, \varrho \in T$ where $\widetilde{\mathcal{X}}$ is a BCK/BCI-algebra. please consult [32], for further details regarding BCK/BCI-algebras.

A "fuzzy set" μ in a "BCK/BCI-algebra" $\widetilde{\mathcal{X}}$ is termed a "fuzzy BCK/BCI-algebra" if it satisfies the condition:

$$(\forall \varpi, \varrho \in \widetilde{\mathcal{X}}) \text{ "}(\mu(\varpi * \varrho) \geq \min\{\mu(\varpi), \mu(\varrho)\})\text{"}.$$

Molodtsov defined a soft set as follows: Consider an initial universe set U and a set of parameters E, where $P(U)$ denotes the power set of U and $\Omega \subset E$.

Definition 1 ([5]). *A pair (ζ, Ω) is termed a soft set over U, where ζ is a function defined by*

$$\zeta : \Omega \to P(U).$$

For further insights, Molodtsov presented "many examples" in [5].

Definition 2. *[33] Let $\widetilde{\mathcal{X}}$ be a BCK/BCI-algebra. Let (ζ, Ω) be called a Soft BCK/BCI-algebra over $\widetilde{\mathcal{X}}$ if $\zeta(\varpi)$ is a BCK/BCI sub-algebra over $\widetilde{\mathcal{X}}$,$\forall \varpi \in \Omega$. $\zeta : \Omega \to P(\varpi)$ and $\varpi \mapsto \zeta(\varpi)$, therefore $\zeta(\varpi)$ is BCK/BCI sub-algebra.*

Example 1. *[33] Let "$\widetilde{\mathcal{X}} = \{0, \alpha, \beta, \gamma, \delta\}$ be a BCK-algebra with the following Cayley table":*
Let (ζ, Ω) be soft set over $\widetilde{\mathcal{X}}$ where $\Omega = \widetilde{\mathcal{X}}$ and $\zeta : \Omega \to P(\widetilde{\mathcal{X}})$ is a set-valued function defined by,

$$\text{"}\zeta(\varpi) = \{\varrho \in \widetilde{\mathcal{X}} : \varpi R \varrho \Leftrightarrow \varrho \in \varpi^{-1} I\}\text{"}$$

For all $\varpi \in \Omega$ "where $I = \{0, \alpha\}$ and $\varpi^{-1} = \{\varrho \in \widetilde{\mathcal{X}} : \varpi \wedge \varrho \in I\}$"

$*$	0	α	β	γ	δ
0	0	0	0	0	0
α	α	0	α	α	α
β	β	β	0	β	β
γ	γ	γ	γ	0	γ
δ	δ	δ	δ	δ	0

we have $\zeta(0) = \zeta(\alpha) = \widetilde{\mathcal{X}}$, $\zeta(\beta) = \{0, \alpha, \beta, \delta\}$, $\zeta(\delta) = \{0, \alpha, \beta, \delta\}$ and $\zeta(\delta) = \{0, \alpha, \beta, \delta\}$ are BCK-subalgebras of $\widetilde{\mathcal{X}}$. "Therefore (ζ, Ω) is a soft BCK-algebra over $\widetilde{\mathcal{X}}$".

Definition 3. *[33] Let "(ζ, Ω) and (ϕ, B) be two soft BCK/BCI-algebras over $\widetilde{\mathcal{X}}$". Then the BCK/BCI-homomorphism $\mu : \Omega \to B$ is "called" a soft BCK/BCI-homomorphism" if*

$$\zeta(\alpha) \subseteq (\phi \circ \mu)(\alpha), \forall \alpha \in \Omega.$$

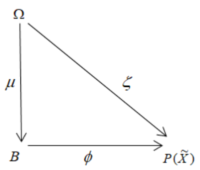

Figure 1:

i.e.

$$\zeta(\alpha) \subseteq (\phi \circ \mu)(\alpha) = \phi(\mu(\alpha), \forall \alpha \in \Omega.$$

5. Category of Soft BCI/BCK-algebra

Definition 4. *To form a category, we need a class of objects: "Soft BCK/BCI-algebras and a class of morphisms: Soft BCK- homomorphisms".*
(I) Composition of maps.
(II) Identity.

Proposition 1. *(Composition of soft-BCK/BCI-homomorphisms is again a soft BCK/BCI-homomorphism). Let (ζ, Ω),(ϕ, B) and $(\varphi, \aleph)$ be any three BCK/BCI-algebras. Let $\mu : \Omega \to B$ and $\hbar : B \to \aleph$ be two soft BCK/BCI-homomorphisms. Then $\hbar \circ \mu : \Omega \to \aleph$ is again a "soft BCK/BCI-homomorphism".*

Proof. Let $\mu : \Omega \to B$ and $\hbar : B \to \aleph$ be two soft BCK/BCI-homomorphisms. Then by definition, we have,

$$\zeta(\alpha) \subseteq (\phi \circ \mu)(\alpha), \forall \alpha \in \Omega$$

and

$$\phi(\beta) \subseteq (\varphi \circ \hbar)(\beta), \forall \beta \in B.$$

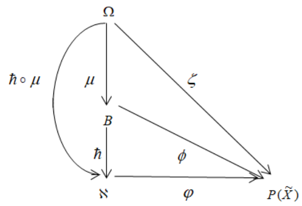

Figure 2:

To show that $\hbar \circ \mu$ is a soft BCK/BCI-homomorphism, we have to prove

$$\text{“}\zeta(\alpha) \subseteq (\varphi \circ (\hbar \circ \mu)(\alpha), \forall \alpha \in \Omega.\text{”}$$

Now,

$$\begin{aligned}
\zeta(\alpha) &\subseteq (\phi \circ \mu)(\alpha) \\
&= (\phi \circ \mu(\alpha)) \\
&= (\phi(\beta)) for some \beta = \mu(\alpha) \in B. \\
&\subseteq \phi(\varphi \circ \hbar)(\beta) \\
&= \varphi(\hbar(\beta)) \\
&= \varphi(\hbar \circ (\mu(\alpha))) \\
&= \varphi((\hbar \circ \mu)(\alpha) \\
&= \varphi \circ (\hbar \circ \mu))(\alpha)
\end{aligned}$$

Consequently, “$\hbar \circ \mu$ is a soft BCK/BCI-homomorphism”.

Definition 5. *Let “(ζ, Ω) be a soft-BCK-algebra over $\widetilde{\mathcal{X}}$”. Then the soft identity BCK/BCI-homomorphism is defined as*

$$\zeta(\alpha) \subseteq (\zeta \circ Id_\Omega)(\alpha).$$

This implies that the following diagram commutes.

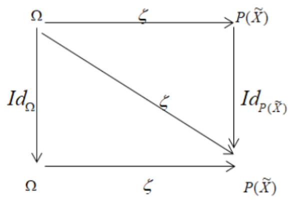

Figure 3:

Then Id_Ω is a soft identity BCK/BCI-homomorphism.
Moreover, $Id_\Omega : (\zeta, \Omega) \to (\zeta, \Omega)$ is a soft BCK/BCI-homomorphism if

$$\zeta(\alpha) \subseteq (\zeta \circ Id_\Omega)(a), \forall \alpha \in \Omega.$$

Now, in view of the above discussion, we have the following.

Definition 6. *The "class of all soft BCK/BCI-algebras together with the class of all soft BCK/BCI-homomorphisms from a category". It is called the "category of soft BCK/BCI-algebra and is denoted by" S_{BCKI}.*

Next, we prove the following results:

Theorem 1. *The category S_{BCKI} has equilizers.*

Proof. We begin this proof with the following diagram

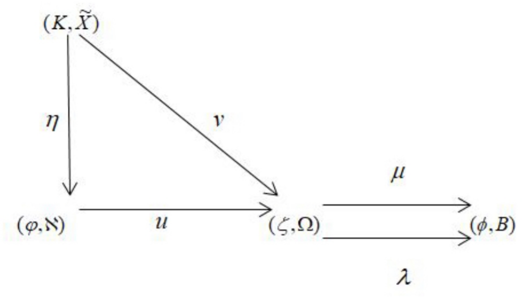

Figure 4:

Let (ζ, Ω) and (ϕ, B) be two soft-BCK/BCI-algebras in S_{BCKI}-objects over $\widetilde{\mathcal{X}}$. Again, let μ and λ be two $S_{BCKI} - morphisms$ from (ζ, Ω) to (ϕ, B). Define $\aleph = \{\alpha \in \Omega : \mu(\alpha) = \lambda(\alpha)\}$, $u : \aleph \to \Omega$ an embedding, and $\varphi = \zeta \circ u$. From assumption, we have $(\varphi, \aleph)$ is a $S_{BCKI} - object$, $\mu \circ u = \lambda \circ u$ and $\varphi(c) = (\zeta \circ u)(c)$, for all $c \in \aleph$. Thus by definition, u is a $S_{BCKI} - morphism$. We will demonstrate that $((\varphi, \aleph), u)$ functions as the equilizer between μ and λ. Let $(K, \widetilde{\mathcal{X}})$ be a S_{BCKI}-object and suppose v is a S_{BCKI}-morphism from $(K, \widetilde{\mathcal{X}})$ to (ζ, Ω) such that

$$\aleph = \{\alpha \in \Omega : \mu(\alpha) = \lambda(\alpha)\}, \aleph \neq \emptyset.$$

Since $\mu(0) = \lambda(0) \Longrightarrow 0 \in \aleph$. Let $\varpi_1, \varpi_2 \in \aleph$. Then $\mu(\varpi_1) = \lambda(\varpi_1)$ and $\mu(\varpi_2) = \lambda(\varpi_2)$. We have

$$\begin{aligned} \mu(\varpi_1 * \varpi_2) &= \mu(\varpi_1) * \mu(\varpi_2) \\ &= \lambda(\varpi_1) * \lambda(\varpi_2) \\ &= \lambda(\varpi_1 * \varpi_2). \end{aligned}$$

This implies that $\varpi_1 * \varpi_2 \in \aleph$. This show that $\aleph$ is a subalgebra of Ω.
Now, define a map $\eta : \widetilde{\mathcal{X}} \to \aleph$ and $\eta = v$. In what follows, Our focus is to demonstrate that η is a S_{BCKI}-morphism from $(K, \widetilde{\mathcal{X}})$ to $(\phi, \aleph)$, and that $v = u \circ \eta$. Firstly by $\mu \circ v = \lambda \circ v$, we get

$$\mu(v(\varpi)) = \lambda(v(\varpi)), \forall \varpi \in \widetilde{\mathcal{X}} \Rightarrow v(\varpi) \in \aleph.$$

Hence $\eta = v$ is well defined.
Again, if $\varphi = \zeta \circ u, \eta = v$ and v being a $S_{BCKI} - morphism$, then form Figure 4, we have

$$K(\varpi) \subseteq \zeta(v(\varpi)) = \zeta(u \circ \eta(\varpi)) = \zeta(u(\eta(\varpi))) = (\zeta \circ u)(\eta(\varpi)) = \varphi(\eta(\varpi))$$

for all $\varpi \in \widetilde{\mathcal{X}} \Longrightarrow \eta$ is a $S_{BCKI} - morphism$ (by definition).
Finally from assumption, we know that $v = u \circ \eta$ and η is unique. Consequently $(\varphi, \aleph)$ is the equilizer of μ and λ is S_{BCKI}.

Proposition 2. *The category S_{BCKI} has a finite product.*

Proof. Let (ζ, Ω) and (ϕ, B) are two $S_{BCI} - objects$. Define three mappings

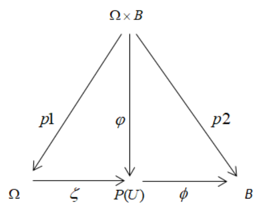

Figure 5:

$$\begin{aligned}
\varphi &: \Omega \times B \to P(U) \\
&\quad (\alpha, \beta) \mapsto \zeta(\alpha) \cap \phi(\beta) \\
p_1 &: \Omega \times B \to \Omega \\
&\quad (\alpha, \beta) \mapsto \alpha \Rightarrow p_1(\alpha, \beta) = \alpha \\
p_2 &: \Omega \times B \to B \\
&\quad (\alpha, \beta) \mapsto \beta
\end{aligned}$$

Foreach $(\alpha, \beta) \in \Omega \times B$.

$$\varphi(\alpha, \beta) = \zeta(\alpha) \cap \phi(\beta) \subseteq \zeta(\alpha) = \zeta(p_1((\alpha, \beta)).$$

This implies that p_1 is an $S_{BCKI} - morphism$. By the some argument, p_2 is a $S_{BCKI} - morphism$. Again for each $S_{BCKI} - objects$ (I, D), suppose that μ and λ are $S_{BCKI} - morphism$ from $(I, D) \to (\zeta, \Omega)$ and (ϕ, B).

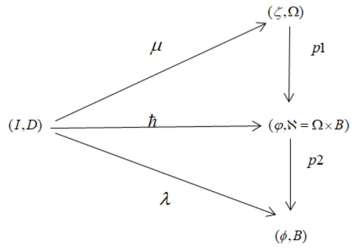

Figure 6:

Then $I(\delta) \subseteq \zeta(\mu(\delta))$ and $I(\delta) \subseteq \phi(\lambda(\delta)), \forall \delta \in D$. Further, define

$$\begin{aligned} \hbar &: D \to \Omega \times B \\ & \quad \delta \mapsto (\mu(\delta), \lambda(\delta)) \\ \hbar(\delta) &= (\mu(\delta), \lambda(\delta)) \end{aligned}$$

for each $\delta \in D$, then we get

$$I(\delta) \subseteq \zeta(\mu(\delta) \cap \phi(\lambda(\delta))) = \varphi(\mu(\delta), \lambda(\delta)) = \varphi(\hbar(\delta) = (\varphi \circ \hbar)(\delta)$$

This implies that $\hbar$ is a $S_{BCI} - morphism$. Finallyforevery $\delta \in D$, we get

$$(p1 \circ \hbar)(\delta) = p1(\hbar(\delta)) = p1(\mu(\delta), \lambda(\delta)) = \mu(\delta)$$

Therefore, $p_1 \circ \hbar = \mu$. Similarity, $p_2 \circ \hbar = \lambda$. Clearly, $\hbar$ is unique. Consequently, $\{(\varphi, \aleph), p_1, p_2\}$ is a finite product of (ζ, Ω) and (ϕ, B).

Proposition 3. *The category S_{BCKI} is a "topological construct".*

Proof. Let $\{(\zeta_i, \Omega_i)\}_{i \in I}$ be a family of S_{BCKI}-objects indexed by a class I, and let $\{(\zeta_i : \Omega \to \Omega_i)\}_{i \in I}$ be a family of mappings. We define asoftset.over.U as follows:

$$\zeta : \Omega \to P(U)$$

$$\alpha \to \cap_{i \in I}(\zeta_i(\mu_i(\alpha))).$$

Then, $(\zeta, \Omega) \in Ob(S_{BCKI})$. It suffices to show that $\{\mu_i : (\zeta, \Omega) \to (\zeta_i, \Omega_i)\}_{i \in I}$ is the unique S_{BCKI} initial lift of $\{\mu_i : \Omega \to \Omega_i\}_{i \in I}$. Now, to "complete the proof, we take the following two cases".

Case 1: We show that $\{\mu_i : (\zeta, \Omega) \to (\zeta_i, \Omega_i)\}_{i \in I}$ is a S_{BCKI} initial.lift.of $\{\mu_i : \Omega \to \Omega_i\}_{i \in I}$. Firstly, we claim that $\{\mu_i : (\zeta, \Omega) \to (\zeta_i, \Omega_i)\}$ is a family of S_{BCKI}-morphisms for every $i \in I$. By the assumption, for each $\alpha \in \Omega$ and $i \in I$, one yields

$$\zeta(\alpha) = \cap_{i \in I}(\zeta_i(\mu_i(\alpha))) \subseteq \zeta_i(\mu_i(\alpha)),$$

Where $\{\mu_i\}_{i\in I}$ is a family of S_{BCKI}-morphisms, Furthermore, suppose that $\phi, B \in Ob(S_{BCKI})$, $\lambda : B \to \Omega$ is a mapping such that $\lambda_i = \mu_i \circ \lambda$ for every $i \in I$, and $\lambda_i : (\phi, B)) \to (\zeta_i, \Omega_i)$ is is a family of S_{BCKI}-morphisms. Then, we can infer that

$$\begin{aligned} \phi(\beta) \subseteq \cap_{i\in I}(\zeta_i(\lambda_i)\beta)) &= \cap_{i\in I}(\zeta_i(\mu_i \circ g)(\beta)) \\ &= \cap_{i\in I}(\zeta_i(\mu_i(g(\beta)))) = \zeta(g(\beta)) \end{aligned}$$

Therefore, λ represents a S_{BCKI}-morphism from ϕ, B to ζ, Ω. As per the definition, it is evident that the family $\{\mu_i : (\zeta, \Omega) \to (\zeta_i, \Omega_i)\}_{i\in I}$ constitutes a S_{BCKI} initial lift of $\{\mu_i : \Omega \to \Omega_i\}_{i\in I}$.

In Case 2, we address the "uniqueness of the initial lift". Assuming that $\{\mu_i : (\widetilde{\zeta}, \Omega) \to (\zeta_i, \Omega_i)\}_{i\in I}$ also stands as a S_{BCKI} initial lift of $\{\mu_i : \Omega \to \Omega_i\}_{i\in I}$, distinct from $\{\mu_i : (\zeta, \Omega) \to (\zeta_i, \Omega_i)\}_{i\in I}$, then $\{\mu_i : (\widetilde{\zeta}, \Omega) \to (\zeta_i, \Omega_i)\}_{i\in I}$ forms a set of S_{BCKI}-morphisms.

It is evident that $\widetilde{\zeta}(\alpha) \subseteq \zeta_i(\mu_i(\alpha))$ forall- $i \in I$ and $\alpha \in \Omega$. Thus, $\widetilde{\zeta}(\alpha) \subseteq \zeta_i(\mu_i(\alpha)) = \zeta(\alpha)$, implying $\widetilde{\zeta} \subseteq \zeta$.

Conversely, for the S_{BCKI}-entities (ζ, Ω) and identity mapping $I\delta_\Omega : \Omega \to \Omega$, given that $\{\mu_i : (\widetilde{\zeta}, \Omega) \to (\zeta_i, \Omega_i)\}_{i\in I}$ serves as a S_{BCKI} initial lift of $\{\mu_i : \Omega \to \Omega_i\}_{i\in I}$, we find that $\mu_i \circ i\delta_\Omega = \mu_i$, where μ_i represents a S_{BCKI}-morphism for all $i \in I$, and $i\delta_\Omega : (\zeta, \Omega) \to (\widetilde{\zeta}, \Omega)$ is also a S_{BCKI}-morphism. Consequently, $\zeta(\alpha) \subseteq \widetilde{\zeta}(i\delta_\Omega(\alpha)) = \widetilde{\zeta}(\alpha)$ for each $\alpha \in \Omega$, indicating $\zeta \subseteq \widetilde{\zeta}$.

In summary, it is established that $\zeta \subseteq \widetilde{\zeta}$. Combining the outcomes of steps (1) and (2), it can be concluded that S_{BCKI} forms a topological construct.

6. Special objects in S_{BCKI}

It is evident that the set $\{0\}$ constitutes a BCI/BCK-algebra. Consequently, it follows trivially that the pair $(\zeta, \{0\})$ represents a "soft BCI/BCK-algebra over $\widetilde{\mathcal{X}}$". Now, for any other object (ϕ, Ω) in S_{BCKI},there is only one morphism from $0 \to \Omega$ i.e, $0 \to 0$ and also there is only morphism $\Omega \to \{0\}$ i.e, $\alpha \to 0, \forall \alpha \in \Omega$.Trivially, the 0-morphism is a soft morphism. Thus S_{BCKI} has zero objects.

Proposition 4. *The empty set (accompanied by an empty function mapping into $P(\widetilde{\mathcal{X}})$) serves as a zero object within S_{BCKI}.*

Proof. For each object $(\zeta, \Omega) \in S_{BCKI}$, there exist unique morphism

The inequalities are satisfied by default. therefore soft BCI/BCK-algebras $((\zeta, \Omega), (\Phi, \Phi))$ and $((\Phi, \Phi), (\zeta, \Omega))$ each contain only one -morphism.

Proposition 5. *The category S_{BCKI} has a terminal object.*

Proof. Define a mapping

$$\begin{gathered} T_{\{\Phi\}} : \Phi \to P(U) \\ \Phi \mapsto U \end{gathered}$$

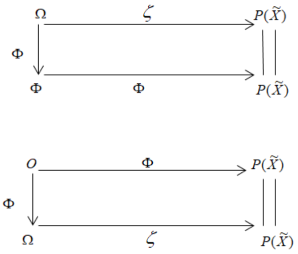

Figure 7:

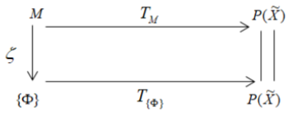

Figure 8:

Trivially $(T_{\{\Phi\}}, \{\Phi\}$ in a soft BCK-algebra for any object (T_M, M), define a map $\mu : M \to \{\Phi\}$ by $m \to \Phi$.

Then, for each $m \in M$, the relationship $T_M(m) \subseteq T_{\{\Phi\}}(\mu(m)) = T_{\{\Phi\}}(\Phi) = U$ holds, indicating that μ functions as a soft BCK-morphism from (T_M, M) to $(T_{\{\Phi\}}, \Phi)$. It is evident that μ is uniquely defined. Hence, $(T_{\{\Phi\}}, \Phi)$ stands as a terminal object in S_{BCKI}.

Proposition 6. *The category S_{BCKI} has an initial object.*

Proof. Similar proof for a terminal object.

Proposition 7. *The category S_{BCKI} has zero objects.*

Proof. Trivially, the empty set Φ forms a BCK-algebra. Then (Φ, Φ) is a soft BCK-algebra $(\Phi : \Phi \to P(U))$. Again, for each object (ζ, Ω) in S_{BCKI}, there exists a unique morphism.

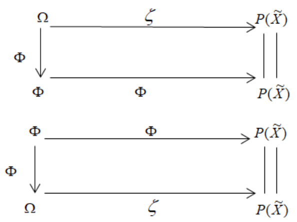

Figure 9:

The inequalities are satisfied by default. Thus there is only one morphism from (ζ, Ω) to (Φ, Φ) and also only one morphism from (Φ, Φ) to (ζ, Ω). Thus, S_{BCK} has no objects.

7. Conclusion

In conclusion, this study delves into the realm of soft BCK/BCI-algebras, introducing novel concepts and elucidating categorical structures such as finite products, equalizers, and co-equalizers within this domain. By demonstrating that the category of soft BCK/BCI-algebras adheres to a topological construct, we provide insights into the underlying organizational principles of these algebraic systems.

Moreover, our exploration reveals the presence of special objects in the category of soft BCK/BCI-algebras, including terminal, initial, and zero objects. These findings contribute to a deeper understanding of the categorical properties and structural nuances inherent in soft BCK/BCI-algebras.

By uncovering and elucidating these categorical structures and properties, this work not only expands the theoretical foundations of soft BCK/BCI-algebras but also sets the stage for further exploration and applications in the broader context of algebraic structures and categorical theory.

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