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Eigenvalue Analysis and Simultaneous Nilpotence in Intuitionistic Fuzzy Matrices

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Abstract. Nilpotent intuitionistic fuzzy matrices are an important tool for analysis of intuitionistic fuzzy matrices. We first examine different nilpotent conditions of such matrices in terms of their eigenvalues in this research work. The notion of nilpotence is generalized to propose simultaneous nilpotence for a finite set of intuitionistic fuzzy matrices. Simultaneous nilpotence is the situation in which an infinite product of a finite number of intuitionistic fuzzy matrices approaches the zero matrix. The basic properties of this extended concept are also formulated.

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1. Introduction

Uncertainty is a key component of most areas of everyday life. Real-world issues in medicine, engineering, industry, and economics are frequently characterized by uncertain or imprecise data. Mathematical methods are not always at hand to overcome such difficulties, which prompted Zadeh [1] to advance the idea of fuzzy set theory a revolutionary instrument for modeling and analyzing uncertainty when classical methods were insufficient. Since then, fuzzy theory and its extensions have made enormous contributions to a variety of mathematical applications, providing successful approaches to manifold real-life problems with uncertainty. To address uncertainty problem, several extensions and variations of fuzzy set theory have been developed, including vague sets, rough sets, intuitionistic fuzzy sets, soft sets, and so on. A Fuzzy Matrix (FM) is a matrix whose entries are in the closed interval $[0, 1]$. FMs were introduced by Kim and Roush [2], and they

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have been widely studied ever since because of their universality of application in science and engineering and, in particular, in solving problems with uncertainty [3]. Meenakshi [4] investageted minus ordering, space ordering, and the Schur complement of FMs and block FMs. Ran and Liu [5], Buckley [6], and Gregory et al. [7] showed that, under the max–min operation, an FM converges to an idempotent matrix or else has finite periodic oscillation. Hashimoto [8] investigated the convergence property of powers of fuzzy transitive matrices. In recent years, numerous researchers have further developed the theory of FMs [9]. While FMs contain membership values alone, Intuitionistic Fuzzy Matrices (IFMs) consists membership and non-membership values and are a more comprehensive framework for uncertainty modeling. IFMs were first introduced by Khan et al. [10], and some of their properties have been studied further in [11]. Convergence of max–min powers of IFMs was researched by Bhowmik and Pal [12], whereas mean powers of convergence were examined by Pradhan and Pal [13]. Lur et al. [14] analyzed the behavior of convergence of IFM powers. Simplify eigenvalue analysis and characterize simultaneous nilpotence conditions in intuitionistic fuzzy matrices through polynomial expansions and kernel methods are given by [15–17]. Various researchers [18–29] have since further studied IFMs and acquired useful results that have continued to enhance their use in practical uncertainty problems. Xin [30] presented the concept of controllable FMs, while subsequently [31] examined the convergence of their powers and established results about nilpotent FMs. Upon this basis, this paper examines nilpotent IFM properties with respect to their eigenvalues using digraph, thus further developing theory regarding IFMs.

Research gap

While many aspects of IFMs have been explored, the eigenvalue properties of nilpotent intuitionistic fuzzy matrices have received little attention. Similarly, the concept of simultaneous nilpotence where an infinite product of matrices converges to a zero matrix has not been systematically studied. This oversight regarding the relationship between eigenvalues and simultaneous nilpotence creates a gap in the theoretical understanding and potential applications of intuitionistic fuzzy matrices, which this study seeks to address.

2. Preliminaries

Definition 1. *[32] An Intuitionistic Fuzzy Set (IFS) A in X is of the form* $A = \{\langle x, \mu_A(x),\ u_A(x)\rangle / x \in X\}$, *where:* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *denotes membership and non-membership function of the member $x \in X$, for all $x \in X : 0 \leq mu_A(x) + \nu_A(x) \leq 1$.*

Briefly we express $\langle x, x'\rangle$ as an IF member with $x + x' \leq 1$. For $\langle x, x'\rangle, \langle y, y'\rangle \in$ IFS, For comparable members, the operation $\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle$ is expressed as

$$\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle = \begin{cases} \langle x, x'\rangle & \text{if } \langle x, x'\rangle > \langle y, y'\rangle, \\ \langle 0, 1\rangle & \text{if } \langle x, x'\rangle \leq \langle y, y'\rangle. \end{cases}$$

Definition 2. *[10] An IFM is represented by* $A = (\langle (x_i, y_j), \mu_A(x_i, y_j),\ u_A(x_i, y_j)\rangle)$ *for* $i = 1, 2.m$ *and* $j = 1, 2, .n$*, where* $\mu_A : X \times Y \to [0, 1]$ *and* $\nu_A : X \times Y \to [0, 1]$ *have the property* $0 \le \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \le 1$*. IFM is a matrix can be written as* $A = (\left\langle a_{ij}, a'_{ij}\right\rangle)$ *such that* $a_{ij} + a'_{ij} \le 1$ *for all* i, j*.*

Definition 3. *[33] The determinant of intutionistic fuzzy matrix is give by*
$$|A| = \left[\left(\bigvee_{\sigma\in S_n} a_{1\sigma(1)} \wedge ... \wedge a_{n\sigma(n)}, \bigwedge_{\sigma\in S_n} a'_{1\ sigma(1)} \vee ... \vee a'_{n\sigma(n)}\right)\right],$$
where S_n *denotes all the permutations groups of the indices* $\{1, 2, ..., n\}$

Definition 4. *Let* $\mathcal{F} = \left\{A^{(1)}, A^{(2)}, ..., A^{(m)}\right\}$ *be a finte set in* F_n*. The IFMs* $\left\{A^{(1)}, A^{(2)}, ..., A^{(m)}\right\}$ *are said to be simultaneously nilpotent if* $\mathcal{F}^p = \{\langle 0, 1\rangle\}$ *for some* $p \in N$*.*

In this paper, the following definition and results are used.
Let $Q = [\left\langle q_{ij}, q'_{ij}\right\rangle]$ and $S = [\left\langle s_{ij}, s'_{ij}\right\rangle]$ be IFMs of order n with elements in $[0, 1] \times [0, 1]$

$Q \vee S = (\left\langle q_{ij} \vee s_{ij}, q'_{ij} \wedge s'_{ij}\right\rangle)$, where $\langle x, x'\rangle \vee \langle y, y'\rangle = max(\langle x, x'\rangle, \langle y, y'\rangle)$,

$Q \wedge S = [(\left\langle q_{ij} \wedge s_{ij}, q'_{ij} \vee s'_{ij}\right\rangle)]$ where $\langle x, x'\rangle \wedge \langle y, y'\rangle = min(\langle x, x'\rangle, \langle y, y'\rangle)$

$Q \overset{c}{\leftharpoondown} S = (\left\langle q_{ij}, q'_{ij}\right\rangle \overset{c}{\leftharpoondown} \left\langle s_{ij}, s'_{ij}\right\rangle)$

$Q \times S = [(\left\langle q_{i1} \wedge s_{1j}, q'_{i1} \vee s'_{1j}\right\rangle) \vee (\left\langle q_{i2} \wedge s_{2j}, q'_{i2} \vee s'_{2j}\right\rangle) \vee \ldots \vee (\left\langle q_{in} \wedge s_{nj}, q'_{in} \vee s'_{nj}\right\rangle)]$,

$Q^{k+1} = Q^k \times Q, \quad k = 1, 2, 3...$

Denote $Q^k = [\left\langle q^k_{ij}, q'^k_{ij}\right\rangle], \quad k = 2, 3, ...$

$$(\left\langle q^k_{ij}, q'^k_{ij}\right\rangle =$$
$$\left\langle \bigvee_{j_1=1}^{n} \bigvee_{j_2=1}^{n} \cdots \bigvee_{j_{k-1}=1}^{n} (q_{ij_1} \wedge q_{j_1 j_2} \wedge ... \wedge q_{j_{k-1}j}), \bigwedge_{j_1=1}^{n} \bigwedge_{j_2=1}^{n} \cdots \bigwedge_{j_{k-1}=1}^{n} (q'_{ij_1} \vee q'_{j_1 j_2} \vee ... \vee q'_{j_{k-1}j}) \right\rangle$$

$Q^0 = I$, (I_n is unit matrix) $Q^T = [\left\langle q_{ji}, q'_{ji}\right\rangle]$ (the transpose)
$\Delta Q = Q \overset{c}{\leftharpoondown} Q^T$, $\nabla Q = Q \wedge Q^T$
$Q \le S$ iff $(\left\langle q_{ij}, q'_{ij}\right\rangle \le \left\langle s_{ij}, s'_{ij}\right\rangle$ for all $i, j \in 1, 2, ..., n)$.
The IFM Q is known as nilpotent if $Q^n = (\langle 0, 1\rangle)$,
Controllable matrix $T = [\left\langle t_{ij}, t'_{ij}\right\rangle] = P \times Q \times P^T$ satisfies $\left\langle t_{ij}, t'_{ij}\right\rangle \ge \left\langle t_{ji}, t'_{ji}\right\rangle$ for $i > j$.

3. Eigenvalue and nilpotence

Definition 5. *Let* $(\langle a_{ij}, a'_{ij}\rangle)$ *be* $n \times n$ *IFM,* $\lambda \in [0, 1]$ *is known as eigenvalue of* $(\langle a_{ij}, a_{ij}\rangle)$ *if* $(\langle a_{ij}, a'_{ij}\rangle) \overset{c}{\leftharpoondown} \langle x, x'\rangle = \lambda \overset{c}{\leftharpoondown} \langle x, x'\rangle$ *for nonzero vector* $\langle x, x'\rangle = [\langle x_j, x'_j\rangle]$ *with* $\langle x_j, x'_j\rangle \in [0, 1]$*, namely,*
$$\bigvee_{j=1} (\langle a_{ij}, a'_{ij}\rangle \wedge \langle x_j, x'_j\rangle) = \lambda \wedge \langle x_i, x'_i\rangle, \text{ for all } i = 1, 2, ..., n.$$
Consider $\sigma(\langle a_{ij}, a'_{ij}\rangle)$ *set of all eigenvalues of* A *and let* $\rho(\langle a_{ij}, a'_{ij}\rangle) = sup\left\{\lambda | \lambda \in \sigma(\langle a_{ij}, a'_{ij}\rangle)\right\}$*.*

We will prove in Theorem 2 that for any IFM $\langle a_{ij}, a'_{ij}\rangle$, there exists a $\lambda \in \sigma(\langle a_{ij}, a'_{ij}\rangle)$. Thus, $\rho((\langle a_{ij}, a'_{ij}\rangle))$ is the largest eigenvalue of $(\langle a_{ij}, a'_{ij}\rangle)$.

Lemma 1. *Let $(\langle a_{ij}, a'_{ij}\rangle) \in F_n$. Then $(\langle a_{ij}, a'_{ij}\rangle)$ has a zero column $\Leftrightarrow \langle 0, 1\rangle \in \sigma((\langle a_{ij}, a'_{ij}\rangle))$*

Proof. Let ith column of $(\langle a_{ij}, a'_{ij}\rangle)$ is 0. And $\langle x, x'\rangle = e^i$, then $\langle x, x'\rangle$ is an eigenvector corresponding to the $\langle 0, 1\rangle$.
Let $\langle x, x'\rangle = (\langle x_i, x'_i\rangle)$ be an eigenvector corresponding to the eigenvalue $\langle 0, 1\rangle$. Suppose that $\langle x_i, x'_i\rangle \neq \langle 0, 1\rangle$. Then

$$(\langle a_{ij}, a'_{ij}\rangle) \overset{c}{\leftharpoondown} \langle x, x'\rangle = \begin{bmatrix} \bigvee_{j=1}^{n} (\langle a_{1j}, a'_{1j}\rangle \wedge \langle x_j, x'_j\rangle \\ \vdots \\ \bigvee_{j=1}^{n} (\langle a_{nj}, a'_{nj}\rangle \wedge \langle x_j, x'_j\rangle \end{bmatrix} = O$$

Thus, $\langle a_{ki}, a'_{ki}\rangle \wedge \langle x_i, x'_i\rangle = \langle 0, 1\rangle$ for all k. Since $\langle x_i, x'_i\rangle \neq \langle 0, 1\rangle$,
we have $\langle a_{ki}, a'_{ki}\rangle = \langle 0, 1\rangle$ for every k. Hence ith column of $(\langle a_{ij}, a'_{ij}\rangle)$ is $\langle 0, 1\rangle$

Lemma 2. *Let $\langle a_{ij}, a'_{ij}\rangle \in F_n$. Then $\rho(\langle a_{ij}, a'_{ij}\rangle)$ is $\langle 0, 1\rangle$ or $\langle 1, 0\rangle$.*

Proof. If $\sigma(\langle a_{ij}, a'_{ij}\rangle) = \langle 0, 1\rangle$, then $\rho(\langle a_{ij}, a'_{ij}\rangle) = \langle 0, 1\rangle$. Otherwise, if $\exists\ \langle 0, 1\rangle \neq \lambda \in \sigma(\langle a_{ij}, a'_{ij}\rangle)$, then for nonzero eigenvector $\langle x, x'\rangle$ we have $(\langle a_{ij}, a'_{ij}\rangle) \overset{c}{\leftharpoondown} \langle x, x'\rangle = \lambda \overset{c}{\leftharpoondown} \langle x, x'\rangle$. For any β with $\lambda \leq \beta \leq \langle 0, 1\rangle$, we have
$\langle a_{ij}, a'_{ij}\rangle \overset{c}{\leftharpoondown} (\lambda \overset{c}{\leftharpoondown} \langle x, x'\rangle) = \lambda \overset{c}{\leftharpoondown} \langle x, x'\rangle = \beta \overset{c}{\leftharpoondown} (\lambda \overset{c}{\leftharpoondown} \langle x, x'\rangle)$.
Hence, $\beta \in \sigma(\langle a_{ij}, a'_{ij}\rangle \Rightarrow \rho(\langle a_{ij}, a'_{ij}\rangle) = \langle 1, 0\rangle$

Lemma 3. *Let $A = (\langle a_{ij}, a'_{ij}\rangle)$ and $B = (\langle b_{ij}, b'_{ij}\rangle) \in F_n$. If $(\langle a_{ij}, a'_{ij}\rangle) \leq (\langle b_{ij}, b'_{ij}\rangle)$, then $\rho(\langle a_{ij}, a'_{ij}\rangle) \leq \rho(\langle b_{ij}, b'_{ij}\rangle)$.*

Proof. By Lemma 2 , $\rho(\langle a_{ij}, a'_{ij}\rangle)$ is either $\langle 0, 1\rangle$ or $\langle 1, 0\rangle$. If $\rho(\langle a_{ij}, a'_{ij}\rangle) = \langle 0, 1\rangle$, the result is trivial. Now, we will show that $\rho(\langle a_{ij}, a'_{ij}\rangle) = \langle 1, 0\rangle \Rightarrow \rho(\langle b_{ij}, b'_{ij}\rangle) = \langle 1, 0\rangle$. Let $(\langle a_{ij}, a'_{ij}\rangle) \overset{c}{\leftharpoondown} (\langle x, x'\rangle) = (\langle x, x'\rangle)$, $(\langle x, x'\rangle) \neq O$. Then $(\langle x, x'\rangle) \leq (A^n) = ((\langle a^n_{ij}, a'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} e$ (by Lemma 2.10 [34]) , where $e = [1, 1, ..., 1]$ and so $(\langle a_{ij}, a'_{ij}\rangle) \leq ((\langle a^n_{ij}, a'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} e \leq (B^n) = ((\langle b^n_{ij}, b'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} e$ (because $(\langle a_{ij}, a'_{ij}\rangle) \leq (\langle b_{ij}, b'_{ij}\rangle)$. Since $\langle x, x'\rangle \neq O$, since $((\langle b^n_{ij}, b'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} e \leq O$. Let $\langle y, y'\rangle = ((\langle b^n_{ij}, b'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} e$. Then $(\langle b_{ij}, b'_{ij}\rangle) \overset{c}{\leftharpoondown} \langle y, y'\rangle = ((\langle b^{n+1}_{ij}, b'^{n+1}_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} e = ((\langle b^n_{ij}, b'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}})e = \langle y, y'\rangle$ (By Lemma 2.9 [34]) and so, $\rho(\langle b_{ij}, b'_{ij}\rangle) = \langle 1, 0\rangle$.

Theorem 1. *Let $(\langle a_{ij}, a'_{ij}\rangle) \in F_n$.*
(1) $(\langle a_{ij}, a'_{ij}\rangle)$ is nilpotent .
(2) $(\langle a_{ij}, a'_{ij}\rangle)^p_{\overset{c}{\leftharpoondown}} = O$ for some integer p.
(3) $\sigma((\langle a_{ij}, a'_{ij}\rangle)) = \{\langle 0, 1\rangle\}$
(4) $\rho((\langle a_{ij}, a'_{ij}\rangle)) = \langle 0, 1\rangle$

Proof. Let (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).In these sequence, (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial. Now we will prove (2) $\Rightarrow$ (3). Suppose $\lambda \in \sigma((\langle a_{ij}, a'_{ij}\rangle))$ is a eigenvalue which is nonzero. Then $(\langle a_{ij}, a'_{ij}\rangle) \overset{c}{\leftharpoondown} \langle x, x'\rangle = \lambda \overset{c}{\leftharpoondown} \langle x, x'\rangle$, where $\langle x, x'\rangle$ is a corresponding eigenvector. Implies that
$O = (((\langle a^p_{ij}, a'^p_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} \langle x, x'\rangle = \lambda^p \overset{c}{\leftharpoondown} \langle x, x'\rangle = \lambda \overset{c}{\leftharpoondown} \langle x, x'\rangle \neq O$,
which is not possible. Now, we will prove that $(d) \Rightarrow (a)$. Assume that $(\langle a^n_{ij}, a'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}} \neq O$.
Then $(((\langle a^n_{ij}, a^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} e \neq O$.
Let $\langle y, y'\rangle = (((\langle a^n_{ij}, a'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}}) \overset{c}{\leftharpoondown} e$.
Then $(\langle a_{ij}, a'_{ij}\rangle) \overset{c}{\leftharpoondown} \langle y, y'\rangle = (((\langle a^{n+1}_{ij}, a^{n+1}_{ij}\rangle)_{\overset{c}{\leftharpoondown}})e = (((\langle a^n_{ij}, a'^n_{ij}\rangle)_{\overset{c}{\leftharpoondown}})e = (\langle y, y'\rangle)$, and so $\rho((\langle a_{ij}, a'_{ij}\rangle)) = \langle 1, 0\rangle$, which is contradiction.

Theorem 2. *Let $(\langle a_{ij}, a'_{ij}\rangle) \in F_n$. Then the set $\sigma((\langle a_{ij}, a'_{ij}\rangle))$ has following properties.*
(1) $(\langle a_{ij}, a'_{ij}\rangle)$ is nilpotent iff $\sigma((\langle a_{ij}, a'_{ij}\rangle)) = \langle 0, 1\rangle$.
(2) $(\langle a_{ij}, a'_{ij}\rangle)$ is non nilpotent, every column of $(\langle a_{ij}, a'_{ij}\rangle)$ is $\neq 0$ iff $\sigma((\langle a_{ij}, a'_{ij}\rangle)) = (0, 1]$
(3) $(\langle a_{ij}, a'_{ij}\rangle)$ is non nilpotent, contains at least one zero column iff $\sigma((\langle a_{ij}, a'_{ij}\rangle)) = [0, 1]$

Proof. Condition (1) has been shown in Theorem 1. (2) By Lemma 1 and condition (1), $\rho((\langle a_{ij}, a'_{ij}\rangle)) = \langle 1, 0\rangle$.
Let $\langle 1, 0\rangle$ is the eigenvalue of $(\langle a_{ij}, a'_{ij}\rangle)$ with nonzero eigenvalue $\langle x, x'\rangle$. For $\langle 0, 1\rangle < \langle \beta, \beta'\rangle \leq \langle 1, 0\rangle$ since,
$(\langle a_{ij}, a'_{ij}\rangle)(\langle \beta, \beta'\rangle\langle x, x'\rangle) = \langle \beta, \beta'\rangle\langle x, x'\rangle = \langle \beta, \beta'\rangle(\langle \beta, \beta'\rangle\langle x, x'\rangle)$.
Thus $[0, 1] \subset \sigma((\langle a_{ij}, a'_{ij}\rangle))$. Every column of $(\langle a_{ij}, a'_{ij}\rangle)$ is $\neq 0$, by Lemma 1 i.e $(0, 1] = \sigma((\langle a_{ij}, a'_{ij}\rangle))$. Inverse follows from Lemma 1 and Properety (1). Proof of (3) similar to that of (2), expect by adding that $\langle 0, 1\rangle \in \sigma((\langle a_{ij}, a'_{ij}\rangle)$ because $(\langle a_{ij}, a'_{ij}\rangle)$ had a zero column. Thus $[0, 1] = \sigma((\langle a_{ij}, a'_{ij}\rangle))$.

Example 1. *Let* $(\langle a_{ij}, a'_{ij}\rangle) = \begin{pmatrix} \langle 0, 1\rangle & \langle 0.5, 0.5\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix}$,
$(\langle b_{ij}, b'_{ij}\rangle) = \begin{pmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix}$
be irreflexive, antisymmetric and w-transitive IFM. Now, let
$(\langle c_{ij}, c'_{ij}\rangle) = \begin{pmatrix} \langle 0, 1\rangle & \langle 0.5, 0.5\rangle \\ \langle 0, 1\rangle & \langle 0.5, 0.5\rangle \end{pmatrix}$.

We note that IFM $(\langle a_{ij}, a'_{ij}\rangle)$ is Nilpotent. Computation shows that $\sigma((\langle a_{ij}, a'_{ij}\rangle)) = \langle 0, 1\rangle$. It also shows by impilication operator $(\langle b_{ij}, b'_{ij}\rangle)$ and $(\langle c_{ij}, c'_{ij}\rangle)$ are not nilpotent IFM, Moreover, $\sigma((\langle b_{ij}, b'_{ij}\rangle)) = (0, 1]$ and $\sigma((\langle c_{ij}, c'_{ij}\rangle)) = [0, 1]$.

Theorem 3. *If $\mathcal{F} = \left\{(\langle a_{ij}, a'_{ij}\rangle)^{(1)}, (\langle a_{ij}, a'_{ij}\rangle)^{(2)}, ..., (\langle a_{ij}, a'_{ij}\rangle)^{(m)}\right\} \subset F_n$. Then*
1. $\mathcal{F}$ is simultaneously nilpotent
2. Each Principal minor of $(\langle m_{ij}, m'_{ij}\rangle)$ is $\langle 0, 1\rangle$ for all $(\langle m_{ij}, m'_{ij}\rangle) \in \bigcup_{k\geq 1} \mathcal{F}^k$
3. The digraph $\Gamma(\mathcal{F})$ is acyclic.

Proof. (1) $\Rightarrow$ (2). let that $\det((\langle m_{ij}, m'_{ij}\rangle)[\alpha]) \neq \langle 0,1\rangle$ for some $(\langle m_{ij}, m'_{ij}\rangle) \in \bigcup_{k\geq 1} \mathcal{F}^k$ and $[\alpha] = \alpha_1, \alpha_2, \alpha_3 ..., \alpha_l \subset 1, 2, 3, ...n$. Then $\exists$ permutation σ on $[\alpha]$ such that $[(\langle m_{ij}, m'_{ij}\rangle)]_{\alpha_i\sigma(\alpha_i)} \neq \langle 0,1\rangle$ for $i = 1, 2, ...l$. Let $(\beta_1, \beta_2 ..., \beta_r)$ cycle in σ. Then
$(\langle m_{ij}, m'_{ij}\rangle)_{\beta_i\beta_{i+1}} \neq \langle 0,1\rangle$ for $i = 1, 2, 3, ...r-1$
and $[(\langle m_{ij}, m'_{ij}\rangle)]\beta_r\beta_1 \neq \langle 0,1\rangle$
Hence, $[(\langle m_{ij}, m'_{ij}\rangle)^r_{\overset{c}{\leftharpoondown}}]_{\beta_i\beta_i} \neq \langle 0,1\rangle\ \forall\ i = 1, 2, ..., r-1$. That is $(\langle m_{ij}, m'_{ij}\rangle)^{nr}_{\overset{c}{\leftharpoondown}} \neq \langle 0,1\rangle$. Thus, $\mathcal{F}^{nq} \neq \{O\}$ for some q, and so $\mathcal{F}^n \neq \{O\}$ that contradicts.
(2) $\Rightarrow$ (3). Assume that $\Gamma(\mathcal{F})$ contains a cycle $\gamma(\upsilon_{\beta_1}, \upsilon_{\beta_2}, ..., \upsilon_{\beta_k}, \upsilon_{\beta_1})$. Then $\exists$ $(\langle a_{ij}, a'_{ij}\rangle)_1, ..., (\langle a_{ij}, a'_{ij}\rangle)_k$ in $\mathcal{F}$ such as
$[(\langle a_{ij}, a'_{ij}\rangle)_i]_{\beta_i\beta_{i+1}} \neq \langle 0,1\rangle$ for $i = 1, 2, ...k-1$. and
$[(\langle a_{ij}, a'_{ij}\rangle)_k]_{\beta_k\beta_1} \neq \langle 0,1\rangle$
That is $[(\langle a_{ij}, a'_{ij}\rangle)_1 \overset{c}{\leftharpoondown} ... \overset{c}{\leftharpoondown} (\langle a_{ij}, a'_{ij}\rangle)_k]_{\beta_1\beta 1} \neq \langle 0,1\rangle$ and hence,
$det((((\langle a_{ij}, a'_{ij}\rangle)_1 \overset{c}{\leftharpoondown} ... \overset{c}{\leftharpoondown} (\langle a_{ij}, a'_{ij}\rangle)_k)[\beta_1] = [(\langle a_{ij}, a'_{ij}\rangle)_1 \overset{c}{\leftharpoondown} ... \overset{c}{\leftharpoondown} (\langle a_{ij}, a'_{ij}\rangle)_k]_{\beta_1\beta_1} \neq \langle 0,1\rangle$.
(3) $\Rightarrow$ (1). Let $(\langle a_{ij}, a'_{ij}\rangle)_1, (\langle a_{ij}, a'_{ij}\rangle)_2 ..., (\langle a_{ij}, a'_{ij}\rangle)_n \in \mathcal{F}$. For every i, j, $\exists\ i_1 = i$, $i_2, ..., i_n, i_{n+1} = j$ as
$[(\langle a_{ij}, a'_{ij}\rangle)_1 \overset{c}{\leftharpoondown} ... \overset{c}{\leftharpoondown} (\langle a_{ij}, a'_{ij}\rangle)_n]_{ij} = [(\langle a_{ij}, a'_{ij}\rangle)_1]_{i_1 i_2} \bigwedge ... \bigwedge [(\langle a_{ij}, a'_{ij}\rangle)_n]_{i_n i_{n+1}}$.
Since $\{i_1, ..., i_{n+1}\} \subset \{1, ..., n\}$, there are $1 \leq r < s \leq n+1$ so that $i_r = i_s$. As $\Gamma(\mathcal{F})$ contains no cycles,
$[(\langle a_{ij}, a'_{ij}\rangle)_r]_{i_r i_{r+1}} \bigwedge ... \bigwedge [(\langle a_{ij}, a'_{ij}\rangle)_{s-1}]_{i_s} = \langle 0,1\rangle$
Hence,
$[(\langle a_{ij}, a'_{ij}\rangle)_1 \overset{c}{\leftharpoondown} ... \overset{c}{\leftharpoondown} (\langle a_{ij}, a'_{ij}\rangle)_n]_{ij} = [(\langle a_{ij}, a'_{ij}\rangle)_1]_{i_1 i_2} \bigwedge ... \bigwedge [(\langle a_{ij}, a'_{ij}\rangle)_n]_{i_n i_{n+1}} = \langle 0,1\rangle$
Therefore, $\mathcal{F}^n = \{O\}$.
Let $\mathcal{F} = \left\{(\langle a_{ij}, a'_{ij}\rangle)^{(1)}, (\langle a_{ij}, a'_{ij}\rangle)^{(2)} ... (\langle a_{ij}, a'_{ij}\rangle)^m\right\} \subset F^n$. It is clear $\Gamma(\mathcal{F})$ consists a directed path having length k iff there is
$(\langle a_{ij}, a'_{ij}\rangle)^{i_1}, (\langle a_{ij}, a'_{ij}\rangle)^{i_2}, ..., (\langle a_{ij}, a'_{ij}\rangle)^{i_k} \in \mathcal{F}$ so that $(\langle a_{ij}, a'_{ij}\rangle)^{i_1} \overset{c}{\leftharpoondown} (\langle a_{ij}, a'_{ij}\rangle)^{i_2}, ... \overset{c}{\leftharpoondown} A^{i_k} \neq \langle 0,1\rangle$. Thus, develop the theorem to find the index $h(\mathcal{F})$ of $\mathcal{F}$, where $h(\mathcal{F})$ is the least positive integer k so that $h(\mathcal{F}^n) = \{O\}$. And If $h(\mathcal{F}) = \{A\}$, then $h(\mathcal{F})$ is denoted by $h((\langle a_{ij}, a'_{ij}\rangle))$.

Theorem 4. *Let* $\mathcal{F} = \left\{(\langle a_{ij}, a'_{ij}\rangle)^{(1)}, (\langle a_{ij}, a'_{ij}\rangle)^{(2)}, ..., (\langle a_{ij}, a'_{ij}\rangle)^{(m)}\right\} \subset F_n$ *be a simultaneously nilpotent set. Then* $h(\mathcal{F}^k) = k \geq 2$ *iff the digraph* $\Gamma(\mathcal{F})$ *contains directed paths having* $k-1$ *length, but none of the directed paths having* k *lengths.*

Proof. Assume $\Gamma(\mathcal{F})$ consists directed paths of $k-1$ length and no directed path of k length. By definition of adjacency matrix of $\Gamma(\mathcal{F})$, the ij^{th} entry of A^l denotes the number of directed paths of length l from $v_i\ to\ v_j$. Thus, if there is a directed path of length $\leq k-1$, then $A^t \neq \langle 0,1\rangle$, for $1 \leq t \leq k-1$ and since there is no directed path of length k, so $A^t \neq \langle 0,1\rangle$ then $\Gamma(\mathcal{F}) = k$. Converse follows by following the above steps in reverse order.

Example 2. *Let*

$$\mathbf{B^{(1)}} = (\langle \mathbf{b_{ij}}, \mathbf{b'_{ij}} \rangle)^{\mathbf{(1)}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} \langle 0,1\rangle & \langle 0.1,0.9\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.1,0.9\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.1,0.9\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.1,0.9\rangle \end{pmatrix} \end{array}$$

$$\mathbf{B^{(2)}} = (\langle \mathbf{b_{ij}}, \mathbf{b'_{ij}} \rangle)^{\mathbf{(2)}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.01,0.99\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.01,0.99\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.01,0.99\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix} \end{array}$$

$$\mathbf{B^{(3)}} = (\langle \mathbf{b_{ij}}, \mathbf{b'_{ij}} \rangle)^{\mathbf{(3)}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.01,0.99\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.01,0.99\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix} \end{array}$$

$$\mathbf{B^{(4)}} = (\langle \mathbf{b_{ij}}, \mathbf{b'_{ij}} \rangle)^{\mathbf{(4)}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.01,0.99\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix} \end{array}$$

$$\mathbf{B^{(5)}} = (\langle \mathbf{b_{ij}}, \mathbf{b'_{ij}} \rangle)^{\mathbf{(5)}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix} \end{array}$$

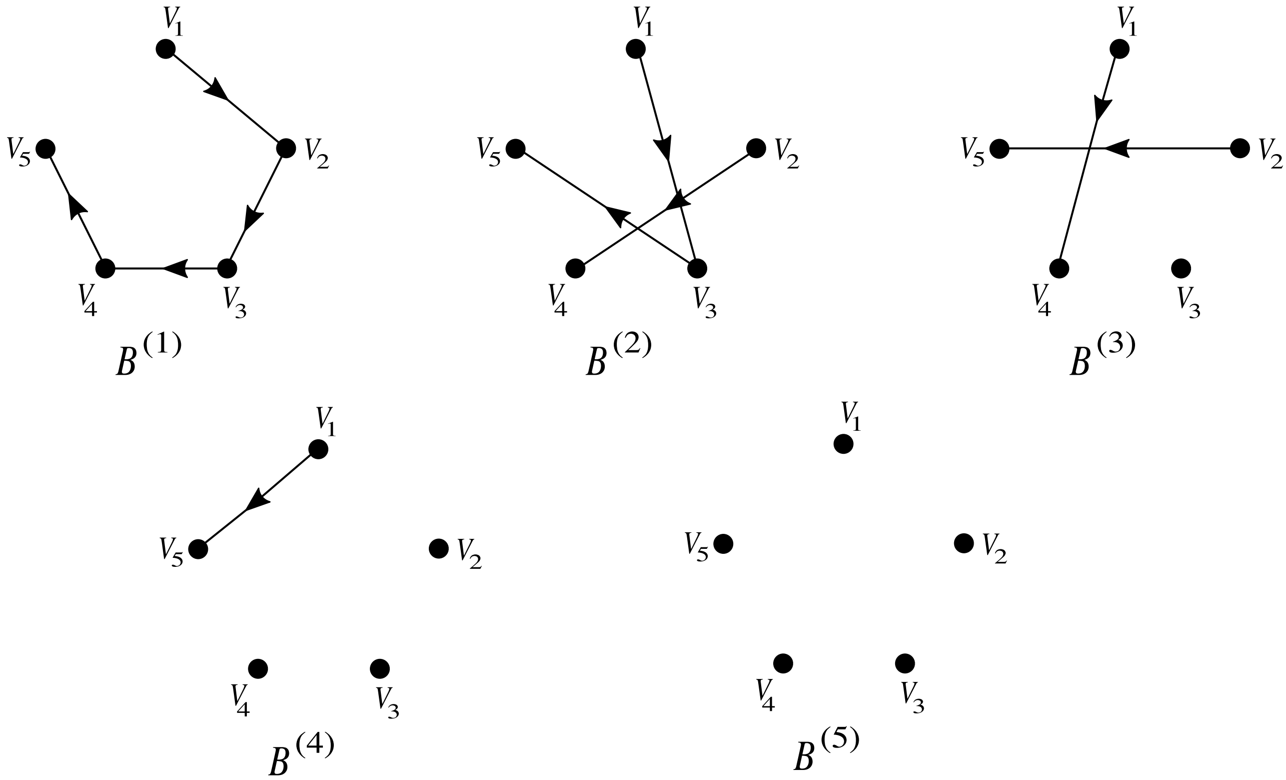

Figure 1: Simultaneous nilpotence.

Note: Zhang [35] studied the index $h(\mathcal{F})$ by using $E_{(\langle a_{ij},a'_{ij}\rangle)} = \Big\{(i,j)|\langle a_{ij},a'_{ij}\rangle \geq \langle 0,1\rangle\Big\}$ is the set of all directed edges in $\Gamma((\langle a_{ij},a'_{ij}\rangle))$,
$E^{(1)}_{(\langle a_{ij},a'_{ij}\rangle)} = \Big\{i|(i,j) \in E_{(\langle a_{ij},a'_{ij}\rangle)}\Big\}$
is the set of all initial vertices of directed edges in $\Gamma((\langle a_{ij},a'_{ij}\rangle))$,
and $E^2_2 = \Big\{j|(i,j) \in E_{(\langle a_{ij},a'_{ij}\rangle)}\Big\}$ is the set of all end vertices of directed edges in $\Gamma((\langle a_{ij},a'_{ij}\rangle))$.
Similarly we can extend with simultaneous nilpotence.
Let $\mathcal{F} = \Big\{(\langle a_{ij},a'_{ij}\rangle)^{(1)}, (\langle a_{ij},a'_{ij}\rangle)^{(2)}, ..., (\langle a_{ij},a'_{ij}\rangle)^{(m)}\Big\} \subset F^{n\times n}$ be a simultaneously nilpotent set .
Define $E_{\mathcal{F}}, E^{(1)}_{\mathcal{F}}, E^{(2)}_{\mathcal{F}}$ by $E_{\mathcal{F}} = \Big\{(i,j) : \langle a_{ij},a'_{ij}\rangle \neq \langle 0,1\rangle \ \ for\ some\ (\langle a_{ij},a'_{ij}\rangle) = [\langle a_{ij},a'_{ij}\rangle] \in \mathcal{F}\Big\}$;
$E^{(1)}_{\mathcal{F}} = \Big\{i : \langle a_{ij},a'_{ij}\rangle \neq \langle 0,1\rangle \ \ for\ \ some\ (\langle a_{ij},a'_{ij}\rangle) = [\langle a_{ij},a'_{ij}\rangle] \in\Big\}$;
$E^{(1)}_{\mathcal{F}} = \Big\{i : \langle a_{ij},a'_{ij}\rangle \neq \langle 0,1\rangle \ \ for\ some\ (\langle a_{ij},a'_{ij}\rangle) = [\langle a_{ij},a'_{ij}\rangle] \in\Big\}$.

Theorem 5. *Let $(\langle a_{ij},a'_{ij}\rangle)$ be IFM . Then*
(1) $(\langle a_{ij},a'_{ij}\rangle)$ is nilpotent IFM.
(2) The digraph $\Gamma((\langle a_{ij},a'_{ij}\rangle))$ is acyclic. (3) Each Principal minor of $(\langle a_{ij},a'_{ij}\rangle)$ is $\langle 0,1\rangle$
(4) Every Principal minor of $(\langle a_{ij},a'_{ij}\rangle)^n$ is $\langle 0,1\rangle$, $n = 1,2,..$

Proof. Proof follows directly follows from theorem 4.

4. Conclusion

This paper explores the issue of nilpotent intuitionistic fuzzy matrices. Since all intuitionistic fuzzy matrices have eigenvalues, we examined their potential sets and concluded that they could be divided into three categories:$\langle 0,1\rangle, (0,1], [0,1],$. It is also demonstrated that nilpotence is specially defined by the eigenvalue $\langle 0,1\rangle$. With finite product of IFMs, the notion of simultaneous nilpotence arises, meaning the infinite products of a finite number of such matrices strongly converge to the zero matrix, this characteristic is further defined using principal minors and directed graphs. The theoretical findings have possible applications in multi-criteria decision-making, stability analysis of fuzzy dynamical systems, network modeling, control theory, and intelligent decision-support systems, where uncertainty and convergence phenomena play a dominant role. Future work could generalize these results to higher-dimensional fuzzy structures, study computational algorithms for determining simultaneous nilpotence, and examine practical applications in medical diagnosis, supply chain management, and machine learning.

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