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Stability Analysis of Fuzzy Generalized Abel Integral Equations Using Homotopy Perturbation Method

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Abstract. In recent years, many more numerical methods were used to solve integral equations due to its fundamental importance in various scientific phenomena. In this paper, note-worthy algorithm is framed for generalized fuzzy Abel integral equation of the first kind based on the homotopy perturbation method. The stability of fuzzy approximate solution of it under the presence of small perturbation function in the initial fuzzy approximation is analyzed. A detailed description of the proof is provided to validate the efficiency of the applied technique and to show that this method provides accurate results for the problem under consideration.

2020 Mathematics Subject Classifications: 03B52, 45E10

Key Words and Phrases: Fuzzy Abel integral equations, parametric forms, stability, homotopy perturbation method, approximate solutions.

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1. Introduction

Fuzzy integral equations are important for studying and solving a large proportion of the problems in many topics in applied mathematics, particularly in relation to physics, geographic, medical, biology, etc. The concept of fuzzy numbers and arithmetic operations on fuzzy numbers was introduced by Zadeh [1]. Further enrichment is made by several authors among which a significant contribution is made by Dubois and Prade [2], who

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have introduced the concept of LR fuzzy number. Moreover, they deduced a computational formula for operations on fuzzy numbers and also presented the concept of integration of fuzzy function in [3]. Ideas of fuzzy mapping and fuzzy function is also discussed in [4]. Later Goetschel and Voxman [5] preferred a Riemann integral type approach, Kaleva [6] chose the definition of integral of fuzzy function using the Lebesgue-type concept for integration. One of the first applications of fuzzy integration was given by Wu and Ma [7]. Many problems in physics like reconstruction of the radially distributed emissivity from the line-of-sight projected intensity, the 3-D image reconstruction from cone beam projections in computerized tomography, etc. lead naturally in the case of radial symmetry, to the study of Abel's type integral equation.

Usually, physical quantities accessible to measurement are quite often related to physically important but experimentally inaccessible ones by Abel's integral equation. Abel integral equation occurs in many branches of scientific fields, such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation. The significance of this topic in most of the applied areas leads to deep exploration and investigation [8–10]. As most of the integral equations that govern the physical world have no closed form solutions in general, the appropriate way is to employ the computational approach to solve them and is a crucial work in scientific research. Various methods for solving the integral equation [11–14] are readily available in the open literature. In the recent years, the homotopy perturbation method (HPM) has been extensively used to solve several linear and nonlinear equations. HPM developed by Ji Huan He [10, 11] is a powerful mathematical tool to investigate a wide variety of problems arising in different fields[15–19]. It is obtained by successfully coupling homotopy theory in topology with perturbation theory. In HPM a complicated problem is easily solved to obtain an analytic or approximate solution. Considering all these specifications, the primary motive of this paper is to take advantage of homotopy perturbation method and to reduce the complexity of the numerical procedure, We aim to develop an algorithm to invert linear fuzzy Abel integrals of the first kind.

Fuzzy integral equations pose difficulties due to parametric fuzzy representations, uncertainty propagation through integrals, and numerical stability issues, while conventional methods become cumbersome because of nonlinear fuzzy arithmetic and complex fuzzy kernels. Homotopy perturbation method provides a systematic and efficient framework to address these issues by transforming the complex fuzzy integral equation into a series of simpler subproblems. Its iterative structure ensures rapid convergence and allows the uncertainty associated with fuzzy parameters to be incorporated seamlessly, thereby simplifying the computational effort while preserving accuracy. The structure of the paper is organized as follows: Section 2, some basic definitions and preliminary results that will be used further are provided. In section 3, introduction to fuzzy Abel integral equations is presented. Section 4 description on the basic idea of homotopy perturbation is given. In section 5, implementation of our proposed applicable algorithm for solving fuzzy Abel integral equation is analyzed. Stability of fuzzy approximate solution is analyzed in section 6. In section 7, application of presented technique to numerical example is illustrated. Finally the conclusion is drawn in section 8.

2. Preliminaries

We now recall some basic definitions that are needed throughout this paper

Definition 1. *[6] A fuzzy number is map* $\nu : \mathbb{R}^1 \to I = [0,1]$ *which satisfies*
i. $\nu(x)$ *is upper semi-continuous on* $\mathbb{R}$,
ii. $\nu(x) = 0$ *outside some interval* $[c,d] \subset \mathbb{R}$,
iii. There exists real numbers a, b *such that* $c \leq a \leq b \leq d$, *where*

(i). $\underline{\nu}(x)$ *is monotonic increasing on* $[c,d]$,

(ii). $\overline{\nu}(x)$ *is monotonic decreasing on* $[b,d]$,

(iii. $\nu(x) = 1, a \leq x \leq b$.

The set of all such fuzzy numbers is represented by R_F.

Definition 2. *[6] Let* V *be a fuzzy set on* R. V *is called a fuzzy interval if:*

i. V *is normal: there exists* $x_0 \in R$ *such that* $V(x_0) = 1$,

ii. V *is convex for all* $x, t \in R$ *and* $0 \leq \lambda \leq 1$, *it holds that* $V(\lambda x + (1-\lambda)t) \geq min\{V(x), V(t)\}$,

iii. V *is upper semi-continuous: for any* $x_0 \in R$, *it hold that* $V(x_0) \geq \lim_{x \to 0^{\pm}} V(x)$,

iv. $[V]^{\alpha} = Cl\{x \in R \mid V(x) > 0$ *is a compact subset of* R.

The $\alpha-$*cut of a fuzzy interval* V *with* $0 < \alpha \leq 1$ *is the crisp set* $[V]^{\alpha} = \{x \in R \mid V(x) > 0\}$. *For a fuzzy set* V, *its* $\alpha-$*cuts are closed intervals in* R. *and they are usually denoted by* $[V]^{\alpha} = [\underline{v}(\alpha), \overline{v}(\alpha)]$.

Definition 3. *[20] An arbitrary fuzzy number in parametric form is represented by an ordered pair of functions* $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, *which satisfies the following requirements:*

i. $\underline{u}(\alpha)$ *is a bounded left-continuous non-decreasing function over* $[0,1]$,

ii. $\overline{u}(\alpha)$ *is a bounded left-continuous non-increasing function over* $[0,1]$,

iii. $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$.

For arbitrary $u = (\underline{u}(\alpha), \overline{u}(\alpha))$, $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ *and* $k \in \mathbb{R}$ *we define addition and multiplication by* k *as*

$$\begin{aligned}
&\underline{\underline{(u+v)}}(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha),\\
&\overline{(u+v)}(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha),\\
&\underline{(ku)}(\alpha) = k\underline{u}(\alpha),\ \overline{(ku)}(\alpha) = k\overline{u}(\alpha),\ k \geq 0,\\
&\underline{(ku)}(\alpha) = k\overline{u}(\alpha),\ \overline{(ku)}(\alpha) = k\underline{u}(\alpha),\ k \geq 0.
\end{aligned}$$

Definition 4. *[21] A fuzzy real valued function* $\tilde{f} : [a, b] \to R_F$ *is said to be continuous in* $x_0 \in [a, b]$ *if for each* $\epsilon > 0$ *there is* $\delta > 0$ *such that* $D(\tilde{f}(x), \tilde{f}(x_0)) < \epsilon$, *whenever* $x \in [a, b]$ *and* $|x - x_0| < \delta$. *We say that* f *is continuous at each* $x_0 \in [a, b]$ *and denote the space of all such functions by* $C_F([a, b])$.

Theorem 1. *[22] Let f (x) be a fuzzy value function on* $[a, \infty]$ *and it is rerpresented by* $(\underline{f}(x, \alpha), \overline{f}(x, \alpha))$. *For any fixed* $\alpha \in [0, 1]$, *assume that* $\underline{f}(x, \alpha)$ *and* $\overline{f}(x, \alpha)$ *are Riemann-integrable on* $[a, b]$ *and for every* $b \geq a$ *and assume there are two positive functions* $\underline{M}(x)$ *and* $\overline{M}(x)$ *such that* $\int\limits_a^b \mid (\underline{f}(x, \alpha) \mid \mathrm{d}x \leq \underline{M}(x)$ *and* $\int\limits_a^b \mid (\overline{f}(x, \alpha) \mid \mathrm{d}x \leq \overline{M}(x)$ *for every* $b \geq a$ *Then* $\tilde{f}(x)$ *is improper Riemann-integrable on* $[a, \infty]$ *and the improper fuzzy Riemann-integral is a fuzzy number. Further, we have*

$$\int\limits_a^\infty \tilde{f}(x)\mathrm{d}x = \left(\int\limits_a^\infty \underline{f}(x, \alpha)\mathrm{d}x, \int\limits_a^\infty \overline{f}(x, \alpha)\mathrm{d}x \right)$$

3. Fuzzy Abel integral equations

The abel integral equation [8] is given by

$$f(x) = \int_a^x \frac{F(t)}{(x-t)^\mu}\mathrm{d}t; a \leq x \leq b \tag{1}$$

where α is a known constant such that $0 < \alpha < 1$, $f(x)$ is a predetermined data function and $F(x)$ is unknown function that will be determined. The expression $(x - t)^{-\mu}$ is called the kernel of the Abel integral equation or simply Abel kernel, that is singular as $t \to x$. If f(x) is a crisp function, then the solution of the Eq.(1) are crisp. However if $f(x)$ is a fuzzy function, these equations may only possess fuzzy solutions.Introducing the parametric forms of $f(x)$ and $F(x)$, we have the parametric forms of fuzzy Abel integral equation as follows:

$$(\underline{f}(x, \alpha), \overline{f}(x, \alpha)) = \left(\int_a^x \frac{\underline{F}(t, \alpha)}{(x-t)^\mu}\mathrm{d}t, \int_a^x \frac{\overline{F}(t, \alpha)}{(x-t)^\mu}\mathrm{d}t \right) \tag{2}$$

where $0 \leq \alpha \leq 1$ and α is a known constant such that $0 < \alpha < 1$, $\tilde{f} = (\underline{f}(x, \alpha), \overline{f}(x, \alpha))$ and $\tilde{F} = (\underline{F}(x, \alpha), \overline{F}(x, \alpha))$ is solution that will be determined. By putting $\alpha = 1/2$ in Eq.(2), we obtain the standard form of the nonlinear Abel fuzzy integral equation as

$$(\underline{f}(x, \alpha), \overline{f}(x, \alpha)) = \left(\int_a^x \frac{\underline{G}(t, \alpha)}{\sqrt{x-t}}\mathrm{d}t, \int_a^x \frac{\overline{G}(t, \alpha)}{\sqrt{x-t}}\mathrm{d}t \right) \tag{3}$$

where the function $(\underline{f}(x, \alpha), \overline{f}(x, \alpha))$ is a given real-valued function where $(\underline{G}(t, \alpha), \overline{G}(t, \alpha)) = (\underline{F}^n(t, \alpha), \overline{F}^n(t, \alpha))$ is a nonlinear function of The unknown function occurs only inside the integral sign for the Abel fuzzy integral Eq.(3).

4. Homotopy perturbation technique

The essential idea of homotopy perturbation method[10, 15, 20] is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0$, the system of equation usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of deformation, the solution of each of which is close to that at the previous stage of deformation. Eventually at $p = 1$, the system takes the original form of the equation and final stage of deformation gives the desired solution. To illustrate HPM, consider the nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{4}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{5}$$

where $A(u) = L(u) + N(u)$, L is a linear operator, N is a nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function. In order to use the HPM, a suitable construction of homotopy is of vital importance. He [23, 24] constructed a homotopy $U : \Omega \times [0, 1]$ that satisfies

$$H(U, p) = (1 - p)[L(U) - L(u_0)] + p[A(U) - f(r)] = 0 \tag{6}$$

or

$$H(U, p) = L(U) - L(u_0) + p[L(u_0) + N(U) - f(r)] = 0 \tag{7}$$

where $r \in \Omega$ and $p \in [0, 1]$ is called homotopy parameter and u_0 is an initial approximation of Eq.(4). It is obvious that

$$H(U, 0) = L(U) - L(u_0) = 0, \quad H(U, 1) = A(u) - f(r) = 0 \tag{8}$$

and the changing process of p from 0 to 1, is just that of $H(U, p)$ from $L(U) - L(u_0)$ to $A(U) - f(r)$ and this deformation is called homotopy in topology. Applying HPM, the solution of Eq.(6) or Eq.(7) can be expressed as a series in p, where $0 \le p \le 1$, is

$$u = u_0 + p^1 u_1 + p^2 u_2 + \dots \tag{9}$$

When $p \to 1$, Eq.(6) or Eq.(7) corresponds to Eq.(4) and becomes the approximate solution of Eq.(4), i.e.,

$$U = lim_{p \to 1} u = \sum_{n=0}^{\infty} = u_n = u_0 + u_1 + u_2 + \dots \tag{10}$$

The above series is convergent for most cases and the rate of convergence depends on [24]. The convergence of the homotopy perturbation method for solving the fuzzy Abel integral equation is ensured under mild conditions on the fuzzy kernel and coefficients. When the fuzzy kernel satisfies a Lipschitz-type condition and the fuzzy parameters are bounded, the successive approximations produced by HPM form a contractive sequence converging uniformly to the exact fuzzy solution. Similar convergence analyses for fuzzy integral equations have been established in related studies ([8, 15, 19]) confirming that HPM yields rapidly convergent series for nonlinear fuzzy systems under these assumptions.

5. Analysis of the proposed algorithm

Our objective of the presented paper is to propose an applicable algorithm to invert the fuzzy Abel integral equation (Eq.(2)) of the first kind with the solution $(\underline{F}(x,\alpha), \overline{F}(x,\alpha))$ by using the homotopy perturbation method (HPM). We now construct the following convex homotopy

$$\begin{cases} \underline{H}(u,p,\alpha) = (1-p)L(u) + p\left(\int_a^x \frac{\underline{F}(t,\alpha)}{(x-t)^{\mu}}\mathrm{d}t - \underline{f}(x,\alpha)\right) = 0 \\ \overline{H}(u,p,\alpha) = (1-p)L(u) + p\left(\int_a^x \frac{\overline{F}(t,\alpha)}{(x-t)^{\mu}}\mathrm{d}t - \overline{f}(x,\alpha)\right) = 0 \end{cases} \tag{11}$$

The operator $L(u)$ represents the linear part of the original fuzzy integral equation. In this work, $L(u)$ is chosen as a simple linear identity operator, that is, $L(u) = u$, which satisfies the property $L(u) = 0$ when $u = 0$. Here the embedding parameter $p \in [0,1]$ can be considered as an expanding parameter [8] to obtain the solution of above equation as

$$\begin{cases} \underline{u}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{u}_i(x,\alpha) \\ \overline{u}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{u}_i(x,\alpha) \end{cases} \tag{12}$$

where $(\underline{u}_i(x,\alpha), \overline{u}_i(x,\alpha))$, $i,j = 1,2,3,\ldots,n$ are the functions to be determined. We use the following numerical scheme to evaluate $(\underline{u}_i(x,\alpha), \overline{u}_i(x,\alpha))$. Substitute Eq.(12) in Eq.(11) and comparing the like powers of p, we obtain

$$p^0 := \begin{cases} \underline{u}_0(x,\alpha) = 0 \\ \overline{u}_0(x,\alpha) = 0 \end{cases} \tag{13}$$

$$p^1 := \begin{cases} \underline{u}_1(x,\alpha) = \underline{f}(x,\alpha) \\ \overline{u}_1(x,\alpha) = \overline{f}(x,\alpha) \end{cases} \tag{14}$$

$$p^2 := \begin{cases} \underline{u}_2(x,\alpha) = \underline{u}_1(x,\alpha) - \int_a^x \frac{\underline{u}_1(x,\alpha)}{(x-t)^{\mu}}\mathrm{d}t \\ \overline{u}_2(x,\alpha) = \overline{u}_1(x,\alpha) - \int_a^x \frac{\overline{u}_1(x,\alpha)}{(x-t)^{\mu}}\mathrm{d}t \end{cases} \tag{15}$$

$$p^3 := \begin{cases} \underline{u}_3(x,\alpha) = \underline{u}_2(x,\alpha) - \int_a^x \frac{\underline{u}_2(x,\alpha)}{(x-t)^\mu}\mathrm{d}t \\ \overline{u}_3(x,\alpha) = \overline{u}_2(x,\alpha) - \int_a^x \frac{\overline{u}_2(x,\alpha)}{(x-t)^\mu}\mathrm{d}t \end{cases} \tag{16}$$

...!

$$p^n := \begin{cases} \underline{u}_n(x,\alpha) = \underline{u}_{n-1}(x,\alpha) - \int_a^x \frac{\underline{u}_{n-1}(x,\alpha)}{(x-t)^\mu}\mathrm{d}t \\ \overline{u}_n(x,\alpha) = \overline{u}_{n-1}(x,\alpha) - \int_a^x \frac{\overline{u}_{n-1}(x,\alpha)}{(x-t)^\mu}\mathrm{d}t \end{cases} \tag{17}$$

and so on. Thus the solution of Eq.(2) is given by

$$\begin{cases} \underline{F}(x,\alpha) = lim_{p\to 1}\underline{u}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{u}_i(x,\alpha) \\ \underline{F}(x,\alpha) = lim_{p\to 1}\overline{u}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{u}_i(x,\alpha) \end{cases} \tag{18}$$

6. Stability analysis

We present in this section, the general stability idea of the proposed applicable algorithm for solving fuzzy Abel integral equation of the first kind Eq.(2). We consider the stability of the solution components $(\underline{u}(x,\alpha), \overline{u}(x,\alpha))$ as given in Eq.(18) under the presence of a small perturbation in the function $(\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ which is used for initial fuzzy approximation as given in Eq.(13) is disturbed with the perturbation function $(\underline{\delta f}(x,\alpha), \overline{\delta f}(x,\alpha))$ where it is an unknown function relative to $(\underline{f}(x,\alpha), \overline{f}(x,\alpha))$.

Theorem 2. *The presence of the small perturbation function $(\underline{\delta f}(x,\alpha), \overline{\delta f}(x,\alpha))$ in the continuous fuzzy function alters the fuzzy approximate solution $(\underline{F}(x,\alpha), \overline{F}(x,\alpha))$ by an equivalent value to the solution of fuzzy Abel integral equations with initial fuzzy approximation equal to the perturbation function $((\underline{\delta f}(x,\alpha), \overline{\delta f}(x,\alpha))$ itself respectively.*

Proof: *Without loss of generality, let us assume $(\underline{F}(x,\alpha), \overline{F}(x,\alpha)) = (\underline{u}(x,\alpha), \overline{u}(x,\alpha))$ as the solution of Eq.(2) under the presence of a small perturbation in form of finite sequences given as follows. For computational convenience we write $(\underline{\delta f}(x,\alpha), \overline{\delta f}(x,\alpha)) = (\underline{\varepsilon}_1(x,\alpha), \overline{\varepsilon}_1(x,\alpha))$ Consequently the iterative scheme becomes*

$$\begin{cases} \underline{\tilde{u}}_0(x,\alpha) = 0 \\ \overline{\tilde{u}}_0(x,\alpha) = 0 \end{cases} \tag{19}$$

$$\begin{cases} \underline{\tilde{u}}_1(x,\alpha) = \underline{f}(x,\alpha) + \underline{\varepsilon}_1(x,\alpha) \\ \overline{\tilde{u}}_1(x,\alpha) = \overline{f}(x,\alpha) + \overline{\varepsilon}_1(x,\alpha) \end{cases} \tag{20}$$

$$\begin{cases} \underline{\tilde{u}}_2(x,\alpha) = \underline{u}_2(x,\alpha) + \underline{\varepsilon}_1(x,\alpha) \\ \overline{\tilde{u}}_2(x,\alpha) = \overline{u}_2(x,\alpha) + \overline{\varepsilon}_1(x,\alpha) \end{cases} \tag{21}$$

...!

$$\begin{cases} \underline{\tilde{u}}_n(x,\alpha) = \underline{u}_n(x,\alpha) + \underline{\varepsilon}_{n-1}(x,\alpha) \\ \overline{\tilde{u}}_n(x,\alpha) = \overline{u}_n(x,\alpha) + \overline{\varepsilon}_{n-1}(x,\alpha) \end{cases} \tag{22}$$

where $(\underline{u}_n(x,\alpha), \overline{u}_n(x,\alpha))$ *is given by Eq.(17) and*

$$\begin{cases} \underline{\varepsilon}_n(x,\alpha) = \underline{\varepsilon}_{n-1}(x,\alpha) - \displaystyle\int_a^x \frac{\underline{\varepsilon}_{n-1}(t,\alpha)}{(x-t)^A}\mathrm{d}t \\ \overline{\varepsilon}_n(x,\alpha) = \overline{\varepsilon}_{n-1}(x,\alpha) - \displaystyle\int_a^x \frac{\overline{\varepsilon}_{n-1}(t,\alpha)}{(x-t)^A}\mathrm{d}t \end{cases} \tag{23}$$

for $n = 2,3,4,\ldots,n$. *The iterative scheme as in Eqs. (19) to (22) is obtained by applying the homotopy perturbation expansion to both the original and perturbed fuzzy integral equations and then taking their difference. Starting with* $\tilde{u}_0 = 0$, *the first perturbed iterate becomes* $\tilde{u}_1 = f + \epsilon_1$, *and the subsequent terms follow* $\tilde{u}_{n+1} = u_n + \epsilon_n$ *(corrected from Eq. (22)). By subtracting the unperturbed and perturbed recursions and using the linearity of the integral operator, the error relation is obtained as shown in Eq. (23).*

$$(\underline{\tilde{F}}(x,\alpha), \overline{\tilde{F}}(x,\alpha)) = (\underline{\tilde{u}}(x,\alpha), \tilde{u}(x,\alpha)) = \left(\lim_{n\to\infty}\sum_{i=0}^{n} \underline{\tilde{u}}_i(x,\alpha), \lim_{n\to\infty}\sum_{i=0}^{n} \tilde{u}_i(x,\alpha)\right) \tag{24}$$

Therefore the inclusion of the small perturbation function term affects the solution by

$$\underline{\delta u}(x,\alpha), \overline{\delta u}(x,\alpha)) = (\underline{\tilde{u}}(x,\alpha) - \underline{u}(x,\alpha), (\overline{\tilde{u}}(x,\alpha) - \tilde{u}(x,\alpha)) \tag{25}$$

$$= \left(\lim_{n\to\infty}\sum_{i=0}^{n} ((\underline{\tilde{u}}_i(x,\alpha) - \underline{u}(x,\alpha))\,, \lim_{n\to\infty}\sum_{i=0}^{n} ((\tilde{u}_i(x,\alpha) - \overline{u}(x,\alpha))\right) \tag{26}$$

$$= \left(\lim_{n\to\infty}\sum_{i=0}^{n}(\underline{\varepsilon}_i(x,\alpha)), \lim_{n\to\infty}\sum_{i=0}^{n}(\overline{\varepsilon}_i(x,\alpha))\right) \tag{27}$$

where $(\underline{\varepsilon}_1(x,\alpha), \overline{\varepsilon}_1(x,\alpha)) = 0$. *From Eq.(27) we conclude that and are linked by the following generalized fuzzy Abel integral equation*

$$\begin{cases} \displaystyle\int_a^x \frac{\underline{\delta u}(t,\alpha)}{(x-t)^\mu}\mathrm{d}t = \underline{\delta f}(x,\alpha) \\ \displaystyle\int_a^x \frac{\overline{\delta u}(t,\alpha)}{(x-t)^\mu}\mathrm{d}t = \overline{\delta f}(x,\alpha) \end{cases} \tag{28}$$

As $(\underline{\delta f}(x,\alpha), \overline{\delta f}(x,\alpha))$ *is an unknown function and by taking the least upper bound for it and let us have*
$\left(\sup_{a\le x\le b}\left|\underline{f}(x,\alpha)\right| < \underline{\varepsilon}, (\sup_{a\le x\le b}\left|\overline{f}(x,\alpha)\right| < \overline{\varepsilon}\right)$*, then Eq.(28) reduces to*

$$\left(\int_a^x \frac{\delta\underline{u}(t,\alpha)}{(x-t)^\mu}\mathrm{d}t, \int_a^x \frac{\overline{\delta u}(t,\alpha)}{(x-t)^\mu}\mathrm{d}t\right) = (\underline{\varepsilon}, \overline{\varepsilon}) \tag{29}$$

Eq. (29) demonstrates that the perturbation in the solution, denoted by $(\delta \underline{u(x,\alpha)}, \overline{\delta u(x,\alpha)})$*, satisfies an Abel-type fuzzy integral equation with* $(\delta \underline{f(x,\alpha)}, \overline{\delta f(x,\alpha)})$ *as its input. By assuming that the supremum of* $|\delta f|$ *and* $|\overline{\delta f}|$ *is bounded by* $(\varepsilon, \overline{\varepsilon})$*, it follows that the resulting deviation in the solution is also bounded within the same order of magnitude. This indicates that small perturbations in the input produce proportionally small deviations in the output, confirming the sensitivity and robustness of the proposed algorithm with respect to input variations. While this does not establish stability in a strict Lyapunov sense, it confirms a form of bounded-input bounded-output behavior under fuzzy conditions, implying that the numerical scheme is stable in the sense of bounded error propagation, thereby it confirms the stability of the proposed applicable algorithm.*

7. Numerical illustration

Example 1:

$$\tilde{f}(x) = \int_a^x \frac{\tilde{F}(t)}{(x-t)^\mu}\mathrm{d}t; \quad 0 \le \alpha \le 1$$

where $\tilde{f} = (\underline{f}(x,\alpha), \overline{f}(x,\alpha)), a = 0, \mu = 1/2,$

$$\begin{cases} \underline{f}(x,\alpha) = \dfrac{4}{3}(ax^{3/2}) \\ \\ \underline{f}(x,\alpha) = \dfrac{4}{3}(2-\alpha)x^{3/2} \end{cases}$$

The exact solution in this case is given by

$$\begin{cases} \underline{u}(x,\alpha) = \underline{F}(x,\alpha) = \alpha x \\ \overline{u}(x,\alpha) = \overline{F}(x,\alpha) = (2-\alpha)x \end{cases}$$

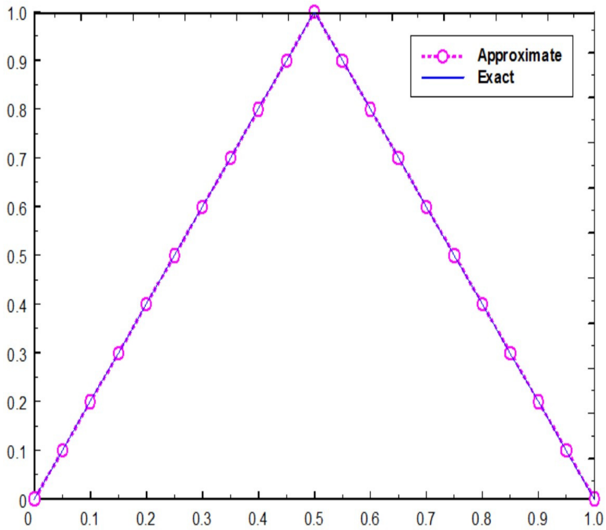
To solve the above equation by HPM, we construct a convex homotopy as follows

$$\begin{cases} \underline{H}(u,p,\alpha) = (1-p)L(u) + p\left(\displaystyle\int_0^x \frac{\underline{F}(t,\alpha)}{(x-t)^A}\mathrm{d}t - \frac{4}{3}(ax^{3/2})\right) = 0 \\ \\ \overline{H}(u,p,\alpha) = (1-p)L(u) + p\left(\displaystyle\int_0^x \frac{\overline{F}(t,\alpha)}{(x-t)^A}\mathrm{d}t - \frac{4}{3}(2-\alpha)x^{3/2}\right) = 0 \end{cases}$$

At this stage, we make use of the proposed numerical technique to find the fuzzy approximate solutions and the error analysis is given by

$$\underline{E}(x,\alpha) = |(\alpha x - \underline{u}(x,\alpha)|$$
$$\overline{E}(x,\alpha) = |(2-\alpha)x - \overline{u}(x,\alpha)|$$

Figure 1. Plots of exact and approximate solutions $(\underline{F}(x,\alpha), \overline{F}(x,\alpha))$



0

Table 1: Comparison between exact & approximate solutions at x= 0.5

α	*Exact solution*		*Approximate Solution*		*Error*	
	$\underline{F}(x,\alpha)$	$\overline{F}(x,\alpha)$	$\underline{F}(x,\alpha)$	$\overline{F}(x,\alpha)$	$\underline{F}(x,\alpha)$	$\overline{F}(x,\alpha)$
0	0	1.000000	0	0.999848	0	0.000152
0.1	0.050000	0.950000	0.049877	0.949717	0.000123	0.000283
0.2	0.100000	0.900000	0.099715	0.899658	0.000285	0.000342
0.3	0.150000	0.850000	0.149685	0.849571	0.000315	0.000429
0.4	0.200000	0.800000	0.199611	0.799546	0.000389	0.000454
0.5	0.250000	0.750000	0.249532	0.749466	0.000468	0.000534
0.6	0.300000	0.700000	0.299453	0.699372	0.000547	0.000628
0.7	0.350000	0.650000	0.349348	0.649174	0.000652	0.000826
0.8	0.400000	0.600000	0.399244	0.599928	0.000776	0.000072
0.9	0.450000	0.550000	0.449915	0.549942	0.000085	0.000058
1	0.500000	0.500000	0.499932	0.499964	0.000068	0.000036

We solved these equations and found the components of the above iterations by using Mathematica program. In this case, fuzzy approximate solutionsis calculated at four iterations and are given in Table 1. Figure 1 shows the graphical illustration of the obtained *approximate solution with the exact solution subject to the initial conditions. We compute the values for x=0.5 and it is worth pointing out that that the approximate solutions are almost nearer to the exact solutions due to the effective convergence of the solution series. In most cases, for the known function series, even the exact solution could be achieved.*

8. Conclusion

In this paper, we intend to propose an applicable algorithm for solving linear Abel fuzzy integral equations of the first kind. It was shown that this technique is easy to implement and produce accurate results. A considerable advantage of this method is that the fuzzy approximate solutions are found easily by using computer programs such as Mathematica. In stability analysis, it is proved that the change in the solution due to the inclusion of small perturbation term in the observable data is the solution of the generalized fuzzy Abel integral equation with the initial fuzzy approximation equal to the perturbation function itself. Numerical results confirm that our suggested method is a viable alternative to the existing numerical scheme for solving the problem under consideration. This algorithm can be further extended to higher order equations with some modifications.

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Statements and declarations

Conflict of interest

The authors have declared that they have no conflicts of interest for the publication of this paper.

Data availability

The authors have not used any associated data for the preparation of this manuscript.

Ethical Approval

The article does not contain any studies with human participants or animals performed by any of the authors.

Authors contributions

The authors confirm sole responsibility for contribution in the idea of the research article. Also all the authors have validated and approved the final manuscript.

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